Lax-Wendroff type second order evolution Galerkin methods for multidimensional hyperbolic systems

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Abstract

The aim of this paper is to present a technique for the construction of higher order genuinely multidimensional finite difference schemes solving systems of conservation laws. We derive simple order conditions guaranteeing that the schemes are p-th order accurate in space and time and apply them to evolution Galerkin (EG) methods for the wave equation system in two space dimensions. The EG methods belong to the class of genuinely multidimensional schemes since they take explicitly into account all of the infinitely many directions of propagation of bicharacteristics. The described technique leads to several new second order corrections to evolution Galerkin methods which are stable up to the CFL number 1.

Key words: genuinely multidimensional schemes, finite differnce methods, numerical diffusion, hyperbolic systems, wave equation, Euler equations, evolution Galerkin schemes

AMS Subject Classification: 35L05, 65M06, 35L45, 35L65, 65M25, 65M15

1 Introduction

In recent years the most commonly used methods for hyperbolic problems were finite volume methods which are based on a quasi dimensional splitting using one-dimensional Riemann solvers. It turns out that in certain cases, e.g. when waves are propagating in directions that are oblique with respect to the mesh, this approach leads to structural deficiencies and large errors in the solution, e.g. see [4], [5], [9], [14], [15] or [22].
The aim of our work is to derive solvers that are less mesh dependent by making more use of the physical propagation properties of solutions. In the same spirit LeVeque [8] has improved the above approach somewhat by computing tangential fluxes at cell edges. The approach that we are following is quite different in the fact that we do not compute combinations of one-dimensional waves at cell interfaces, but consider the multidimensional propagation of the states in each cell by the method of characteristics. We construct

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methods which take into account all of the infinitely many directions of propagation of bicharacteristics. Using the bicharacteristic or Monge cones, called Mach cones in gasdynamics, we consider integral equations for solutions to hyperbolic systems of differential equations as the basis for deriving our schemes. These include a time integration from which we derive *approximate evolution operators* by quadrature. This is quite analogous to the derivation of Runge-Kutta formulas for the computation of numerical solutions of ordinary differential equations by applying quadrature rules to the corresponding integral equation. For higher order quadrature rules this involves in both cases a procedure for calculating intermediate stages. This basic approach was pioneered by Ostkamp [21], [22] and extended in our previous papers [13], [14], [15].

The use of bicharacteristics in numerical schemes is not very new at all, see, e.g., [2], [3], [11], [12], [14] and the references therein. In preceeding schemes bicharacteristics were used to trace nodal information backwards in time, mostly in Lagrange-Galerkin or semi-Lagrangian, Euler-Characteristic-Galerkin and "modified characteristics" frameworks. The values of the physical states on the previous time levels were interpolated to the base of the bicharacteristics in various ways.

Our approach is completely different in nature. We are pursuing the use of the approximate evolutions derived by integration of bicharacteristics in the context of the so-called evolution Galerkin methods. The concept of the evolution Galerkin methods is the following: In a time step the physical states, i.e. numerical data, are initially given as a function in a suitable finite element space, preferably as piecewise constants on mesh cells. These data are evolved to the next time level using an approximate evolution as outlined above and then projected back onto the finite element space. In our previous papers [13], [14], [15] we have derived and studied first order schemes obtained in this way from a theoretical as well as computational point of view. In parallel papers [16] and [17] we are pursuing a more refined and possibly simpler approach in which we use the approximate evolutions to calculate fluxes in a finite volume scheme.

In this manner we obtain various genuinely multidimensional numerical schemes for solving systems of hyperbolic equations. Our schemes reflect the multidimensional character of hyperbolic systems that do not decouple by making neither a reduction to the one-dimensional nor the scalar case.

It is very common to obtain higher order scheme by using some recovery procedure within a first order scheme. This approach that is easily implemented in finite volume methods was pursued by us in [16] and [17]. But it leads to larger stencils, i.e. twenty five point schemes. In the present paper we study an alternative approach, which gives some insight into the finite difference nature of our schemes and their numerical dissipation matrices. We consider the approximation of second order derivatives in the truncation error following the Lax-Wendroff approach. Such a correction approach was used also by LeVeque [7], [8] and Fey [19] in order to constract second order variants of their first order multidimensional methods. The second order methods which are developed in this paper use multidimensional evolution Galerkin scheme of first order with second order correction terms.

2 Evolution Galerkin schemes

We consider a general symmetric hyperbolic system in d space dimensions

$$\underline{U}_t + \sum_{k=1}^{d} \underline{\underline{A}}_k \underline{U}_{x_k} = 0, \;\; \underline{x} = (x_1, \ldots, x_d)^T \in I\!R^d, \tag{2.1}$$

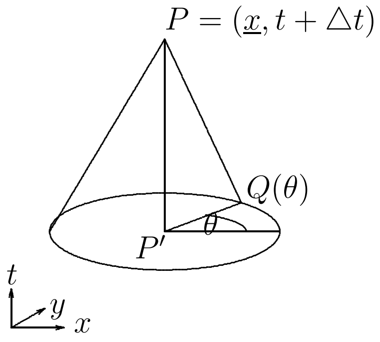

Figure 1: Bicharacteristic along the Mach cone through P and $Q(\underline{\theta})$

where the symmetric coefficient matrices $\underline{\underline{A}}_k = \underline{\underline{A}}_k(\underline{U})$, $k = 1, \ldots, d$ are elements of $I\!R^{m\times m}$ and the dependent variables are $\underline{U} = (u_1, \ldots, u_m)^T \in I\!R^m$. Let us denote by $E(s) : (H^k(I\!R^d))^m \to (H^k(I\!R^d))^m$ the exact evolution operator associated with a time step s for the system (2.1), i.e.

$$\underline{U}(\cdot, t+s) = E(s)\underline{U}(\cdot, t). \tag{2.2}$$

We suppose that S_h is a finite element space consisting of piecewise polynomials of order r. Let $\underline{U}^n$ be an approximation in the space S_h to the exact solution $\underline{U}(\cdot, t_n)$ at a time $t_n > 0$ and take $E_\triangle : S_h \to (H^k(I\!R^d))^m$ to be a suitable approximation to the exact evolution operator $E(\Delta t)$. We denote by $P_h : (H^k(I\!R^d))^m \to S_h$ the L^2-projection onto S_h. Then we can define an evolution Galerkin method in the following manner.

Definition 2.1 *Starting from some initial value $\underline{U}^0$ at time $t = 0$, the evolution Galerkin method (EG) is recursively defined by means of*

$$\underline{U}^{n+1} = P_h E_\triangle \underline{U}^n. \tag{2.3}$$

For simplicity we assume constant time steps Δt, i.e. $t_n = n\Delta t$. The method is uniquely determined by the approximate evolution operator $E_\triangle$ and the projection P_h.

In what follows we deal for simplicity with a particular hyperbolic problem, namely a wave equation system. We first describe several approximate evolution operator for it.

