SIAM J. NUMER. ANAL.
Vol. 36, No. 5, pp. 1528–1548

ERROR ESTIMATES FOR A COMBINED FINITE VOLUME–FINITE ELEMENT METHOD FOR NONLINEAR CONVECTION–DIFFUSION PROBLEMS*

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Abstract. The subject of this paper is the analysis of error estimates of the combined finite volume–finite element (FV–FE) method for the numerical solution of a scalar nonlinear conservation law equation with a diffusion term. Nonlinear convective terms are approximated with the aid of a monotone finite volume scheme considered over the finite volume mesh dual to a triangular grid, whereas the diffusion term is discretized by piecewise linear conforming triangular finite elements. Under the assumption that the exact solution possesses some regularity properties and the triangulations are of a weakly acute type, with the aid of the discrete maximum principle and a priori estimates, error estimates of the method are proved.

Key words. nonlinear convection-diffusion equation, monotone finite volume schemes, finite element method, numerical integration, discrete maximum principle, a priori estimates, error estimates, compressible Navier–Stokes equations

AMS subject classifications. 65M12, 65M60, 35K60, 76M10, 76M25

PII. S0036142997314695

1. Introduction.

Convection-diffusion processes appear in many areas of science and technology; e.g., fluid dynamics, heat and mass transfer, hydrology, and environmental protection. This is the reason that the numerical solution of convection-diffusion problems attracts a number of specialists. From an extensive literature devoted to linear problems let us mention the papers [1], [2], [3], [4], [7], [20], [23], [32], [34], [35], [40], [41], [42], and [43], monographs [22], [33], and [36], and the references therein. One approach to the solution of nonlinear problems can be found in [22, Chap. 5].

In the theory of weak solutions for partial differential equations in divergence form there are two roughly equivalent formulations in common use, namely, the functional formulations involving integration against smooth test functions versus the finite volume type over arbitrary control volumes. The former corresponds to energy methods and leads naturally to FE discretizations for elliptic and parabolic, i.e., diffusive, problems. The latter corresponds in a natural way to the physical formulation of the basic laws of conservation of mass, momentum, and energy in fluid mechanics leading directly to the well-known FV methods. However, it is not mandatory to adhere to these paths of discretization in their respective regimes of common use. The finite (control) volume method is known as the box method for elliptic problems [21], the finite element method is, on the other hand, applied to convection [23]. Often, the

*Received by the editors January 6, 1997; accepted for publication (in revised form) April 2, 1998; published electronically September 1, 1999. This research was supported by grant 201/96/0313 of the Czech Grant Agency and grant Wa 633/6-1 of Deutsche Forschungsgemeinschaft (DFG).

http://www.siam.org/journals/sinum/36-5/31469.html

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control volume approach is used in the framework of the FE methods for obtaining upwinding (see, e.g., [3], [4], [34], [37]). The FV upwind flux vector splitting schemes using numerical flux and based on the approximate solution of the Riemann problem represent a powerful tool for the numerical solution of nonlinear conservation laws, including the Euler equations describing inviscid flow (see, e.g., [27] or [11, Chap. 7] and the references therein).

In the work reported here we investigate the approach by trying to have the best of both worlds, i.e., the combination of finite volumes for inviscid conservation laws with finite elements for the diffusion. Our main goal is to develop a robust theoretically based numerical method for the solution of viscous compressible flow applied on unstructured meshes. In [13] we proposed numerical schemes for the solution of viscous gas flow based on the combination of the FV method for the discretization of inviscid convective terms and the FE method applied to the approximation of viscous terms. The numerical computations for the system of compressible viscous flow [15], [12], [6] have demonstrated that the combined FV–FE method is feasible and produces numerical results which are very promising. Unfortunately, the state of the art of theoretical analysis of these equations does not allow for a numerical analysis of the full problem. As is commonly done, we have to make a compromise by considering a simplified model which contains, as much as possible, of the flavor of the original problem while also allowing a numerical analysis. We confine our considerations to a scalar nonlinear conservation law equation with a diffusion term, which is the simplest prototype of the Navier–Stokes system describing viscous compressible flow. Nonetheless, numerical analysis for appropriate model problems such as this do enhance the confidence with which the method is applied to realistic flows.

In [14], the convergence of a combined FV–FE scheme was investigated. Here we will be concerned with the continuation of results from [14]. We will present the analysis of error estimates of the combined FV–FE scheme applied to an initial-boundary value problem for a scalar nonlinear conservation law equation with a diffusion term. The nonlinear convective terms are discretized by a monotone finite volume scheme on the barycentric finite volume mesh dual to a triangular grid of weakly acute type, whereas the diffusion term is approximated with the aid of conforming piecewise linear finite elements. With the use of results from [14], under some assumptions on the regularity of the exact solution of the continuous problem, we prove error estimates of the method.

The basic tools used in the investigation of error estimates to be presented here are the discrete maximum principle and a priori estimates of approximate solutions. The discrete maximum principle, implying the L^∞-estimate, is necessary for the control of nonlinear fluxes, since no growth conditions are imposed on them. That is why the analysis requires the use of triangulations of a weakly acute type. (It would also be possible to start from a Delaunay–Voronoi pair.) Moreover, for the estimate of the FV error in nonlinear inviscid fluxes, the inverse assumption is necessary. Both of these assumptions are quite common in a number of works, where numerical methods preserving the inverse monotonicity of the continuous problem are treated [5], [9], [22], [34], [35], [36, para. 3.1]. Finally, we suppose a certain regularity of the exact solution (much weaker than that from, e.g., [22, Chap. 5]).

The analysis leading to the error estimates in this paper must be seen as a first step to more general results. We are aware that some restrictive assumptions are in practical computations ignored and, therefore, call for further efforts in the future. Comments concerning open problems are given at the end of the paper.

An important issue in error estimates and, more generally, numerical methods for convection-diffusion equations involving a small diffusion parameter ν is their robustness in the singular limit $\nu \to 0$ (see, e.g., [36]). In view of [24], avoiding the exponential growth of error constants (resulting from the use of Gronwall's lemma) has become an interesting research problem, though this may be feasible only for special problems. These topics are well beyond the scope of this paper. Also we point out that the numerical method studied here was designed for fixed positive values for viscosity and heat conductivity. This is in some sense in agreement with the well-known fact that for very small viscosity the flow becomes turbulent and models used for turbulence modeling cause the increase of the magnitude of the diffusion parameters by adding the so-called turbulent viscosity and turbulent heat conductivity. So the singular limit is not of urgent interest here.

2. Formulation of the problem. We will denote by $\mathbb{R}^n$ the n-dimensional Euclidean space equipped with the norm $|\cdot|$. By x_1, x_2, and t we denote the Cartesian coordinates of points $x \in \mathbb{R}^2$ and time, respectively. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. (Hence, its boundary $\partial\Omega$ is Lipschitz-continuous.) In the space-time cylinder $Q_T = \Omega \times (0,T)$ $(0 < T < \infty)$ we will consider the following *initial-boundary value problem*:

Find $u : \overline{Q}_T \to \mathbb{R},\ u = u(x,t),\ x \in \Omega,\ t \in [0,T]$, such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \nu\,\Delta\, u = g \quad \text{in } Q_T, \tag{2.1}$$

$$u|_{\partial\Omega \times (0,T)} = 0, \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{2.3}$$

where $\nu > 0$ is a given constant and $f_s : \mathbb{R} \to \mathbb{R},\ s = 1,2,\ g : Q_T \to \mathbb{R},\ u^0 : \Omega \to \mathbb{R}$ are given functions. Further assumptions for these functions will be given below. In the theory of conservation laws the functions f_s are called the *fluxes of the quantity* u in the directions $x_s, s = 1,2,$ g represents the *density of sources* and ν is the *diffusion coefficient*.

Equation (2.1) is the simplest prototype of the Navier–Stokes system describing viscous gas flow. However, we meet such equations in other areas as well, such as hydrology, oil recovery, traffic flow, and two phase flow (see, e.g., [30]).

In the following we will be concerned with the concept of a weak solution. We use the standard notation $L^p(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, and $L^p(0,T;X)$ (provided X is a Banach space, $k \ge 1$, $1 \le p \le \infty$) for the Lebesgue, Sobolev, and Bochner spaces. By $C([0,T];X)$ we denote the space of all continuous mappings of $[0,T]$ into X. (See, e.g., [29], [11, paragraphs 2.7, 8.2].) By $\|\cdot\|_X$ we denote the norm of a space X. The symbol $W_0^{1,p}(\Omega)$ will denote the space of all functions from $W^{1,p}(\Omega)$ with zero traces on $\partial\Omega$ and we set

$$V = H_0^1(\Omega) = W_0^{1,2}(\Omega). \tag{2.4}$$

In the space $H^1(\Omega)$, beside its norm

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (|u|^2 + |\nabla u|^2)\,\mathrm{d}x \right)^{1/2} \tag{2.5}$$

we will often work with the seminorm

$$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\,\mathrm{d}x \right)^{1/2}, \tag{2.6}$$

which is a norm on V equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$: there exist constants c_1, $c_2 > 0$ such that

$$c_1\|v\|_{H^1(\Omega)} \le |v|_{H^1(\Omega)} \le c_2\|v\|_{H^1(\Omega)} \quad \forall\, v \in V. \tag{2.7}$$

Further, we set

$$(u, v) = \int_\Omega uv\,\mathrm{d}x, \quad u,\, v \in L^2(\Omega), \tag{2.8}$$

for the scalar product on $L^2(\Omega)$ and

$$(\!(u, v)\!) = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x, \quad u,\, v \in H^1(\Omega), \tag{2.9}$$

for the scalar product on V inducing the norm $|\cdot|_{H^1(\Omega)}$ on V.