3 Wave equation system and approximate evolution operators

It will be convenient to consider the wave equation and write it as a first order system in the following form

$$\left.\begin{array}{l} \phi_t + c(u_x + v_y) = 0 \\ u_t + c\phi_x = 0 \\ v_t + c\phi_y = 0 \end{array}\right\} \tag{3.1}$$

using the unknown functions ϕ, u, v. Consider a cone with the apex $P = (x, y, t + \Delta t)$ and the base points $Q = Q(\theta) = (x + c\Delta t\cos\theta, y + c\Delta t\sin\theta, t)$ parametrized by the angle $\theta \in [0, 2\pi]$, see Figure 1.

Denote $P' = (x, y, t)$ a center of the base of the cone. The lines from $Q(\theta)$ to P generating the mantle of the so-called bicharacteristic cone are called bicharacteristics, see e.g. [15]

for more details. Using the theory of bicharacteristics it can be shown that the solution (ϕ, u, v) at the point P is determined by its values on the base and on the mantle of the characteristic cone. Exact representation formulae were derived by Bulter [2], Prasad *et.al.* [23], [24] for the special case of the above system. Later this procedure was extended to the general theory of linear hyperbolic problems by Ostkamp [22]. In what follows we will define three EG schemes for the wave equation system. The first two EG schemes were derived using the approach of Butler [2] and Prasad [24]. The EG1 scheme uses the rectangle rule for time approximation, whereas in the EG2 scheme the trapezoidal rule is used, see Lukáčová, Morton and Warnecke [15]. The third EG scheme is based on the more general approach using the theory for hyperbolic problems, see Ostkamp [21], [22] and Lukáčová, Morton and Warnecke [14], [15].

3.1 Approximate Evolution Operator for the EG1 Scheme

$$\phi_P = \frac{1}{2\pi} \int_0^{2\pi} [\phi_Q - 2u_Q \cos\theta - 2v_Q \sin\theta] \mathrm{d}\theta + O(\Delta t^2) \tag{3.2}$$

$$u_P = \frac{1}{\pi} \int_0^{2\pi} [-\phi_Q \cos\theta + u_Q(3\cos^2\theta - 1) + 3v_Q \sin\theta\cos\theta] \mathrm{d}\theta + O(\Delta t^2) \tag{3.3}$$

$$v_P = \frac{1}{\pi} \int_0^{2\pi} [-\phi_Q \sin\theta + 3u_Q \sin\theta\cos\theta + v_Q(3\sin^2\theta - 1)] \mathrm{d}\theta + O(\Delta t^2) \tag{3.4}$$

3.2 Approximate Evolution Operator for the EG2 Scheme

$$\phi_P = \frac{1}{\pi} \int_0^{2\pi} [\phi_Q - u_Q \cos\theta - v_Q \sin\theta] \mathrm{d}\theta - \phi_{P'} + O(\Delta t^3) \tag{3.5}$$

$$u_P = \frac{1}{\pi} \int_0^{2\pi} [-\phi_Q \cos\theta + u_Q(2\cos^2\theta - \tfrac{1}{2}) + 2v_Q \sin\theta\cos\theta] \mathrm{d}\theta + O(\Delta t^3) \tag{3.6}$$

$$v_P = \frac{1}{\pi} \int_0^{2\pi} [-\phi_Q \sin\theta + 2u_Q \sin\theta\cos\theta + v_Q(2\sin^2\theta - \tfrac{1}{2})] \mathrm{d}\theta + O(\Delta t^3) \tag{3.7}$$

3.3 Approximate Evolution Operator for the EG3 Scheme

$$\phi_P = \frac{1}{2\pi} \int_0^{2\pi} (\phi_Q - 2u_Q \cos\theta - 2v_Q \sin\theta) d\theta + O(\Delta t^2) \tag{3.8}$$

$$u_P = \frac{1}{2} u_{P'} + \frac{1}{2\pi} \int_0^{2\pi} (-2\phi_Q \cos\theta + u_Q(3\cos^2\theta - 1) + 3v_Q \sin\theta\cos\theta) d\theta + O(\Delta t^2) \tag{3.9}$$

$$v_P = \frac{1}{2} v_{P'} + \frac{1}{2\pi} \int_0^{2\pi} (-2\phi_Q \sin\theta + 3u_Q \sin\theta\cos\theta + v_Q(3\sin^2\theta - 1))d\theta + O(\Delta t^2). \tag{3.10}$$

The equations (3.2)–(3.4), (3.5)–(3.7) and (3.8)–(3.10) define three approximate evolution operators for the wave equation system. Using the L^2 - projection onto a space of piecewise constant functions we obtain three numerical schemes which will be referred to as the EG1, EG2 and EG3 schemes, respectively. The remaining integrals and the space integrals coming from the projection step are computed exactly, i.e. no further numerical quadrature is used. The finite difference formulations of these schemes are given in the Appendix 1. These schemes were studied from a theoretical as well as the computational point of view in [14], where we showed their stability up to some CFL number less then 1 and proved that they are of first order. Numerical experiments demonstrated that the schemes perform very well with respect to the preservation of vorticity and rotationality as well as the resolution of symmetric structures and shocks.

4 The second order corrections - general systems

In this section we present a simple description of the general algorithm of constructing higher order correction terms for multidimensional systems. The aim is to keep the stencil as compact as possible. We want to restrict ourselves to nine point schemes. This leaves a sufficient number of degrees of freedom to construct second order schemes. The approach will then be applied in order to derive higher order EG-methods based on our first order EGM.

For simplicity we are only considering rectangular meshes. Let Δx, Δy be mesh size parameters resp. in the x-, y- directions. Moreover, we assume $\mathcal{O}(\Delta x) = \mathcal{O}(\Delta y) =: \mathcal{O}(h)$. The computational domain is covered with points $x_{kl} := [k\Delta x, l\Delta y]$. We denote by Ω_{kl} the rectangular mesh cell $[(k-\frac{1}{2})\Delta x, (k+\frac{1}{2})\Delta x] \times [(l-\frac{1}{2})\Delta y, (l+\frac{1}{2})\Delta y]$ and χ_{kl} the corresponding characteristic function, $k, l \in \mathbb{Z}$. We work with the usual L^2-projection P_h onto spaces S_h of piecewise constant step functions with respect to our meshes. It is given by integral averages

$$P_h \underline{U} = \sum_{k,l \in \mathbf{Z}} \left(\frac{1}{\Delta x \Delta y} \int_{\Omega_{kl}} \underline{U}(x,y) dx dy \right) \chi_{kl} \qquad \underline{U} \in (L^2(\mathbb{R}^d))^m. \tag{4.1}$$

The approximate evolution operators E_Δ given in Sections 3.1-3.3 together with the L^2-projection give the resulting *finite difference* formulae for each EG scheme. These can be put into the following form

$$\underline{U}_{kl}^{n+1} = \underline{U}_{kl}^n + \sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{C}}_{ij} \underline{U}_{k+i\, l+j}^n, \tag{4.2}$$

where the entries of the symmetric **finite difference matrices** $\underline{\underline{C_{ij}}}$ are calculated by inserting the respective approximate evolution E_Δ into the projection. The resulting triple integrals are simplified if the order of integration is switched. Afterwards it is relatively easy to compute these integrals exactly for any discrete basis functions (e.g., piecewise constants, piecewise linears, etc.). In the Appendix 1 examples are given in terms of the **stencil matrices** $\underline{\underline{\alpha}}, \underline{\underline{\beta}}, \underline{\underline{\gamma}}$. They are related to the finite difference matrices by

$$\underline{\underline{C}}_{ij} = \left(\begin{array}{ccc} \alpha_{ij}^1 & \beta_{ij}^1 & \gamma_{ij}^1 \\ \\ \alpha_{ij}^2 & \beta_{ij}^2 & \gamma_{ij}^2 \\ \\ \alpha_{ij}^3 & \beta_{ij}^3 & \gamma_{ij}^3 \end{array} \right). \tag{4.3}$$

In the following we give various conditions that have to be appropriately satisfied by any numerical scheme considered for solving a two-dimensional hyperbolic system of equations (2.1), $d = 2$.

Firstly, the schemes should be **conservative**. This means that we require

$$\sum_{k,l} \underline{U}_{k,l}^{n+1} = \sum_{k,l} \underline{U}_{k,l}^{n}. \tag{4.4}$$

Using (4.2) and (4.4) we obtain the following necessary and sufficient condition

$$\sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{C}}_{ij} = \underline{\underline{0}}. \tag{4.5}$$

Note that each EG-scheme we deal with, i.e. EG1, EG2, EG3, satisfies (4.5) and thus these schemes are conservative. It is a well-known fact that solving a hyperbolic problem by an explicit scheme of the form (4.2) we have to take a time step Δt small enough to satisfy the so-called CFL stability condition. Let us introduce the Courant numbers $\nu_x = c\Delta t/\Delta x$, $\nu_y = c\Delta t/\Delta y$, where c is a maximal eigenvalue of the matrix pencil $\underline{\underline{A}}(\underline{n}) := \sum_{k=1}^{2} n_k \underline{\underline{A}}_k$ for any unit vector $\underline{n} = (n_1, n_2)^T \in I\!R^2$. Then the CFL condition implies a step size restriction of the following type

$$\max(\nu_x, \nu_y) \leq \nu_{max}, \tag{4.6}$$

where ν_{max} is a number from the interval $(0, 1]$.

Now we will give simple **order conditions** for the numerical scheme (4.2) to be of p-th order, where $p \in \mathbb{N}$, $p \geq 1$. Let us rewrite (4.2) in the following form.

$$\frac{1}{\Delta t} \left(\underline{U}_{k,l}^{n+1} - \underline{U}_{k,l}^{n} \right) = \frac{1}{\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{C}}_{ij} \underline{U}_{k+i,l+j}^{n}. \tag{4.7}$$

We insert nodal values of an exact solution of the system (2.1) and expand both sides of (4.7) into a Taylor series, one with respect to Δt and the other with respect to $\Delta x, \Delta y$. This gives for any $p \in \mathbb{N}$, $p \geq 1$

$$\frac{1}{\Delta t} \left(\underline{U}_{k,l}^{n+1} - \underline{U}_{k,l}^{n} \right) = (\underline{U}_t)_{k,l}^n + \frac{\Delta t}{2!} (\underline{U}_{tt})_{k,l}^n + \ldots + \frac{(\Delta t)^{p-1}}{p!} (\underline{U}_{t^p})_{k,l}^n + \mathcal{O}(\Delta t^p) \tag{4.8}$$

and

$$\begin{aligned} \frac{1}{\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{C}}_{ij} \underline{U}_{k+i,l+j}^{n} \quad = \quad & \frac{\Delta x}{\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} i \underline{\underline{C}}_{ij} (\underline{U}_x)_{k,l}^n + \frac{\Delta y}{\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} j \underline{\underline{C}}_{ij} (\underline{U}_y)_{k,l}^n \\ & + \frac{\Delta x^2}{2\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} i^2 \underline{\underline{C}}_{ij} (\underline{U}_{xx})_{k,l}^n + \frac{\Delta y^2}{2\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} j^2 \underline{\underline{C}}_{ij} (\underline{U}_{yy})_{k,l}^n \end{aligned}$$

$$
\begin{aligned}
&+\frac{\Delta x \Delta y}{\Delta t} \sum_{i=-1}^{1} \sum_{j=-1}^{1} ij \underline{\underline{C}}_{ij} (\underline{U}_{xy})^n_{k,l} + \ldots \\
&+\sum_{r=0}^{p} \frac{\Delta x^r \Delta y^{(p-r)}}{p! \Delta t} \left(\sum_{i=-1}^{1} \sum_{j=-1}^{1} i^r j^{(p-r)} \underline{\underline{C}}_{ij} (\underline{U}_{x^r, y^{p-r}})^n_{k,l} \right) \\
&+\mathcal{O}\left(\frac{\Delta x^{p+1} + \Delta y^{p+1}}{\Delta t}\right).
\end{aligned} \tag{4.9}
$$

Moreover, let us define the operator $\underline{\underline{L}} = \underline{\underline{A}}_1 \partial_x + \underline{\underline{A}}_2 \partial_y$ for the system (2.1). Then we have for any smooth solution the higher order equations

$$
\begin{aligned}
\underline{U}_t &= -\underline{\underline{L}}\,\underline{U} \\
\underline{U}_{tt} &= (-\underline{\underline{L}})^2 \underline{U} \\
&\cdots \\
\underline{U}_{t^p} &= (-\underline{\underline{L}})^p \underline{U}
\end{aligned} \tag{4.10}
$$

satisfied. Substituting (4.10) into (4.8) and then comparing the coefficients for each derivative of $\underline{U}$ in (4.8) and (4.9) we obtain the order conditions for the numerical scheme (4.2). Let us define for each $\ell,\ m \in \mathbb{N}$ the following matrices

$$
\underline{\underline{Q}}_{x^\ell, y^m} := \sum_{i=-1}^{1} \sum_{j=-1}^{1} i^\ell j^m \underline{\underline{C}}_{ij}, \tag{4.11}
$$

$$
\underline{\underline{P}}_x := -\frac{\Delta t}{\Delta x} \underline{\underline{A}}_1, \tag{4.12}
$$

$$
\underline{\underline{P}}_y := -\frac{\Delta t}{\Delta y} \underline{\underline{A}}_2. \tag{4.13}
$$

The scheme (4.2) is of **first order**, iff

$$
\underline{\underline{P}}_x = \underline{\underline{Q}}_x \quad \text{and} \quad \underline{\underline{P}}_y = \underline{\underline{Q}}_y. \tag{4.14}
$$