Provided the functions f_s, $s = 1, 2$, g, and u^0 are sufficiently regular (e. g., $f_s \in C^1(\mathbb{R})$), the *classical solution* of our problem can be defined as a function $u \in C^2(\overline{Q}_T)$ satisfying (2.1)–(2.3).

In what follows, similarly as in [14], we will assume that the following *assumptions on the data* are satisfied:

$$f_s \in C^1(\mathbb{R}), \quad f_s(0) = 0, \quad s = 1, 2, \tag{2.10}$$

$$g \in C([0, T]; W^{1,q}(\Omega)) \quad \text{for some } q > 2, \tag{2.11}$$

$$u^0 \in W_0^{1,p}(\Omega) \quad \text{for some } p > 2. \tag{2.12}$$

In view of the form of (2.1), the assumption that $f_s(0) = 0$ is not a restriction.

Now we derive the weak formulation of problem (2.1)–(2.3). Let us assume that u is a classical solution. Multiplying (2.1) by an arbitrary test function $v \in V$, integrating over Ω, using Green's theorem, and interchanging integration over Ω with differentiation with respect to t, we obtain the identity

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\,v\,\mathrm{d}x - \int_\Omega \sum_{s=1}^{2} f_s(u(t))\,\frac{\partial v}{\partial x_s}\,\mathrm{d}x + \nu\int_\Omega \nabla u(t)\cdot\nabla v\,\mathrm{d}x \\ &\quad = \int_\Omega g(t)\,v\,\mathrm{d}x \quad \forall\ v \in V \text{ and all } t \in (0, T]. \end{aligned} \tag{2.13}$$

Here, for a given $t \in [0, T]$, $u(t)$ denotes the function $u(\cdot, t) : \Omega \to \mathbb{R}$ and $\nabla u(t) = \nabla u(\cdot, t) = (\partial u(\cdot, t)/\partial x_1,\, \partial u(\cdot, t)/\partial x_2)$. Let us set

$$b(\varphi, v) = -\int_\Omega \sum_{s=1}^{2} f_s(\varphi)\,\frac{\partial v}{\partial x_s}\,\mathrm{d}x \quad \text{for any } \varphi \in L^\infty(\Omega), \quad v \in V. \tag{2.14}$$

Identity (2.13) and the above notation lead us to the following concept.

DEFINITION 2.1. *We say that a function u is a weak solution of problem* (2.1)–(2.3) *if it satisfies the following conditions:*

(2.15)

(a) $u \in L^2(0, T; V) \cap L^\infty(Q_T)$,

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(u(t), v) + b(u(t), v) + \nu(\!(u(t), v)\!) = (g(t), v) \quad \forall\, v \in V,$ *in the sense of distributions on* $(0, T)$,

(c) $u(0) = u^0$.

Identity (2.15(b)) is (2.13) rewritten with the aid of the above notation. In view of (2.11), $g \in L^2(0,T;V^*)$. (V^* denotes the dual to V.) Using assumption (2.10) and conditions (2.15(a)–(b)), we find that u has the derivative u' defined almost everywhere (a.e.) in $(0,T)$ and $u' \in L^2(0,T;V^*)$. This immediately implies that u is absolutely continuous on $[0,T]$ and hence, $u \in C([0,T],V^*)$ and we see that also condition (2.15(c)) makes sense.

In [14] it was shown that *problem* (2.15) *has a unique solution.*

3. Discretization. The discretization of problem (2.1)–(2.3) will be carried out with the aid of a generally unstructured triangular mesh.

By $\mathcal{T}_h$ we will denote a triangulation of Ω with the following properties: $T \in \mathcal{T}_h$ are closed triangles and

(3.1) (a) $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$,

(b) if T_1, $T_2 \in \mathcal{T}_h$, then $T_1 \cap T_2 = \emptyset$ or T_1 and T_2 have a common side or T_1 and T_2 have a common vertex.

The triangulation $\mathcal{T}_h$ is called a *basic mesh*. Let $\mathcal{P}_h = \{P_i; i \in J\}$ be the set of all vertices of all $T \in \mathcal{T}_h$. (J is a suitable index set.) We set $\overset{\circ}{J} = \{i \in J; P_i \in \Omega\}$ for the set of the indices of all interior vertices.

By $h(T)$ and $\theta(T)$ we denote the length of the longest side and the magnitude of the smallest angle, respectively, of the triangle $T \in \mathcal{T}_h$, and put

$$h = \max_{T \in \mathcal{T}_h} h(T), \quad \theta_h = \min_{T \in \mathcal{T}_h} \theta(T). \tag{3.2}$$

Now let us construct the *dual mesh* $\mathcal{D}_h = \{D_i; i \in J\}$ over the basic mesh $\mathcal{T}_h$. The *dual finite volume* D_i associated with a vertex $P_i \in \mathcal{P}_h$ is a *closed polygon* obtained in the following way: We join the center of gravity of every triangle $T \in \mathcal{T}_h$ that contains the vertex P_i with the midpoint of every side of T containing P_i. If $P_i \in \mathcal{P}_h \cap \partial\Omega$, then we complete the obtained contour by the straight segments joining P_i with the midpoints of boundary sides (i. e., sides which are subsets of $\partial\Omega$) that contain P_i. In this way we get the boundary ∂D_i of the finite volume D_i. (See Figure 3.1.) It is obvious that

$$\overline{\Omega} = \bigcup_{i \in J} D_i. \tag{3.3}$$

The interiors of D_i, $i \in J$, are mutually disjoint.

If for two different finite volumes D_i and D_j their boundaries contain a common segment, we call them *neighbors*. Then we put

$$\Gamma_{ij} = \Gamma_{ji} = \partial D_i \cap \partial D_j. \tag{3.4}$$

The set Γ_{ij} consists of one or two straight segments Γ_{ij}^{α}: $\Gamma_{ij} = \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^{\alpha}$, where $\beta_{ij} = 2$ for D_i or $D_j \subset \Omega$ and $\beta_{ij} = 1$, if both D_i and D_j are adjacent to $\partial\Omega$. (See Figure 3.1.)

For $i \in J$, let $s(i) = \{j \in J; D_j$ is a neighbor of $D_i\}$. If $P_i \in \mathcal{P}_h \cap \partial\Omega$, then we denote by $\Gamma_{i,-1}^{\alpha}$, $\alpha = 1, 2 =: \beta_{i,-1}$, the segments that form $\partial D_i \cap \partial\Omega$. In this case we set $S(i) = s(i) \cup \{-1\}$; otherwise (for $P_i \in \mathcal{P}_h \cap \Omega$) we put $S(i) = s(i)$. Obviously, for

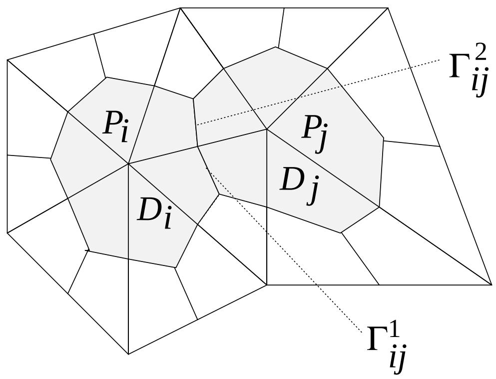

FIG. 3.1. *Dual finite volumes.*

every $D_i \in \mathcal{D}_h$ we have

$$\partial D_i = \bigcup_{j\in S(i)} \Gamma_{ij} = \bigcup_{j\in S(i)} \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^{\alpha}. \tag{3.5}$$

The open segments obtained by removing the endpoints from Γ_{ij}^{α} are mutually disjoint.

Moreover, we introduce the following *notation*: $|D_i|$ = area of $D_i \in \mathcal{D}_h$, $|T|$ = area of $T \in \mathcal{T}_h$, $\mathbf{n}_{ij}^{\alpha} = (n_{1ij}^{\alpha}, n_{2ij}^{\alpha})$ = unit outer normal to ∂D_i on the segment Γ_{ij}^{α}, ℓ_{ij}^{α} = length of Γ_{ij}^{α}, $\ell_{ij} = |\Gamma_{ij}|$ = length of Γ_{ij}, $|\partial D_i|$ = length of ∂D_i. Moreover, let us consider a partition of the time interval $(0, T)$ formed by time instants $t_k = k\tau, k = 0, 1, \dots$, where $\tau > 0$ is a (sufficiently small) time step.