If moreover the following conditions hold the numerical scheme (4.2) is of **second order**

$$
\begin{aligned}
&\underline{\underline{P}}_x^2 = \underline{\underline{Q}}_{xx}, \quad \underline{\underline{P}}_y^2 = \underline{\underline{Q}}_{yy}, \\
&\left(\underline{\underline{P}}_x \underline{\underline{P}}_y + \underline{\underline{P}}_y \underline{\underline{P}}_x \right)/2 = \underline{\underline{Q}}_{xy}.
\end{aligned} \tag{4.15}
$$

For a **third order** scheme we have to additionally satisfy the following conditions

$$
\begin{aligned}
&\underline{\underline{P}}_x^3 = \underline{\underline{Q}}_{xxx}, \quad \underline{\underline{P}}_y^3 = \underline{\underline{Q}}_{yyy}, \\
&(\underline{\underline{P}}_x \underline{\underline{P}}_x \underline{\underline{P}}_y + \underline{\underline{P}}_x \underline{\underline{P}}_y \underline{\underline{P}}_x + \underline{\underline{P}}_y \underline{\underline{P}}_x \underline{\underline{P}}_x)/3 = \underline{\underline{Q}}_{xxy}, \\
&(\underline{\underline{P}}_y \underline{\underline{P}}_y \underline{\underline{P}}_x + \underline{\underline{P}}_y \underline{\underline{P}}_x \underline{\underline{P}}_y + \underline{\underline{P}}_x \underline{\underline{P}}_y \underline{\underline{P}}_y)/3 = \underline{\underline{Q}}_{xyy}.
\end{aligned} \tag{4.16}
$$

In a similar manner general higher order conditions may be derived. We use equivalently both notations $\underline{\underline{Q}}_{xx}$ and $\underline{\underline{Q}}_{x^2}$, similarly for y. We say, that the scheme (4.2) is of **order** $p \in \mathbb{N},\ p \geq 1$, iff for each $s = 1, \ldots, p$ the following conditions hold

$$
\binom{s}{\ell}^{-1} \sum_{\substack{\delta_i = x; \\ \#\{\delta_i\} = \ell}} \sum_{\substack{\delta_j = y; \\ \#\{\delta_j\} = m}} \underline{\underline{P}}_{\delta_1} \cdot \ldots \cdot \underline{\underline{P}}_{\delta_s} = \underline{\underline{Q}}_{x^\ell, y^m}, \tag{4.17}
$$

where $\ell + m = s,\ \ell, m \in \mathbb{N}$ and $\#\{\delta_i\}$ denotes the cardinal number of the set $\{\delta_i; \delta_i = x, i \in \{1, \ldots, s\}\ \}$. An analogous notation is used for y. In fact we have in (4.17) the sums over all combinations of the products of ℓ matrices $\underline{\underline{P}}_x$ and m matrices $\underline{\underline{P}}_y$. If in the above sums $\ell = 0$ or $m = 0$, then the corresponding summation is not taken into account.

In [25] Roe provided simple conditions which determine the accuracy of a numerical scheme for the solution of the one-dimensional linear equation. A generalization for the multi-dimensional scalar advection equation has been given by Billet and Toro [1]. We have independently derived similar order of accuracy conditions (4.17) for general first order systems in two dimensions. Moreover, it is easy to see that the generalization to d-dimensional systems (2.1) is straightforward. Note that the matrices $\underline{\underline{Q}}_{xx}, \underline{\underline{Q}}_{yy}, \underline{\underline{Q}}_{xy}$ give the **coefficients of numerical diffusion**. Together they define the third order **tensor of numerical diffusion** for the scheme (4.2).

New second order numerical schemes will be constructed in the following form

$$\underline{U}^{n+1}_{k,l} = \underline{U}^{n}_{k,l} + \sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{C}}_{ij} \underline{U}^{n}_{k+i,l+j} + \sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{D}}_{ij} \underline{U}^{n}_{k+i,l+j}, \tag{4.18}$$

where matrices $\underline{\underline{D}}_{ij}$ give the second order correction terms. These second order correction matrices have to satisfy some conditions. First, they must not spoil conservativity and the first order of the original numerical scheme (4.2). On the order hand they must have the right numerical diffusion that will give second order. This can be mathematically written in the following way.

$$\begin{aligned}
&\sum_{i=-1}^{1} \sum_{j=-1}^{1} \underline{\underline{D}}_{ij} = \underline{\underline{0}} \\
&\sum_{i=-1}^{1} \sum_{j=-1}^{1} i\underline{\underline{D}}_{ij} = \underline{\underline{0}}, \qquad \sum_{i=-1}^{1} \sum_{j=-1}^{1} j\underline{\underline{D}}_{ij} = \underline{\underline{0}} \\
&\sum_{i=-1}^{1} \sum_{j=-1}^{1} i^2\underline{\underline{D}}_{ij} = \underline{\underline{P}}^2_x - \underline{\underline{Q}}_{xx} \\
&\sum_{i=-1}^{1} \sum_{j=-1}^{1} j^2\underline{\underline{D}}_{ij} = \underline{\underline{P}}^2_x - \underline{\underline{Q}}_{yy} \\
&\sum_{i=-1}^{1} \sum_{j=-1}^{1} ij\underline{\underline{D}}_{ij} = (\underline{\underline{P}}_x\underline{\underline{P}}_y + \underline{\underline{P}}_x\underline{\underline{P}}_y)/2 - \underline{\underline{Q}}_{xy}\ .
\end{aligned} \tag{4.19}$$

Similarly as the matrices $\underline{\underline{C}}_{ij}$ the matrices $\underline{\underline{D}}_{ij}$ can be defined with the help of some correction stencil matrices $\underline{\underline{\hat{\alpha}}}^1, \ldots, \underline{\underline{\hat{\gamma}}}^3$, i.e.

$$\underline{\underline{D}}_{ij} = \begin{pmatrix} \hat{\alpha}^1_{ij} & \hat{\beta}^1_{ij} & \hat{\gamma}^1_{ij} \\ \hat{\alpha}^2_{ij} & \hat{\beta}^2_{ij} & \hat{\gamma}^2_{ij} \\ \hat{\alpha}^3_{ij} & \hat{\beta}^3_{ij} & \hat{\gamma}^3_{ij} \end{pmatrix}. \tag{4.20}$$

The entries $\hat{\alpha}^1_{ij}, \ldots, \hat{\gamma}^3_{ij}$ depend on the first order scheme we are using and on the system of equations we want to solve. In the next section we will derive correction matrices $\underline{\underline{D}}_{ij}$ for the EG methods solving the wave equation system (3.1).

5 Correction terms for the wave equation system

Going back to the wave equation system (3.1) we can explicitly write down the matrices $\underline{\underline{P}}_x$, $\underline{\underline{P}}_y$ and their products up to any order. In what follows we restrict ourselves for simplicity to the order condition $p = 2$, however correction terms up to any order $p \in \mathbb{N}$ can be derived using the above order conditions (4.17), provided the stencil is taken to be large enough. Moreover, it is easy to verify that the nine point schemes (4.2) do not have enough degrees of freedom in order to construct correction terms for $p > 2$. Generally we would need to enlarge the stencils for higher orders. For each equation and variable we have nine degrees of freedom in a nine point stencil. The conservativity takes one, first order two and second order three more. For third order we would need additional four degrees of freedom, whereas we are left with three degrees of freedom only for each equation and variable. This leaves 27 degrees of freedom, to construct second order schemes from any first order scheme for system (3.1). Further, taking into account the symmetry of the system we are dealing with, we have a reduction by another 3×3 degrees of freedom and end up with 18 degrees of freedom. In what follows we will show that taking into account certain properties of the wave equation system as well as of the EG schemes we can construct second order schemes from only two degrees of freedom, cf. (5.11). Now we have for the wave equation system (3.1)

$$\underline{\underline{P}}_x := \begin{pmatrix} 0 & -\nu_x & 0 \\ -\nu_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{P}}_y := \begin{pmatrix} 0 & 0 & -\nu_y \\ 0 & 0 & 0 \\ -\nu_y & 0 & 0 \end{pmatrix},$$

$$\underline{\underline{P}}_x^2 := \begin{pmatrix} \nu_x^2 & 0 & 0 \\ 0 & \nu_x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{P}}_y^2 := \begin{pmatrix} \nu_y^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu_y^2 \end{pmatrix},$$

$$\frac{1}{2}\left(\underline{\underline{P}}_x\underline{\underline{P}}_y + \underline{\underline{P}}_y\underline{\underline{P}}_x\right) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu_x\nu_y/2 \\ 0 & \nu_x\nu_y/2 & 0 \end{pmatrix}.$$

On the other hand, using (4.3) and the stencil matrices $\underline{\underline{\alpha}}^1, \ldots, \underline{\underline{\gamma}}^3$, that are given for each of the EG-schemes in the Appendix 1, the entries of the matrices $\underline{\underline{Q}}_{x^\ell,y^m}$ can be computed. Thus we find that all EG-schemes we have dealt with in [14], i.e. EG1, EG2, EG3, satisfy condition (4.14), which again confirms the fact that they are the first order schemes. Moreover, for example the Lax-Wendroff scheme, see e.g. [20] for the precise formulation, obviously satisfies the order condition (4.15).

Now we will derive second order correction terms for the EG1, EG2 and EG3 schemes. They are conservative first order schemes, i.e. satisfy (4.4), and (4.14) but not (4.15). Let us write numerical diffusion matrices for each scheme.

The EG1 scheme:

$$\underline{\underline{Q}}_{xx} = \begin{pmatrix} \frac{2}{\pi}\nu_x & 0 & 0 \\ 0 & \frac{4}{\pi}\nu_x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{Q}}_{yy} = \begin{pmatrix} \frac{2}{\pi}\nu_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{4}{\pi}\nu_y \end{pmatrix}, \tag{5.1}$$

$$\underline{\underline{Q}}_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3\nu_x\nu_y/4 \\ 0 & 3\nu_x\nu_y/4 & 0 \end{pmatrix}.$$

The EG2 scheme:

$$\underline{\underline{Q}}_{xx} = \begin{pmatrix} \frac{4}{\pi}\nu_x & 0 & 0 \\ 0 & \frac{10}{3\pi}\nu_x & 0 \\ 0 & 0 & \frac{2}{3\pi}\nu_x \end{pmatrix}, \quad \underline{\underline{Q}}_{yy} = \begin{pmatrix} \frac{4}{\pi}\nu_y & 0 & 0 \\ 0 & \frac{2}{3\pi}\nu_y & 0 \\ 0 & 0 & \frac{10}{3\pi}\nu_y \end{pmatrix}, \tag{5.2}$$

$$\underline{\underline{Q}}_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu_x\nu_y/2 \\ 0 & \nu_x\nu_y/2 & 0 \end{pmatrix}.$$

The EG3 scheme:

$$\underline{\underline{Q}}_{xx} = \begin{pmatrix} \frac{2}{\pi}\nu_x & 0 & 0 \\ 0 & \frac{2}{\pi}\nu_x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{Q}}_{yy} = \begin{pmatrix} \frac{2}{\pi}\nu_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\pi}\nu_y \end{pmatrix}, \tag{5.3}$$

$$\underline{\underline{Q}}_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3\nu_x\nu_y/8 \\ 0 & 3\nu_x\nu_y/8 & 0 \end{pmatrix}.$$

In order to construct a second order finite difference scheme we need exactly the right amount of numerical diffusion placed in the correct position, as they are prescribed in the matrices $\underline{\underline{P}}_x^2$, $\underline{\underline{P}}_y^2$, $(\underline{\underline{P}}_x\underline{\underline{P}}_y + \underline{\underline{P}}_x\underline{\underline{P}}_y)/2$ if we want (4.15) to hold. We want to point out that the EG3 scheme, the scheme constructed using the theoretical background of

bicharacteristics for hyperbolic systems, has the smallest amount of numerical diffusion, which is moreover placed in the correct positions. Our aim is now to find some correction matrices $\underline{\underline{D}}_{ij}$ for each EG-scheme such that the schemes become second order.
From the second order conditions (4.19) we obtain the following equalities

$$\sum_{i,j} i^2 \underline{\underline{D}}_{ij} = \begin{pmatrix} \nu_x^2 - \sum_{ij} i^2\alpha_{ij}^1 & 0 & 0 \\ 0 & \nu_x^2 - \sum_{ij} i^2\beta_{ij}^2 & 0 \\ 0 & 0 & -\sum_{ij} i^2\gamma_{ij}^3 \end{pmatrix}, \tag{5.4}$$

$$\sum_{i,j} j^2 \underline{\underline{D}}_{ij} = \begin{pmatrix} \nu_y^2 - \sum_{ij} j^2\alpha_{ij}^1 & 0 & 0 \\ 0 & -\sum_{ij} j^2\beta_{ij}^2 & 0 \\ 0 & 0 & \nu_y^2 - \sum_{ij} j^2\gamma_{ij}^3 \end{pmatrix}, \tag{5.5}$$

$$\sum_{i,j} ij \underline{\underline{D}}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu_x\nu_y/2 - \sum_{i,j} ij\gamma_{ij}^2 \\ 0 & \nu_x\nu_y/2 - \sum_{i,j} ij\beta_{ij}^3 & 0 \end{pmatrix}. \tag{5.6}$$

Remember that in principle we could have an 18 parameter family of second order schemes to determine at this point. But, the wave equation system (3.1) has some additional symmetries in the nature of treating the u and v variables. While it is not absolutely necessary that a first order scheme treats u and v in the same manner, it is simpler to do so. Moreover the first order EG schemes that we are considering do have such symmetries. As we see below, this structure leads to only 2 degrees of freedom for the corrections.
Note that for any EG scheme we have only one possibility to define $\underline{\underline{\hat{\gamma}}}^2$ and $\underline{\underline{\hat{\beta}}}^3$, since $\sum_{i,j=-1}^1 ij\gamma_{ij}^2 = \sum_{i,j=-1}^1 ij\beta_{ij}^3 =: q_{xy}$. Thus we have

$$\underline{\underline{\hat{\gamma}}}^2 = \underline{\underline{\hat{\beta}}}^3 = \begin{pmatrix} -(\nu_x\nu_y/8 - q_{xy}/4) & 0 & \nu_x\nu_y/8 - q_{xy}/4 \\ 0 & 0 & 0 \\ \nu_x\nu_y/8 - q_{xy}/4 & 0 & -(\nu_x\nu_y/8 - q_{xy}/4) \end{pmatrix}. \tag{5.7}$$