Let us define the following spaces over the grids $\mathcal{T}_h$ and $\mathcal{D}_h$:

$$\begin{aligned} X_h &= \{v_h \in C(\overline{\Omega}); v_h|_T \text{ is linear for each } T \in \mathcal{T}_h\} \subset H^1(\Omega),\\ V_h &= \{v_h \in X_h; v_h = 0 \text{ on } \partial\Omega\},\\ Z_h &= \{w \in L^2(\Omega); w|_{D_i} = \text{const for each } D_i \in \mathcal{D}_h\},\\ Y_h &= \{w \in Z_h; w = 0 \text{ on } D_i \in \mathcal{D}_h \text{ for each } P_i \in \mathcal{P}_h \cap \partial\Omega\}. \end{aligned} \tag{3.6}$$

By r_h we denote the operator of the *Lagrange interpolation*: If $v : \mathcal{P}_h \to \mathbb{R}$, then

$$r_h v \in X_h, \quad (r_h v)(P_i) = v(P_i), \quad P_i \in \mathcal{P}_h. \tag{3.7}$$

Furthermore, we define the so-called *lumping operator* $L_h : C(\overline{\Omega}) \to Z_h$: For $v \in C(\overline{\Omega})$ we define $L_h v$ in such a way that

$$L_h v|_{D_i} = v(P_i), \quad i \in J. \tag{3.8}$$

Obviously, $L_h(V_h) = Y_h$.

In order to derive the discrete problem corresponding to (2.15), we put

$$\begin{aligned} (u, v)_h &= \int_\Omega r_h(uv)\,\mathrm{d}x, && u, v \in C(\overline{\Omega}),\\ \|u\|_h &= (u, u)_h^{1/2}, && u \in C(\overline{\Omega}). \end{aligned} \tag{3.9}$$

Moreover, we will construct the approximation b_h of the form b with the aid of the *finite volume approach* (e.g., [11, paragraph 7.3], [13], [27], [31]). For this purpose we introduce a suitable numerical flux $H : \mathbb{R}^2 \times \mathcal{S} \to \mathbb{R}$, where $\mathcal{S} = \{\mathbf{n} \in \mathbb{R}^2; |\mathbf{n}| = 1\}$.

Next we use the following assumptions.

3.1. Properties of the numerical flux. (a) $H = H(y, z, \mathbf{n})$ is *locally Lipschitz-continuous* with respect to y, z : for any $M > 0$ there exists $\tilde{c}_1(M) > 0$ such that

$$|H(y, z, \mathbf{n}) - H(y^*, z^*, \mathbf{n})| \le \tilde{c}_1(M)\,(|y - y^*| + |z - z^*|) \quad \forall\, y,\, y^*,\, z,\, z^* \in [-M, M],\ \forall\, \mathbf{n} \in \mathcal{S}.$$

(b) H is *consistent*:

$$H(u, u, \mathbf{n}) = \mathcal{F}(u, \mathbf{n}) := \sum_{s=1}^{2} f_s(u)\, n_s \quad \forall\, u \in \mathbb{R},\ \forall\, \mathbf{n} = (n_1, n_2) \in \mathcal{S}.$$

(c) H is *conservative*:

$$H(y, z, \mathbf{n}) = -H(z, y, -\mathbf{n}) \quad \forall\, y,\, z \in \mathbb{R},\ \forall\, \mathbf{n} \in \mathcal{S}.$$

(d) H is *monotone* in the following sense: For a given fixed number $M > 0$ the function $H(y, z, \mathbf{n})$ is nonincreasing with respect to the second variable z on the set $\mathcal{M}_M = \{(y, z, \mathbf{n}); y,\, z \in [-M, M], \mathbf{n} \in \mathcal{S}\}$.

(The symbol $\mathcal{F}(u, \mathbf{n})$ defined above denotes the flux of the quantity u in the direction $\mathbf{n}$.)

3.2. Examples. (a) The Lax–Friedrichs scheme has the numerical flux

$$H(u, v, \mathbf{n}) = \frac{1}{2}\left(\mathcal{F}(u, \mathbf{n}) + \mathcal{F}(v, \mathbf{n}) - \frac{1}{2\lambda}(v - u)\right),$$

where $\lambda > 0$ is in general different for different Γ_{ij} and is chosen so that by condition (d) from section 3.1

(b) the Engquist–Osher scheme has the numerical flux

$$H(u, v, \mathbf{n}) = \frac{1}{2}\left(\mathcal{F}(u, \mathbf{n}) + \mathcal{F}(v, \mathbf{n}) - \int_u^v |F(q, \mathbf{n})| dq\right),$$

where $F(u, \mathbf{n}) = \sum_{s=1}^{2} f_s'(u)\, n_s$.

Now we are ready to introduce the approximation b_h. Using (3.3), Green's theorem, (3.5), (2.10), and the definition of the space V_h, for u, $v \in V_h$ we write

$$\begin{aligned}
(3.10)\quad \int_\Omega \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\, v\, \mathrm{d}x &\approx \int_\Omega \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\, L_h\, v\, \mathrm{d}x \\
&= \sum_{i \in J} v(P_i) \int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\, \mathrm{d}x \\
&= \sum_{i \in J} v(P_i) \int_{\partial D_i} \sum_{s=1}^{2} f_s(u)\, n_s\, \mathrm{d}S \\
&= \sum_{i \in J} v(P_i) \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} \int_{\Gamma_{ij}^\alpha} \sum_{s=1}^{2} f_s(u)\, n_{sij}^\alpha\, \mathrm{d}S \\
&= \sum_{i \in J} v(P_i) \sum_{j \in s(i)} \sum_{\alpha=1}^{\beta_{ij}} \int_{\Gamma_{ij}^\alpha} \sum_{s=1}^{2} f_s(u)\, n_{sij}^\alpha\, \mathrm{d}S \\
&\approx \sum_{i \in J} v(P_i) \sum_{j \in s(i)} \sum_{\alpha=1}^{\beta_{ij}} H(u(P_i), u(P_j), \mathbf{n}_{ij}^\alpha)\, \ell_{ij}^\alpha =: b_h(u, v).
\end{aligned}$$

(Here $\mathbf{n} = (n_1, n_2)$ denotes the unit outer normal to ∂D_i.) This leads us to the following *semi-implicit scheme* for the numerical solution of problem (2.1)–(2.3).

3.3. Discrete problem. We define the *approximate solution* of (2.1)–(2.3) as functions u_h^k, $t_k \in [0,T]$, given by the conditions

$$\text{(3.11)}\quad \begin{aligned} &\text{(a) } u_h^0 = r_h u^0 \ (\in V_h),\\ &\text{(b) } u_h^{k+1} \in V_h, \quad t_k \in [0,T),\\ &\text{(c) } \frac{1}{\tau}(u_h^{k+1} - u_h^k, v_h)_h + b_h(u_h^k, v_h) + \nu(\!(u_h^{k+1}, v_h)\!) = (g^{k+1}, v_h)_h\\ &\qquad \forall\, v_h \in V_h,\ t_k \in [0,T). \end{aligned}$$

Here we set $g^{k+1} = g(\cdot, t_{k+1})$. The function u_h^k is the approximate solution at time t_k.

It is easy to establish the following basic properties of the discrete problem.

LEMMA 3.1. (1) *The bilinear forms* $(\cdot,\cdot)_h$ *and* $(\!(\cdot,\cdot)\!)$, *defined in* (3.9) *and* (2.9), *respectively, are scalar products on* V_h.

(2) *For each* $u \in X_h$, $b_h(u,\cdot)$ *is a linear form defined on* V_h.

(3) *If* $i \in J$ *and* $T \in \mathcal{T}_h$ *is a triangle with the vertex* $P_i \in \mathcal{P}_h$, *then*

$$\text{(3.12)}\qquad |T \cap D_i| = \frac{1}{3}|T|.$$

(4) *The approximation* $(\cdot,\cdot)_h$ *of the* L^2*-scalar product can be defined with the aid of numerical integration using the vertices* P_1^T, P_2^T, P_3^T *of* $T \in \mathcal{T}_h$ *as the integration points:*

$$\text{(3.13)}\quad \begin{aligned} (u,v)_h &= \sum_{T\in\mathcal{T}_h} |T| \sum_{n=1}^{3} u(P_n^T)\, v(P_n^T)/3 = \int_\Omega (L_h u)\,(L_h v)\,\mathrm{d}x, \quad u,\, v \in C(\overline{\Omega}),\\ \|u\|_h &= \|L_h u\|_{L^2(\Omega)}, \quad u \in C(\overline{\Omega}). \end{aligned}$$

(5) *Problem* (3.11(b)–(c)) *has a unique solution.*

In [14] the convergence of approximate solutions to the exact weak solution was proved for $h, \tau \to 0$ in suitable spaces. The aim of this paper is to derive error estimates. To this end we will consider a family $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ ($h_0 > 0$) of triangulations of the domain Ω.

In what follows we shall need a number of various constants. By $c, c_1, c_2, \ldots, \hat{c}_1, \hat{c}_2, \ldots, \tilde{c}, \ldots$ we denote constants independent of h, τ, ν, whereas $C, C_1, \ldots$ will denote constants that are independent of h, τ, but depend on ν. Moreover, c will be used as a generic constant attaining in general different values at different places.

3.4. Assumptions. (a) Let the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ be *regular*, i. e., there exists $\vartheta_0 > 0$ such that

$$\text{(3.14)}\qquad \theta_h \ge \vartheta_0 > 0 \quad \forall\, h \in (0, h_0).$$

See also (3.2).

(b) The triangulations $\mathcal{T}_h$ are of *weakly acute type*. This means that the magnitude of all angles of all $T \in \mathcal{T}_h$, $h \in (0,h_0)$, is less than or equal to $\pi/2$.

(c) The triangulations $\mathcal{T}_h$ satisfy the *inverse assumption*:

$$\text{(3.15)}\qquad h \le c_3 h_T \quad \forall\, T \in \mathcal{T}_h, \forall h \in (0, h_0),$$

with a constant $c_3 > 0$ independent of $T \in \mathcal{T}_h$ and h. Then the following inverse estimate holds (see [8, Theorem 3.2.6]):

$$\|v_h\|_{H^1(\Omega)} \le c_4 h^{-1} \|v_h\|_{L^2(\Omega)}, \quad v_h \in X_h,\, h \in (0, h_0). \tag{3.16}$$

In view of [8, Remark 3.1.3], assumptions (a) and (c) from section 3.4 imply the existence of a constant $c_5 > 0$, such that

$$h^2 \le c_5 |T|, \quad T \in \mathcal{T}_h,\ h \in (0, h_0). \tag{3.17}$$

In our further considerations we suppose that assumptions (2.10)–(2.12), (3.1), and (a)–(c) from section 3.4 are satisfied and that the numerical flux H has properties from section 3.1. Estimates of the error between the exact and approximate solutions will be obtained in several steps.

4. A priori estimates of approximate solutions. First we will summarize some important results.

LEMMA 4.1. *There exist constants* c, $\hat{c}_1$, $\hat{c}_2 > 0$ *such that for any* $h \in (0, h_0)$ *we have*

(a) $\hat{c}_1 \|v\|_{L^2(\Omega)} \le \|L_h v\|_{L^2(\Omega)} = \|v\|_h \le \hat{c}_2 \|v\|_{L^2(\Omega)}, \quad v \in X_h,$
(b) $\|v - L_h v\|_{L^2(\Omega)} \le c\, h \|v\|_{H^1(\Omega)}, \quad v \in X_h,$
(c) $|(u, v) - (u, v)_h| \le c\, h^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \quad u,\, v \in X_h,$
(d) $|(g^k, v) - (g^k, v)_h| \le c\, h \|g\|_{C([0,T]; W^{1,q}(\Omega))} \|v\|_{H^1(\Omega)}, \quad v \in V_h, \quad t_k \in [0, T].$
(e) *If* $M > 0$, *then there exists a constant* $\tilde{c}_2 = \tilde{c}_2(M)$ *such that*
$|b(z, v) - b_h(z, v)| \le \tilde{c}_2\, h \|z\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},$
$z \in V_h,\ \|z\|_{L^\infty(\Omega)} \le M,\ v \in V_h,\ h \in (0, h_0).$

Proof. Assertions (a)–(d) can be carried out with the aid of a standard finite element technique (for (e), see [14]). ☐

By virtue of (2.11) and (2.12), $u^0 \in C(\overline{\Omega})$ and $g \in C(\overline{Q}_T)$. Hence, there exist constants $\tilde{M}$ and $\tilde{K}$ such that

$$\|u^0\|_{L^\infty(\Omega)} \le \tilde{M}, \quad \|g\|_{L^\infty(Q_T)} \le \tilde{K}. \tag{4.1}$$

Let us put

$$M = \tilde{M} + T\, \tilde{K}. \tag{4.2}$$

The application of the discrete maximum principle yields the following theorem (see [14, Theorem 4.1]).

THEOREM 4.2. *If* $\tau > 0$ *and* $h \in (0, h_0)$ *satisfy the condition*

$$\tau\, \tilde{c}_1(M)\, |\partial D_i| \le |D_i|, \quad i \in J, \tag{4.3}$$

where $\tilde{c}_1(M)$ *is the constant from the section* 3.1, *assertion* (a), *and if* (4.1) *and* (4.2) *hold, then*

$$\|u_h^k\|_{L^\infty(\Omega)} \le M \quad \text{for each} \quad t_k \in [0, T]. \tag{4.4}$$

4.1. Remark. It is possible to show that (3.14), (3.12), and (3.17) imply the existence of a constant $c_6 > 0$ such that

$$|D_i| \,/\, |\partial D_i| \geq c_6\, h \quad \forall\, i \in J,\ \forall\, h \in (0, h_0). \tag{4.5}$$

In what follows, we will have to consider the stability condition

$$0 < \tau \leq c_6\, \tilde{c}_1(M)^{-1} h, \tag{4.6}$$

which is a typical CFL condition used in the numerical solution of conservation laws [27]. Obviously, (4.5) and (4.6) yield (4.3). As we see, $\tau = O(h)$. Because of our further considerations we also introduce an "*inverse stability assumption*" $h = O(\tau)$. This means that we consider the condition

$$h \leq c_7 \tau \tag{4.7}$$

with a constant c_7 independent of h and τ. (We can meet similar nonstandard conditions also in other works analyzing numerical methods for evolution problems, e.g., [36, paragraphs 4.2, 5.1] or [28]).

Furthermore, by [14] and the proof of Theorem 4.2, we have the following.

THEOREM 4.3. *Let* (4.1) *and* (4.2) *hold. Then there exists a constant* $c^* > 0$ *independent of* h, τ, *and* ν *such that*

$$\begin{aligned} &\text{(a)}\ \max_{t_k \in [0,T]} \|u_h^k\|_{L^2(\Omega)} \leq c^* \nu^{-1/2}, \\ &\text{(b)}\ \sum_{k=1}^{m} \|u_h^k - u_h^{k-1}\|^2_{L^2(\Omega)} \leq c^* \nu^{-1}, \quad t_m \in (0, T], \\ &\text{(c)}\ \tau \sum_{k=0}^{m} \|u_h^k\|^2_{H^1(\Omega)} \leq c^* \nu^{-2}, \quad t_m \in [0, T] \end{aligned} \tag{4.8}$$

$\forall\ h \in (0, h_0)$ *and* $\tau > 0$ *satisfying condition* (4.6).

Now we derive the estimate of the approximate solution in the $H^1(\Omega)$-norm.

THEOREM 4.4. *Let assumptions* (a)–(c) *from section* 3.4 *be satisfied. Then there exists a constant* $C_1 > 0, C_1 = O(\nu^{-3/2})$, *independent of* h, τ *such that*

$$\|u_h^k\|_{H^1(\Omega)} \leq C_1, \qquad t_k \in [0, T], \tag{4.9}$$

for $h \in (0, h_0)$ *and* $\tau > 0$ *satisfying* (3.15) *and* (4.6).

Proof. Let $\tau > 0$ and $h \in (0, h_0)$ satisfy conditions (3.15) and (4.6). Since $(\cdot,\cdot)_h$ and $(\!(\cdot,\cdot)\!)$ are scalar products on V_h, we can define the mapping $A_h : V_h \to V_h$ such that for each $\varphi_h \in V_h$,

$$(A_h \varphi_h, v_h)_h = (\!(\varphi_h, v_h)\!) \quad \forall\, v_h \in V_h. \tag{4.10}$$

Substituting $v_h := A_h u_h^k$ in (3.11(c)) with $k := k - 1$ and using (4.10), we find that

$$(\!(u_h^k - u_h^{k-1}, u_h^k)\!) + \tau b_h(u_h^{k-1}, A_h u_h^k) + \tau\nu(A_h u_h^k, A_h u_h^k)_h = \tau(g^k, A_h u_h^k)_h. \tag{4.11}$$

Now, the relation

$$(\!(z - v, z)\!) = \frac{1}{2}\{|z|^2_{H^1(\Omega)} - |v|^2_{H^1(\Omega)} + |z - v|^2_{H^1(\Omega)}\},$$

Lemma 4.1, the continuous embedding $W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$, and assumption (2.11) imply that

$$\begin{aligned}(4.12)\qquad & |u_h^k|^2_{H^1(\Omega)} - |u_h^{k-1}|^2_{H^1(\Omega)} + |u_h^k - u_h^{k-1}|^2_{H^1(\Omega)} + 2\tau\nu\|A_h u_h^k\|_h^2 \\ & \le c\tau\|g\|_{C([0,T];W^{1,q}(\Omega))}\|A_h u_h^k\|_h - 2\tau b_h(u_h^{k-1}, A_h u_h^k).\end{aligned}$$

Further, we can write

$$(4.13)\qquad |b_h(u_h^{k-1}, A_h u_h^k)| \le |b(u_h^{k-1}, A_h u_h^k)| + |b_h(u_h^{k-1}, A_h u_h^k) - b(u_h^{k-1}, A_h u_h^k)|.$$