Further $\underline{\underline{\hat{\gamma}}}^3$ can be easily computed from $\underline{\underline{\hat{\beta}}}^2$. If ν_x and ν_y are exchanged in $\underline{\underline{\hat{\gamma}}}^3$, then it holds that

$$\underline{\underline{\hat{\gamma}}}^3 = (\underline{\underline{\hat{\beta}}}^2)^T. \tag{5.8}$$

Thus we only need to determine the form of matrices $\underline{\underline{\hat{\alpha}}}^1$ and $\underline{\underline{\hat{\beta}}}^2$ in general. However, there are infinitely many possiblities to define $\underline{\underline{\hat{\alpha}}}^1$ and $\underline{\underline{\hat{\beta}}}^2$, such that they satisfy all of the above conditions (4.19), as we will show later. For $\underline{\underline{\hat{\alpha}}}^1$ the only non-zero conditions are

$$\sum_{i,j=-1}^1 i^2\hat{\alpha}_{ij}^1 = \nu_x^2 - \sum_{i,j=-1}^1 i^2\alpha_{ij}^1, \quad \sum_{i,j=-1}^1 j^2\hat{\alpha_{ij}}^1 = \nu_y^2 - \sum_{i,j=-1}^1 j^2\alpha_{ij}^1. \tag{5.9}$$

Similarly, we get for $\underline{\underline{\hat{\beta}}}^2$ the following non-zero conditions

$$\sum_{i,j=-1}^1 i^2\hat{\beta}_{ij}^2 = \nu_x^2 - \sum_{i,j=-1}^1 i^2\beta_{ij}^2, \quad \sum_{i,j=-1}^1 j^2\hat{\beta_{ij}}^2 = -\sum_{i,j=-1}^1 j^2\beta_{ij}^2. \tag{5.10}$$

The general forms of the correction stencil matrices $\underline{\underline{\hat{\alpha}}}^1$ and $\underline{\underline{\hat{\beta}}}^2$ read

$$\underline{\underline{\hat{\alpha}}}^1 = \begin{pmatrix} b/4 & c/2 & b/4 \\ a/2 & -a-c-b & a/2 \\ b/4 & c/2 & b/4 \end{pmatrix}, \quad \underline{\underline{\hat{\beta}}}^2 = \begin{pmatrix} d/4 & f/2 & d/4 \\ e/2 & -e-f-d & e/2 \\ d/4 & f/2 & d/4 \end{pmatrix}, \tag{5.11}$$

where $a+b = \sum_{i,j=-1}^{1} i^2 \hat{\alpha_{ij}}^1$, $c+b = \sum_{i,j=-1}^{1} j^2 \hat{\alpha_{ij}}^1$, $e+d = \sum_{i,j=-1}^{1} i^2 \hat{\beta_{ij}}^2$, $d+f = \sum_{i,j=-1}^{1} j^2 \hat{\beta_{ij}}^2$, $a,b,c,d,e,f \in I\!R$. The sums on the RHS are prescribed by conditions (5.9), (5.10) and we end with two degrees of freedom by specifying, for example, b and d as parameters.

5.1 Some examples of the second order EG schemes

In what follows we give some examples of correction stencil matrices such that all necessary conditions are satisfied. These schemes were also tested numerically and the results are presented in the next section. For simplicity we assume that $\nu_x = \nu_y =: \nu$. We choose the following parameters $b = \nu^2 - 2\nu/\pi$, $b = -2\nu/\pi$, $b = 0$, or $b = \nu^2$ and $d = -4\nu/\pi$ or $d = 0$. These give the correction matrices (5.12) - (5.15) and (5.16) - (5.17), respectively.

EG1 second order corrected schemes

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} \nu^2/4 - \nu/2\pi & 0 & \nu^2/4 - \nu/2\pi \\ 0 & -\nu^2 + 2\nu/\pi & 0 \\ \nu^2/4 - \nu/2\pi & 0 & \nu^2/4 - \nu/2\pi \end{pmatrix} \tag{5.12}$$

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} -\nu/2\pi & \nu^2/2 & -\nu/2\pi \\ \nu^2/2 & -2\nu^2 + 2\nu/\pi & \nu^2/2 \\ -\nu/2\pi & \nu^2/2 & -\nu/2\pi \end{pmatrix} \tag{5.13}$$

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} 0 & \nu^2/2 - \nu/\pi & 0 \\ \nu^2/2 - \nu/\pi & -2\nu^2 + 4\nu/\pi & \nu^2/2 - \nu/\pi \\ 0 & \nu^2/2 - \nu/\pi & 0 \end{pmatrix} \tag{5.14}$$

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} \nu^2/4 & -\nu/\pi & \nu^2/4 \\ -\nu/\pi & -\nu^2 + 4\nu/\pi & -\nu/\pi \\ \nu^2/4 & -\nu/\pi & \nu^2/4 \end{pmatrix} \tag{5.15}$$

$$\underline{\underline{\hat{\beta}}}^2 := \begin{pmatrix} -\nu/\pi & 2\nu/\pi & -\nu/\pi \\ \nu^2/2 & -\nu^2 & \nu^2/2 \\ -\nu/\pi & 2\nu/\pi & -\nu/\pi \end{pmatrix} \tag{5.16}$$

$$\underline{\underline{\hat{\beta}}}^2 := \begin{pmatrix} 0 & 0 & 0 \\ \nu^2/2 - 2\nu/\pi & -\nu^2 + 4\nu/\pi & \nu^2/2 - 2\nu/\pi \\ 0 & 0 & 0 \end{pmatrix} \tag{5.17}$$

Finally, we can say that using previous definitions of correction matrices $\underline{\underline{\hat{\alpha}}}^1, \underline{\underline{\hat{\beta}}}^2$, and $\underline{\underline{\hat{\beta}}}^3, \underline{\underline{\hat{\gamma}}}^2$, $\underline{\underline{\hat{\gamma}}}^3$, cf. (5.7), (5.8), we obtain together 4×2, i.e. eight possibilities for the second order correction matrices $\underline{\underline{D}}_{ij}$. Thus, we have derived eight second order numerical schemes based on the EG1 method. In what follows we will denote them by the EG1-A1, EG1-B1, ..., EG1-D2 depending on the parameter b, i.e. A-D, and on the parameter d, i.e. 1-2. Similar construction will be done for other schemes.