Let $z, v \in V_h, \|z\|_{L^\infty(\Omega)} \le M$. Then (2.14), assumption (2.10), and Green's theorem imply that

$$\begin{aligned}(4.14)\qquad |b(z,v)| &= \left|\int_\Omega \sum_{s=1}^2 f_s(z)\frac{\partial v}{\partial x_s}dx\right| = \left|\int_\Omega \sum_{s=1}^2 \frac{\partial f_s}{\partial x_s}(z) v dx\right| \\ &= \left|\int_\Omega \sum_{s=1}^2 f_s'(z)\frac{\partial z}{\partial x_s} v dx\right| \le \tilde{c}_3|z|_{H^1(\Omega)}\|v\|_{L^2(\Omega)},\end{aligned}$$

with $\tilde{c}_3 = \tilde{c}_3(M)$. Now, by (4.4), (4.13), (4.14), Lemma 4.1(e) (i.e., the consistency of the form b_h), and inequalities (3.16) and (2.7), we have

$$\begin{aligned}(4.15)\qquad & |b_h(u_h^{k-1}, A_h u_h^k)| \le \tilde{c}_3|u_h^{k-1}|_{H^1(\Omega)}\|A_h u_h^k\|_{L^2(\Omega)} \\ & +\tilde{c}_2 h\|u_h^{k-1}\|_{H^1(\Omega)}\|A_h u_h^k\|_{H^1(\Omega)} \le c|u_h^{k-1}|_{H^1(\Omega)}\|A_h u_h^k\|_{L^2(\Omega)}.\end{aligned}$$

This, along with (4.12), (3.13), Lemma 4.1(a), and Young's inequality, gives the estimates

$$\begin{aligned}& |u_h^k|^2_{H^h(\Omega)} - |u_h^{k-1}|^2_{H^1(\Omega)} + |u_h^k - u_h^{k-1}|^2_{H^1(\Omega)} + 2\hat{c}_1^2\tau\nu\|A_h u_h^k\|^2_{L^2(\Omega)} \\ & \le c\tau(\hat{c}_2\|g\|_{C([0,T],W^{1,q}(\Omega))} + |u_h^{k-1}|_{H^1(\Omega)})\|A_h u_h^k\|_{L^2(\Omega)} \\ & \le \tau\nu\hat{c}_1^2\|A_h u_h^k\|^2_{L^2(\Omega)} + \frac{c\tau}{\nu}(\|g\|^2_{C([0,T];W^{1,q}(\Omega))} + |u_h^{k-1}|^2_{H^1(\Omega)}).\end{aligned}$$

Hence,

$$\begin{aligned}(4.16)\qquad & |u_h^k|^2_{H^1(\Omega)} - |u_h^{k-1}|^2_{H^1(\Omega)} + |u_h^k - u_h^{k-1}|^2_{H^1(\Omega)} + \tau\nu\hat{c}_1^2\|A_h u_h^2\|^2_{L^2(\Omega)} \\ & \le \frac{c\tau}{\nu}(1 + |u_h^{k-1}|^2_{H^1(\Omega)}).\end{aligned}$$

The summation of (4.16) over $k = 1, \dots, m,\ t_m \in (0,T]$, and the estimate (4.8(c)) yield

$$\begin{aligned}(4.17)\qquad & |u_h^m|^2_{H^1(\Omega)} + \sum_{k=1}^m |u_h^k - u_h^{k-1}|^2_{H^1(\Omega)} + \tau\nu\hat{c}_1^2\sum_{k=1}^m \|A_h u_h^k\|^2_{L^2(\Omega)} \\ & \le \frac{cT}{\nu} + \frac{c\tau}{\nu}\sum_{k=1}^m |u_h^{k-1}|^2_{H^1(\Omega)} \le \frac{cT}{\nu} + \frac{c}{\nu^3},\end{aligned}$$

which together with (2.7) already implies (4.9). ◻

5. Truncation error. We start from the following assumptions.

5.1. Regularity of the exact solution. Let us suppose that the exact solution $u : (0, T) \to V$ of problem (2.15) satisfies the conditions

$$u \in L^\infty(0, T; H^{1+\mu}(\Omega)),\quad u' \in L^\infty(0, T; L^2(\Omega)),\quad u'' \in L^\infty(0, T; V^*), \tag{5.1}$$

with some $\mu \in (1/4, 1]$. (Concerning the bound for μ, see the remark in section 6.1.) For $\varepsilon \in (0, 1)$, the symbol $H^{1+\varepsilon}(\Omega)$ denotes the Sobolev–Slobodetskii space of functions with "noninteger derivatives" (see, e.g., [29]). By u' and u'' we denote the first and second derivatives of the mapping $u : (0, T) \to V$.

The above assumptions imply that $u \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; V^*) \cap L^\infty(Q_T)$. We set $\tilde{M} := \|u\|_{L^\infty(Q_T)} < \infty$. In what follows we will denote $u^k = u(t_k) = u(\cdot, t_k)$. Let us investigate the *truncation error*.

LEMMA 5.1. *Under assumptions* (5.1), *for* $t_k \in [0, T)$ *we have*

$$|(u^{k+1} - u^k, v) - \tau(u'(t_{k+1}), v)| \le c\tau^2 \|v\|_{H^1(\Omega)}, \qquad v \in V, \tag{5.2}$$

$$\begin{aligned} &|b(u^{k+1}, v) - b(u^k, v)| \\ &\le 2 \max_{\xi \in [-\tilde{M}, \tilde{M}], s=1,2} |f_s'(\xi)| \, \|u^{k+1} - u^k\|_{L^2(\Omega)} |v|_{H^1(\Omega)}, \qquad v \in V, \end{aligned} \tag{5.3}$$

$$\|u^{k+1} - u^k\|_{L^2(\Omega)} \le c\tau \tag{5.4}$$

with $c = c(u)$.

Proof. (a) The proof of (5.2) is based on the following result (see [11, para. 8.2] or [18]): If $\eta : (0, T) \to V^*$ is such that $\eta, \eta' \in L^1(0, T; V^*)$ and $v \in V$, then $\langle \eta', v \rangle \in L^1(0, T)$ and

$$\int_{t_1}^{t_2} \langle \eta'(t), v \rangle dt = \langle \eta(t_2) - \eta(t_1), v \rangle, \qquad t_1, t_2 \in [0, T].$$

Here $\langle \varphi, v \rangle$ denotes the value of a functional $\varphi \in V^*$ at a point $v \in V$. A similar result holds, if $\eta, \eta' \in L^1(0, T; L^2(\Omega))$ and the duality $\langle \cdot, \cdot \rangle$ is replaced by the L^2-scalar product $(\cdot, \cdot)$. Let $v \in V$. Then

$$\begin{aligned} (u^{k+1} - u^k, v) - \tau(u'(t_{k+1}), v) &= (u(t_{k+1}) - u(t_k), v) - \int_{t_k}^{t_{k+1}} (u'(t), v) dt \\ &\quad + \int_{t_k}^{t_{k+1}} (u'(t) - u'(t_{k+1}), v) dt. \end{aligned} \tag{5.5}$$

Taking into account that

$$(u(t_{k+1}) - u(t_k), v) = \int_{t_k}^{t_{k+1}} (u'(t), v) dt$$

and $L^2(\Omega) \hookrightarrow V^*$, we see that

$$\begin{aligned} (u^{k+1} - u^k, v) - \tau(u'(t_{k+1}), v) &= \int_{t_k}^{t_{k+1}} (u'(t) - u'(t_{k+1}), v) dt \\ &= \int_{t_k}^{t_{k+1}} \langle u'(t) - u'(t_{k+1}), v \rangle dt. \end{aligned} \tag{5.6}$$

Since $u'' \in L^\infty(0,T;V^*)$,

$$\langle u'(t) - u'(t_{k+1}), v\rangle = \int_{t_{k+1}}^{t} \langle u''(\vartheta), v\rangle d\vartheta \tag{5.7}$$

and thus,

$$\int_{t_k}^{t_{k+1}} \langle u'(t) - u'(t_{k+1}), v\rangle dt = \int_{t_k}^{t_{k+1}} \left(\int_{t_{k+1}}^{t} \langle u''(\vartheta), v\rangle d\vartheta \right) dt.$$

This, (5.5)–(5.7) and the assumption that $u'' \in L^\infty(0,T;V^*)$ imply that

$$|(u^{k+1} - u^k, v) - \tau(u'(t_{k+1}), v)| \le \tau^2 \|u''\|_{L^\infty(0,T;V^*)} \|v\|_{H^1(\Omega)},$$

which yields (5.2).

(b) By (2.14) and the Cauchy inequality, for $v \in V$ we get

$$\begin{aligned} |b(u^{k+1}, v) - b(u^k, v)| &= \left| \int_\Omega \sum_{s=1}^{2} (f_s(u^{k+1}) - f_s(u^k)) \frac{\partial v}{\partial x_s} dx \right| \\ &\le \int_\Omega \left\{ \sum_{s=1}^{2} \int_0^1 |f_s'(u^k + \vartheta(u^{k+1} - u^k))| d\vartheta \right\} |u^{k+1} - u^k| \left| \frac{\partial v}{\partial x_s} \right| dx \\ &\le 2 \max_{\xi \in [-\tilde{M}, \tilde{M}], s=1,2} |f_s'(\xi)| \|u^{k+1} - u^k\|_{L^2(\Omega)} |v|_{H^1(\Omega)}. \end{aligned}$$

(c) Finally, since $u' \in L^\infty(0,T;L^2(\Omega))$,

$$\begin{aligned} \|u^{k+1} - u^k\|_{L^2(\Omega)} &= \|u(t_{k+1}) - u(t_k)\|_{L^2(\Omega)} \\ &= \left\| \int_{t_k}^{t_{k+1}} u'(t) dt \|_{L^2(\Omega)} \le \tau \|u' \right\|_{L^\infty(0,T;L^2(\Omega))}, \end{aligned}$$

which yields (5.4). ∎

COROLLARY 5.2. *From assertions* (b) *and* (c) *of Lemma* 5.1 *it follows that*

$$|b(u^{k+1}, v) - b(u^k, v)| \le c\tau |v|_{H^1(\Omega)}, \qquad v \in V, \tag{5.8}$$

where $c = c(u)$.

Using the above results, we get the estimate of the truncation error.

THEOREM 5.3. *Under assumptions* (5.1) *we have*

$$\begin{aligned} &(u_h^{k+1} - u^{k+1}, v_h) - (u_h^k - u^k, v_h) \\ &+\tau[b(u_h^k, v_h) - b(u^k, v_h)] + \tau\nu(\!(u_h^{k+1} - u^{k+1}, v_h)\!) \\ &= \varepsilon_1(\tau, u; v_h) + \varepsilon_2(\tau, h, u_h^k, u_h^{k+1}; v_h), \qquad v_h \in V_h,\ t_k \in [0,T), \end{aligned} \tag{5.9}$$

where u_h^k *and* u^k *denote the approximate solution and the exact solution, respectively, of problem* (2.15) *at time* $t = t_k$, *and*

$$|\varepsilon_1(\tau, u; v_h)| \le c\tau^2 \|v_h\|_{H^1(\Omega)}, \qquad v_h \in V_h, \tau > 0,\ c = c(u), \tag{5.10}$$

$$|\varepsilon_2(\tau, h, u_h^k, u_h^{k+1}; v_h)| \le c\tau h \left[(\|u_h^k\|_{H^1(\Omega)} + \|u_h^{k+1}\|_{H^1(\Omega)}) \frac{h}{\tau} + 1 \right] \|v_h\|_{H^1(\Omega)}, \tag{5.11}$$

$v_h \in V_h$, $h \in (0, h_0)$, *and* $\tau > 0$ *satisfy* (4.6), $c = c(M)$, $t_k \in [0,T)$.

Proof. From (5.1) it follows that in (2.15(b)) we can write $\frac{d}{dt}(u(t),v) = (u'(t),v)$ for $v \in V$. Hence, the exact solution u satisfies at $t = t_{k+1}$ the relation

$$(5.12)\quad \begin{aligned}&(u^{k+1}-u^k,v)+\tau b(u^k,v)+\tau\nu(\!(u^{k+1},v)\!) = \tau(g^{k+1},v)\\ &+[(u^{k+1}-u^k,v)-\tau(u'(t_{k+1}),v)]+\tau[b(u^k,v)-b(u^{k+1},v)], \qquad v\in V.\end{aligned}$$

Setting now $v := v_h \in V_h$ and subtracting (5.12) from (3.11(c)) multiplied by τ, we find that

$$\begin{aligned}&(u_h^{k+1}-u^{k+1},v_h)-(u_h^k-u^k,v_h)+\tau[b(u_h^k,v_h)-b(u^k,v_h)]+\tau\nu(\!(u_h^{k+1}-u^{k+1},v_h)\!)\\ &=\varepsilon_1(\tau,u;v_h)+\varepsilon_2(\tau,h,u_h^k,u_h^{k+1};v_h),\end{aligned}$$

where

$$\begin{aligned}&\varepsilon_1(\tau,u;v_h) = -[(u^{k+1}-u^k,v_h)-\tau(u'(t_{k+1}),v_h)]-\tau[b(u^k,v_h)-b(u^{k+1},v_h)],\\ &\varepsilon_2(\tau,h,u_h^k,u_h^{k+1};v_h) = \underbrace{\tau[(g^{k+1},v_h)_h-(g^{k+1},v_h)]}_{\sigma(1)}+\underbrace{[(u_h^{k+1},v_h)-(u_h^{k+1},v_h)_h]}_{\sigma(2)}\\ &-\underbrace{[(u_h^k,v_h)-(u_h^k,v_h)_h]}_{\sigma(3)}+\underbrace{\tau[b(u_h^k,v_h)-b_h(u_h^k,v_h)]}_{\sigma(4)}.\end{aligned}$$

Due to (5.2) and (5.8),

$$|\varepsilon_1(\tau,u;v_h)| \le c\tau^2\|v_h\|_{H^1(\Omega)},$$

which is (5.10). Further, from Lemma 4.1 and Theorem 4.2 we deduce that

$$\begin{aligned}&|\sigma(1)| \le c\tau h\|v_h\|_{H^1(\Omega)},\\ &|\sigma(2)|+|\sigma(3)| \le ch^2(\|u_h^k\|_{H^1(\Omega)}+\|u_h^{k+1}\|_{H^1(\Omega)})\|v_h\|_{H^1(\Omega)},\\ &|\sigma(4)| \le \tilde{c}_2(M)\tau h\|u_h^k\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}.\end{aligned}$$

These estimates immediately imply (5.11). ∎

LEMMA 5.4. *The form b is locally Lipschitz-continuous: For any $\hat{M} > 0$ there exists a constant $\tilde{c}_4 = \tilde{c}_4(\hat{M})$ such that*

$$(5.13)\quad \begin{aligned}&|b(z,v)-b(\tilde{z},v)| \le \tilde{c}_4\|z-\tilde{z}\|_{L^2(\Omega)}|v|_{H^1(\Omega)},\\ &z,\tilde{z},v\in H^1(\Omega),\ z,\tilde{z}\in L^\infty(\Omega),\ \|z\|_{L^\infty(\Omega)},\ \|\tilde{z}\|_{L^\infty(\Omega)} \le \hat{M}.\end{aligned}$$

Proof. By (2.14), using assumption (2.10) and the Cauchy inequality, we find that for $z, \tilde{z}, v$ with the above properties we have

$$\begin{aligned}|b(z,v)-b(\tilde{z},v)| &= \left|\int_\Omega\left(\int_0^1\sum_{s=1}^2 f_s'(\tilde{z}+t(z-\tilde{z}))dt\right)(z-\tilde{z})\frac{\partial v}{\partial x_s}dx\right|\\ &\le 2\max_{\xi\in[-\hat{M},\hat{M}],s=1,2}|f_s'(\xi)|\,\|z-\tilde{z}\|_{L^2(\Omega)}|v|_{H^1(\Omega)},\end{aligned}$$

which is (5.13). ∎

6. Error estimates. Because of our further considerations we introduce several results and concepts. Let us consider the following problem: Given $\gamma \in V^*$, find $z : \Omega \to \mathbb{R}$ such that

$$\text{(a) } z \in V, \qquad \text{(b) } (\!(z, v)\!) = \langle \gamma, v \rangle \quad \forall\, v \in V. \tag{6.1}$$

This problem has a unique solution z due to the Riesz representation theorem. Further, we define the *Ritz projection* $P_h : V \to V_h$. If $\varphi \in V$, then

$$\text{(a) } P_h\varphi \in V_h, \qquad \text{(b) } (\!(P_h\varphi, v_h)\!) = (\!(\varphi, v_h)\!) \quad \forall v_h \in V_h. \tag{6.2}$$

Obviously,

$$|P_h\varphi|_{H^1(\Omega)} \le |\varphi|_{H^1(\Omega)}, \quad \varphi \in V. \tag{6.3}$$

From the abstract error estimate

$$\|P_h\varphi - \varphi\|_{H^1(\Omega)} \le c \inf_{\varphi_h \in V_h} \|\varphi - \varphi_h\|_{H^1(\Omega)}, \quad \varphi \in V, \quad h \in (0, h_0),$$

(which can be obtained in a standard way), for $\varphi \in H^{1+\varepsilon}(\Omega) \cap V$ with $\varepsilon \in (0, 1]$, we find that (see also [10])

$$\|P_h\varphi - \varphi\|_{H^1(\Omega)} \le c\|\varphi - r_h\varphi\|_{H^1(\Omega)} \le ch^\varepsilon\|\varphi\|_{H^{1+\varepsilon}(\Omega)}, \qquad h \in (0, h_0). \tag{6.4}$$

Here c is independent of φ and h (but depends on ε).