EG2 second order corrected schemes

Note that for the EG2 scheme

$$\underline{\underline{Q}}_{xy} = (\underline{\underline{P}}_x \underline{\underline{P}}_y + \underline{\underline{P}}_x \underline{\underline{P}}_y)/2,$$

and therefore the only necessary correction stencil matrices are $\underline{\underline{\hat{\alpha}}}^1$, $\underline{\underline{\hat{\beta}}}^2$, $\underline{\underline{\hat{\gamma}}}^3$. Moreover, $\underline{\underline{\hat{\gamma}}}^3 = (\underline{\underline{\hat{\beta}}}^2)^T$. Thus, as before, choosing the following parameters $b = \nu^2 - 4\nu/\pi$, $b = -4\nu/\pi$, $b = 0$, or $b = \nu^2$ and $d = -2\nu/3\pi$ or $d = 0$ we obtain the correction matrices $\underline{\underline{\hat{\alpha}}}^1$ and $\underline{\underline{\hat{\beta}}}^2$.

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} \nu^2/4 - \nu/\pi & 0 & \nu^2/4 - \nu/\pi \\ 0 & -\nu^2 + 4\nu/\pi & 0 \\ \nu^2/4 - \nu/\pi & 0 & \nu^2/4 - \nu/\pi \end{pmatrix} \tag{5.18}$$

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} -\nu/\pi & \nu^2/2 & -\nu/\pi \\ \nu^2/2 & -2\nu^2 + 4\nu/\pi & \nu^2/2 \\ -\nu/\pi & \nu^2/2 & -\nu/\pi \end{pmatrix} \tag{5.19}$$

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} 0 & \nu^2/2 - 2\nu/\pi & 0 \\ \nu^2/2 - 2\nu/\pi & -2\nu^2 + 8\nu/\pi & \nu^2/2 - 2\nu/\pi \\ 0 & \nu^2/2 - 2\nu/\pi & 0 \end{pmatrix} \tag{5.20}$$

$$\underline{\underline{\hat{\alpha}}}^1 := \begin{pmatrix} \nu^2/4 & -2\nu/\pi & \nu^2/4 \\ -2\nu/\pi & -\nu^2 + 8\nu/\pi & -2\nu/\pi \\ \nu^2/4 & -2\nu/\pi & \nu^2/4 \end{pmatrix} \tag{5.21}$$

$$\underline{\underline{\hat{\beta}}}^2 := \begin{pmatrix} -\nu/6\pi & 0 & -\nu/6\pi \\ \nu^2/2 - 4\nu/3\pi & -\nu^2 + 10\nu/3\pi & \nu^2/2 - 4\nu/3\pi \\ -\nu/6\pi & 0 & -\nu/6\pi \end{pmatrix} \tag{5.22}$$

$$
\underline{\underline{\hat{\beta}}}^2 := \begin{pmatrix} 0 & -\nu/3\pi & 0 \\ \nu^2/2 - 5\nu/3\pi & -\nu^2 + 4\nu/\pi & \nu^2/2 - 5\nu/3\pi \\ 0 & -\nu/3\pi & 0 \end{pmatrix} \tag{5.23}
$$

In this way we obtain again eight possibilities for the definition of the second order schemes based on the first order EG2 scheme, which we denote, similarly as above, by the EG2-A1, ..., EG2-D2.

EG3 second order corrected schemes

The correction stencil matrices $\underline{\underline{\hat{\gamma}}}^2$, $\underline{\underline{\hat{\gamma}}}^3$ and $\underline{\underline{\hat{\beta}}}^3$ are already uniquely defined by (5.7) and the (5.8). Since the first equation of the EG3 scheme equals the first equation of the EG1 scheme the correction matrix $\underline{\underline{\hat{\alpha}}}^1$ is already determined by (5.12), (5.13), (5.14) and (5.15). Further, taking $d = -2\nu/\pi$ and $d = 0$, we obtain the following choices of matrix $\underline{\underline{\hat{\beta}}}^2$, respectively,

$$
\underline{\underline{\hat{\beta}}}^2 := \begin{pmatrix} -\nu/2\pi & \nu/\pi & -\nu/2\pi \\ \nu^2/2 & -\nu^2 & \nu^2/2 \\ -\nu/2\pi & \nu/\pi & -\nu/2\pi \end{pmatrix}, \tag{5.24}
$$

$$
\underline{\underline{\hat{\beta}}}^2 := \begin{pmatrix} 0 & 0 & 0 \\ \nu^2/2 - \nu/\pi & 2\nu/\pi - \nu^2 & \nu^2/2 - \nu/\pi \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.25}
$$

6 Numerical experiments

The goal of this section is to test numerically accuracy and stability of the second order order EG schemes derived above. We consider the initial value problem for the wave equation with the initial values

$$
\phi(\underline{x}, 0) = -\frac{1}{c}(\sin 2\pi x + \sin 2\pi y), \;\; u(\underline{x}, 0) = 0 = v(\underline{x}, 0).
$$

In this case the exact solution is known:

$$
\phi(\underline{x}, t) = -\frac{1}{c} \cos 2\pi ct (\sin 2\pi x + \sin 2\pi y), \tag{6.1}
$$

$$
u(\underline{x}, t) = \frac{1}{c} \sin 2\pi ct \cos 2\pi x, \tag{6.2}
$$

$$
v(\underline{x}, t) = \frac{1}{c} \sin 2\pi ct \cos 2\pi y. \tag{6.3}
$$

This is the same problem as we used in our recent paper [14], where the behaviour of the first order schemes was studied. The computational domain is taken $[-1, 1] \times [-1, 1]$ and in order to omit any influence of boundary data approximation we take the periodic boundary conditions. In Appendix 2 tables of the errors for the second order schemes are given for meshes of 40×40, $80 \times 80, \ldots, 640 \times 640$ cells, together with the experimental order of convergence (EOC) computed from two meshes of sizes N_1 and N_2

$$EOC = \ln \frac{\|\underline{U}_{N_1}(T) - \underline{U}^n_{N_1}\|}{\|\underline{U}_{N_2}(T) - \underline{U}^n_{N_2}\|} / \ln\left(\frac{N_2}{N_1}\right).$$

In all cases the results are obtained for a CFL-number ν of 0.45 and an end time $T = 0.2$. As we will see in Tables 1-24 the schemes have similar error behaviour, are of second order, although some of them unstable. Nevertheless, by the construction described above we obtain a number of the second order EG schemes whose stability will be analyzed further.