On the basis of [19, Remark 2.4.6], we conclude that the following regularity result is valid for the weak solution of the Poisson problem with the homogeneous Dirichlet condition in the polygonal domain Ω: Let z be a weak solution of the problem $\Delta z = f \in L^2(\Omega)$, $z|_{\partial\Omega} = 0$, i.e., z satisfies (6.1(a)–(b)), where $\langle \gamma, v \rangle = (f, v)$. Let ω be the maximal angle of all reentrant corners of $\partial\Omega$ (i.e., the corners with interior angles $\alpha_i \in (\pi, 2\pi)$) and $\Lambda = \pi/\omega (\in (1/2, 1))$. Then $z \in H^{1+\varepsilon}(\Omega)$ for every $\varepsilon \in (0, \Lambda)$. In the case of no reentrant corners we can take $\varepsilon = 1$. Moreover, for every such ε there exists a constant $c_\varepsilon > 0$ such that

$$\|z\|_{H^{1+\varepsilon}(\Omega)} \le c_\varepsilon\|f\|_{L^2(\Omega)}, \quad f \in L^2(\Omega). \tag{6.5}$$

The following lemma is a generalization of the result well known in the finite element circle for $\varphi \in H^2(\Omega), \Omega$ polygonal and convex [8, para. 3.2].

LEMMA 6.1. *There exists $\alpha \in (1/2, 1]$ such that for any $\varepsilon \in (0, \alpha]$ and $\varphi \in H^{1+\varepsilon}(\Omega)$*

$$\|P_h\varphi - \varphi\|_{L^2(\Omega)} \le ch^{2\varepsilon}\|\varphi\|_{H^{1+\varepsilon}(\Omega)}, \quad h \in (0, h_0). \tag{6.6}$$

If the polygonal domain Ω is convex, then $\alpha = 1$. The constant c is independent of φ and h, but depends on ε.

Proof. We use the well-known Aubin–Nitsche duality method. Let $\varphi \in H^{1+\varepsilon}(\Omega) \cap V$ and $\varphi_h = P_h\varphi$. This means that

$$\varphi_h \in V_h, \qquad (\!(\varphi_h, v_h)\!) = (\!(\varphi, v_h)\!) \qquad \forall\, v_h \in V_h. \tag{6.7}$$

Then, by [8, Theorem 3.2.4],

$$\|\varphi_h - \varphi\|_{L^2(\Omega)} \le c\|\varphi_h - \varphi\|_{H^1(\Omega)} \sup_{f\in L^2(\Omega)} \left\{ \frac{1}{\|f\|_{L^2(\Omega)}} \inf_{\psi_h \in V_h} \|\sigma_f - \psi_h\|_{H^1(\Omega)} \right\}, \tag{6.8}$$

where for $f \in L^2(\Omega)$ we denote by σ_f the solution to the dual problem

$$\sigma_f \in V, \qquad (\!(v, \sigma_f)\!) = (f, v) \qquad \forall\ v \in V. \tag{6.9}$$

As mentioned above, there exists $\alpha \in (1/2, 1]$ such that $\sigma_f \in H^{1+\varepsilon}(\Omega) \forall \varepsilon \in (0, \alpha]$ and all $f \in L^2(\Omega)$. Further, we use the estimates (6.4) for φ and σ_f and (6.5) with $z := \sigma_f$. We find that

$$\|\varphi_h - \varphi\|_{H^1(\Omega)} \le ch^\varepsilon \|\varphi\|_{H^{1+\varepsilon}(\Omega)}, \tag{6.10}$$

$$\begin{aligned} \inf_{\psi_h \in V_h} \|\sigma_f - \psi_h\|_{H^1(\Omega)} &\le \|\sigma_f - r_h\sigma_f\|_{H^1(\Omega)} \le ch^\varepsilon \|\sigma_f\|_{H^{1+\varepsilon}(\Omega)} \\ &\le ch^\varepsilon \|f\|_{L^2(\Omega)}. \end{aligned} \tag{6.11}$$

Inequalities (6.8), (6.10), and (6.11) already yield the estimate (6.6). □

Now we denote by

$$e^k = u_h^k - u^k \tag{6.12}$$

the *error of the method* at time $t = t_k$. Our goal is to estimate e^k in a suitable norm in terms of h.

Let us recall assumption (2.12), i.e., $u^0 \in W_0^{1,p}(\Omega)$ with some $p > 2$. By [25, Theorem 2.19], $H^{1+\beta}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ with $\beta = 1 - 2/p \in (0, 1)$.

Now we will formulate the *main result.*

THEOREM 6.2. *Let assumptions* (2.10)–(2.12), (3.1), (a)–(d) *from section* 3.1, (a)–(c) *from section* 3.4, (4.1), *and* (4.2) *be satisfied. Further, let* $\{u_h^k\}_{t_k = k\tau\in[0,T]}$ *be the approximate solution of problem* (2.15) *obtained with the aid of the discrete problem* (3.11). *Let the exact solution* u *of* (2.15) *satisfy conditions* (5.1) *and let* $\varepsilon = \min(\mu, \alpha)$. *Moreover, on the basis of the above note we assume that* $u^0 \in H^{1+\beta}(\Omega)$ *with* $\beta \in (0, 1)$. *Then there exist constants* $C_1, \dots, C_4 > 0$ *independent of* h, τ *and a constant* $c > 0$ *independent of* h, τ, ν *such that*

$$\max_{t_k\in[0,T]} \|e^k\|_{L^2(\Omega)} \le [C_1 h^{1+\beta} + C_2 h + C_3 h^\varepsilon + C_4 h^{2\varepsilon - 1/2}] \exp\left(\frac{cT}{\nu}\right), \tag{6.13}$$

$$\begin{aligned} &\left(\nu\tau \sum_{t_k\in[0,T]} |e^k|^2_{H^1(\Omega)}\right)^{1/2} \\ &\quad \le [C_1 h^{1+\beta} + C_2 h + C_3 h^\varepsilon + C_4 h^{2\varepsilon-1/2}] \exp\left(\frac{cT}{\nu}\right) \nu^{-1/2}, \end{aligned} \tag{6.14}$$

provided $h \in (0, h_0), \tau \in (0, T)$ *satisfy conditions* (4.6) *and* (4.7). *Moreover,*

$$C_1 = O(1), C_2 = O(\nu^{-3/2}), C_3 = O(\nu), C_4 = O(\nu^{1/2}). \tag{6.15}$$

6.1. Remark. Since $\mu \in (1/4, 1]$ (see (5.1)) and $\alpha \in (1/2, 1]$ (cf. Lemma 6.1), we have the following behavior of the error estimates (6.13), (6.14). If $\mu \in (1/2, 1)$, then also $\varepsilon = \min(\mu, \alpha) \in (1/2, 1)$ and $2\varepsilon - 1/2 \in (1/2, 3/2)$. On the other hand, if $\mu \in (1/4, 1/2]$, then $\varepsilon = \mu$ and $2\varepsilon - 1/2 \in (0, 1/2]$. In the case of a convex polygonal domain we have $\alpha = 1$. If also $\mu = 1$, then the error of the method is of order $O(h)$.

Proof of Theorem 6.2. Let $h \in (0, h_0), \tau > 0$ satisfy conditions (4.6) and (4.7). From (5.9) and (6.12) we obtain the relation

$$(e^{k+1}, v_h) - (e^k, v_h) + \tau\nu(\!(e^{k+1}, v_h)\!) = -\tau[b(u_h^k, v_h) - b(u^k, v_h)] \\ + \varepsilon_1(\tau, u; v_h) + \varepsilon_2(\tau, h, u_h^k, u_h^{k+1}; v_h).$$

Let us set $v_h := P_h e^{k+1}$. Denoting by $I : V \to V$ the identity operator ($I\varphi = \varphi$ for $\varphi \in V$), we get

$$\begin{aligned} (6.16)\ &(e^{k+1}, e^{k+1}) - (e^k, e^{k+1}) + \tau\nu(\!(e^{k+1}, e^{k+1})\!) \\ &= -\tau[b(u_h^k, P_h e^{k+1}) - b(u^k, P_h e^{k+1})] \\ &\quad + \varepsilon_1(\tau, u; P_h e^{k+1}) + \varepsilon_2(\tau, h, u_h^k, u_h^{k+1}; P_h e^{k+1}) \\ &\quad + (e^{k+1}, (I - P_h)e^{k+1}) - (e^k, (I - P_h)e^{k+1}) + \tau\nu(\!(e^{k+1}, (I - P_h)e^{k+1})\!). \end{aligned}$$

From (6.12) and (6.2) it follows that $(I - P_h)e^{k+1} = P_h u^{k+1} - u^{k+1}$. Hence, by (6.4) and Lemma 6.1,

$$\begin{aligned} (6.17)\qquad &\text{(a)}\ \|(I - P_h)e^{k+1}\|_{L^2(\Omega)} \le ch^{2\varepsilon}\|u^{k+1}\|_{H^{1+\varepsilon}(\Omega)}, \\ &\text{(b)}\ \|(I - P_h)e^{k+1}\|_{H^1(\Omega)} \le ch^{\varepsilon}\|u^{k+1}\|_{H^{1+\varepsilon}(\Omega)}. \end{aligned}$$

It follows from Lemma 5.4 (where we set $\hat{M} = \max(M, \tilde{M})$) and (6.3) that

$$\begin{aligned} (6.18)\qquad &|b(u_h^k, P_h e^{k+1}) - b(u^k, P_h e^{k+1})| \\ &\le c\|e^k\|_{L^2(\Omega)}|P_h e^{k+1}|_{H^1(\Omega)} \le c\|e^k\|_{L^2(\Omega)}|e^{k+1}|_{H^1(\Omega)}. \end{aligned}$$

Furthermore, by (6.17) and the Cauchy inequality, we have

$$|(e^{k+1}, (I - P_h)e^{k+1}) - (e^k, (I - P_h)e^{k+1})| \le ch^{2\varepsilon}\|e^{k+1} - e^k\|_{L^2(\Omega)}\|u^{k+1}\|_{H^{1+\varepsilon}(\Omega)},$$

$$(\!(e^{k+1}, (I - P_h)e^{k+1})\!) \le ch^{\varepsilon}|e^{k+1}|_{H^1(\Omega)}\|u^{k+1}\|_{H^{1+\varepsilon}(\Omega)}. \tag{6.19}$$

Now, from (6.16)–(6.19), (5.10), (5.11), and (6.3) we obtain the estimate

$$\begin{aligned} &\|e^{k+1}\|^2_{L^2(\Omega)} - \|e^k\|^2_{L^2(\Omega)} + \|e^{k+1} - e^k\|^2_{L^2(\Omega)} + 2\tau\nu|e^{k+1}|^2_{H^1(\Omega)} \\ &\le c\tau\|e^k\|_{L^2(\Omega)}|e^{k+1}|_{H^1(\Omega)} + c\tau^2|e^{k+1}|_{H^1(\Omega)} \\ &\quad + c\tau h\left[(\|u_h^k\|_{H^1(\Omega)} + \|u_h^{k+1}\|_{H^1(\Omega)})\frac{h}{\tau} + 1\right]|e^{k+1}|_{H^1(\Omega)} \\ &\quad + ch^{2\varepsilon}\|e^{k+1} - e^k\|_{L^2(\Omega)}\|u^{k+1}\|_{H^{1+\varepsilon}(\Omega)} + c\tau h^{\varepsilon}\nu|e^{k+1}|_{H^1(\Omega)}\|u^{k+1}\|_{H^{1+\varepsilon}(\Omega)}, \end{aligned}$$

where $c > 0$ is a constant independent of h, τ, ν. Taking into account conditions (5.1), (4.6), (4.7), estimate (4.9), and using Young's inequality, we find that

$$\begin{aligned} &\|e^{k+1}\|^2_{L^2(\Omega)} - \|e^k\|^2_{L^2(\Omega)} + \|e^{k+1} - e^k\|^2_{L^2(\Omega)} + 2\tau\nu|e^{k+1}|^2_{H^1(\Omega)} \\ &\le \frac{c\tau}{\nu}\|e^k\|^2_{L^2(\Omega)} + \frac{\tau\nu}{4}|e^{k+1}|^2_{H^1(\Omega)} + \frac{c\tau h^2}{\nu} + \frac{\tau\nu}{4}|e^{k+1}|^2_{H^1(\Omega)} + \frac{c\tau h^2}{\nu}(1 + \nu^{-3}) \\ &\quad + \frac{\tau\nu}{4}|e^{k+1}|^2_{H^1(\Omega)} + c\tau h^{4\varepsilon-1} + \|e^{k+1} - e^k\|^2_{L^2(\Omega)} + c\tau\nu h^{2\varepsilon} + \frac{\tau\nu}{4}|e^{k+1}|^2_{H^1(\Omega)}. \end{aligned}$$

Hence,

$$\|e^{k+1}\|^2_{L^2(\Omega)} - \|e^k\|^2_{L^2(\Omega)} + \tau\nu|e^{k+1}|^2_{H^1(\Omega)} \le \frac{c\tau}{\nu}\|e^k\|^2_{L^2(\Omega)} + \tau c\left[\frac{h^2}{\nu}(1+\nu^{-3}) + \nu h^{2\varepsilon} + h^{4\varepsilon-1}\right]. \tag{6.20}$$

This implies that

$$\|e^{k+1}\|^2_{L^2(\Omega)} \le A\|e^k\|^2_{L^2(\Omega)} + \tau B, \qquad t_k \in [0,T), \tag{6.21}$$

where

$$A = 1 + \frac{c\tau}{\nu}, \quad B = c\left[\frac{h^2}{\nu}(1+\nu^{-3}) + \nu h^{2\varepsilon} + h^{4\varepsilon-1}\right]. \tag{6.22}$$

By induction over $k = 0, 1, \dots,$ from (6.21) we easily deduce that

$$\|e^k\|^2_{L^2(\Omega)} \le A^k\|e^0\|^2_{L^2(\Omega)} + \tau B\frac{A^k-1}{A-1}, \qquad t_k \in [0,T]. \tag{6.23}$$

Since $A \le \exp(c\tau/\nu)$, it follows from (6.23) that

$$\|e^k\|^2_{L^2(\Omega)} \le \exp\left(\frac{ct_k}{\nu}\right)\|e^0\|^2_{L^2(\Omega)} + c\left[\frac{h^2}{\nu}(1+\nu^{-3}) + \nu h^{2\varepsilon} + h^{4\varepsilon-1}\right]\left[\exp\left(\frac{ct_k}{\nu}\right) - 1\right]\nu, \qquad t_k \in [0,T]. \tag{6.24}$$

Taking into account that $u^0 \in H^{1+\beta}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, with $\beta = 1 - 2/p \in (0,1)$, by virtue of [10, Theorem 2.27], we have

$$\|e^0\|^2_{L^2(\Omega)} = \|u^0 - r_h u^0\|^2_{L^2(\Omega)} \le ch^{2(1+\beta)}\|u^0\|^2_{H^{1+\beta}(\Omega)}. \tag{6.25}$$

From this and (6.24) we obtain the estimate

$$\|e^k\|_{L^2(\Omega)} \le c\exp\left(\frac{ct_k}{2\nu}\right)h^{1+\beta} + c\Bigg[h(1+\nu^{-3})^{1/2} + \nu h^{\varepsilon} + \sqrt{\nu}h^{2\varepsilon-1/2}\Bigg]\left[\exp\left(\frac{ct_k}{\nu}\right) - 1\right]^{1/2} \qquad t_k \in [0,T], \tag{6.26}$$

which already yields (6.13) and (6.15) .

In order to prove the error estimate (6.14), we sum up (6.20) over $k = 0, \dots, m-1$ for $t_m \in (0,T]$ and use (6.24), (6.25). Then we get

$$\begin{aligned}
&\|e^m\|^2_{L^2(\Omega)} + \nu\tau\sum_{k=0}^{m-1}|e^{k+1}|^2_{H^1(\Omega)} \le \|e^0\|^2_{L^2(\Omega)} + \frac{cm\tau}{\nu}\max_{k=0,\dots,m-1}\|e^k\|^2_{L^2(\Omega)} \\
&+cm\tau\left[\frac{h^2}{\nu}(1+\nu^{-3}) + \nu h^{2\varepsilon} + h^{4\varepsilon-1}\right] \\
&\le ch^{2(1+\beta)}\left(1 + \frac{1}{\nu}\exp\left(\frac{ct_m}{\nu}\right)\right) + \frac{ct_m}{\nu}\left[h^2\left(1+\nu^{-3}\right) + \nu^2 h^{2\varepsilon} + \nu h^{4\varepsilon-1}\right]\exp\left(\frac{ct_m}{\nu}\right),
\end{aligned}$$

which immediately implies (6.14). □

6.2. Concluding remarks. (a) The above results can be extended to the case when $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain and p, q from (2.11) and (2.12) are greater than three. The maximum principle can be applied in this case on the basis of the results from [26].

(b) For the sake of simplicity, in this paper we considered the homogeneous Dirichlet boundary condition and assumed that the domain Ω was polygonal. With the aid of the theory of finite element variational crimes developed in [16] and [17], the theoretical analysis presented here can be generalized to the case of nonhomogeneous mixed Dirichlet–Neumann boundary conditions on a piecewise-smooth boundary $\partial\Omega$. Furthermore, it is possible to consider a nonlinear diffusion term in (2.1), provided that some assumptions of monotonicity or pseudomonotonicity are satisfied (cf. [16], [17]).

(c) There are several open questions and problems connected with our investigation: the proof of error estimates for other combined FV–FE schemes (fully explicit or implicit schemes, the method of fractional steps, schemes on other meshes; cf. [12]), the study of higher order schemes, the derivation of a posteriori error estimates, the development of adaptive mesh refinement techniques, and generalization to systems of equations.

(d) Particularly important, but rather difficult, is investigating the behavior of the error in dependence on coefficient ν and obtaining uniform estimates with respect to ν. Our estimates depend on ν (see the behavior of the constants $C_1, \dots, C_4$ from (6.14); moreover, the constant $c = c(u)$ from Lemma 5.1 depends on the norms $\|u'\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|u''\|_{L^\infty(0,T;V^*)}$, which give an implicit and, unfortunately, unknown dependence on ν). Therefore, estimates (6.21) are not robust. This is the drawback of a number of works dealing with numerical schemes for singularly perturbed problems. The uniform convergence with respect to ν has been obtained in very few works analyzing simple problems under rather special assumptions when complete analytic behavior of solutions is known (cf., e.g., [39], [1], [2], [38]; for further citations, see [36]).

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