The next important question besides the accuracy of a scheme is its stability. Due to the linearity of the system we are dealing with it is possible, and most convenient, to use Fourier analysis to establish the stability of the EG schemes. It should be pointed out that this is nontrivial because of the approximations to the evolution operator that have been made: if it were possible to use the exact evolution operator unconditional stability would follow immediately. In [14] we proved that the first order EG schemes are stable on $[0, \nu_{max}]$, for some $\nu_{max} > 0$, however analytical formulae for ν_{max} are still unknown for the general 2D case. In fact, the application of standard stability analysis techniques to schemes which solve systems in two and more dimensions is notoriously hard, due to the complexity of algebraic expressions. In the previous sections we were able to derive simple order conditions for the finite difference schemes solving multi-dimensional systems of conservation laws. One would want to have similar simple conditions to determine the stability restrictions on schemes. To our best knowledge there are no general conditions in the literature even for the scalar multi-dimensional advection equation.
The simplest approach known to the authors for obtaining practically reliable indication on the stability of a scheme when the algebra associated with standard techniques becomes intractable is a numerical sampling, see, e.g., [8], [18], [1] and the references therein. Another more sophisticated approach is to use quantifier eliminating algebra packages to derive a proof of stability estimates. However, such an approach also does not solve the problem in general, see [6], [10].
In what follows we estimate the interval of stability with the aim of numerical computations. The methods are strongly stable on a particular grid if $\rho(\underline{\underline{T}}(\xi, \eta)) \leq 1$ for all ξ and $\eta \in [0, 2\pi]$, where ρ is the spectral radius and $\underline{\underline{T}}(\xi, \eta) = \underline{\underline{Id}} + \sum_{k,l=-1}^{1} \underline{\underline{C}}_{kl} \exp(-ih(k\xi + l\eta))$ is the amplification matrix. Here we denote by $\underline{\underline{Id}}$ the unit matrix. We can test the stability of methods by computing $\max_{\xi,\eta} \rho(\underline{\underline{T}}(\xi, \eta))$ over a discrete set of points ξ,η in $[0, 2\pi] \times [0, 2\pi]$. Doing this for different values of ν and observing at which value the maximum exceeds 1, it is possible to estimate the stability limit ν_{max}.
The tables below show spectral radii for each of the second order EG schemes, which we derived, except the D versions, which are unconditionally unstable. We can note that the EG2-C1 and the EG3-A2 have the stability up to the CFL number $\nu_{max} = 1$. Moreover the EG1-A2 and the EG2-C2, whose CFL numbers are 0.98 and 0.99, respectively, are also practically stable up to 1.

7 Conclusion

In the present paper we have derived simple order conditions for finite difference schemes solving the multi-dimensional systems of conservation laws to be of p-th order accuracy.

ν	$\max_{\xi,\eta} \rho(T(\xi,\eta))$ EG1-A1	EG1-B1	EG1-C1	EG1-A2	EG1-B2	EG1-C2
0.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.40	**1.1535**	**1.1535**	**1.1535**	1.0000	1.0000	1.0000
0.50	1.7282	1.7282	1.7282	1.0000	1.0000	1.0000
0.60	2.9212	2.9212	2.9212	1.0000	**1.5095**	1.0000
0.70	2.9212	2.9212	2.9212	1.0000	2.1186	1.0000
0.80	3.5395	3.5395	3.5395	1.0000	2.7823	1.0000
0.90	4.1723	4.1723	4.1723	1.0000	3.5005	**1.2087**
1.0	4.8197	4.8197	4.8197	**1.003**	4.2732	1.7267
1.1	5.4816	5.4816	5.4816	1.4200	5.1005	2.2994

Table 25: Amplification factors for the second order EG1's-schemes

ν	$\max_{\xi,\eta} \rho(T(\xi,\eta))$ EG2-A1	EG2-B1	EG2-C1	EG2-A2	EG2-B2	EG2-C2
0.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.40	1.0000	**1.2697**	1.0000	1.0000	**1.2697**	1.0000
0.50	1.0000	1.9099	1.00000	1.00000	1.9099	1.0000
0.60	**1.1390**	2.5790	1.0000	**1.1390**	2.5790	1.0000
0.70	1.3173	3.2773	1.0000	1.3173	3.2773	1.0000
0.80	1.4446	4.0046	1.0000	1.4446	4.0046	1.0000
0.90	1.5210	4.7610	1.0000	1.5210	4.7610	1.0000
1.0	1.5464	5.5464	1.0000	1.5464	5.5464	**1.0001**
1.1	1.5210	6.3610	**1.4200**	1.5210	6.3610	1.4200

Table 26: Amplification factors for the second order EG2's-schemes

For the special case of the wave equation system in 2D we applied these conditions to genuinely multi-dimensional first order EG-schemes, i.e. EG1, EG2 and EG3. In this manner correction terms of second order were constructed for each EG-scheme and a number of multi-dimensional formally second order EG-schemes were derived. It turned out by numerical experiments that some of the schemes are unstable. The second order behaviour of the others was confirmed by numerical examples. Unfortunately, it is not possible to find similar simple conditions to determine stability restriction for our schemes. Nevertheless, by a relatively easy numerical sampling technique we were able to give a good indication of stability conditions.

In [16] and [17] second order FVEG-schemes based on recovery were constructed. These twenty five point schemes are less compact and their CFL number is strictly less than 1. By the technique described here we constructed several new nine point second order EG-schemes which are stable up to the CFL number 1.

These new second order EG-schemes are based on the Lax-Wendroff procedure by replacing time derivatives by spatial ones. Therefore they inherit from the Lax-Wendroff scheme similar error behaviour concerning dispersion and some limiting procedure should

ν	$\max_{\xi,\eta}\rho(T(\xi,\eta))$ EG3-A1	EG3-B1	EG3-C1	EG3-A2	EG3-B2	EG3-C2
0.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	1.0000	1.0000	1.00000	1.00000	1.0000	1.0000
0.60	**1.0187**	**1.5095**	**1.0187**	1.0000	**1.5095**	1.0000
0.70	1.4506	2.1186	1.4506	1.0000	2.1186	1.0000
0.80	1.9097	2.7823	1.9097	1.0000	2.7823	1.0000
0.90	2.3962	3.5005	2.3962	1.0000	3.5005	**1.2087**
1.0	2.9099	4.2732	2.9099	1.0000	4.2732	1.7268
1.1	3.4508	5.1005	3.4508	**1.4200**	5.1005	2.2994

Table 27: Amplification factors for the second order EG3's-schemes

be used to resolve discontinuities without oscillations.

8 Appendix 1

We can rewrite the numerical scheme (4.2) in the following way:

$$\begin{aligned} \phi^{n+1}_{k,l} &= \phi^n_{k,l} + \sum_{i,j=-1}^{1}\alpha^1_{i,j}\phi^n_{k+i,l+j} + \sum_{i,j=-1}^{1}\beta^1_{i,j}u^n_{k+i,l+j} + \sum_{i,j=-1}^{1}\gamma^1_{i,j}v^n_{k+i,l+j} \\ u^{n+1}_{k,l} &= u^n_{k,l} + \sum_{i,j=-1}^{1}\alpha^2_{i,j}\phi^n_{k+i,l+j} + \sum_{i,j=-1}^{1}\beta^2_{i,j}u^n_{k+i,l+j} + \sum_{i,j=-1}^{1}\gamma^2_{i,j}v^n_{k+i,l+j} \\ v^{n+1}_{k,l} &= v^n_{k,l} + \sum_{i,j=-1}^{1}\alpha^3_{i,j}\phi^n_{k+i,l+j} + \sum_{i,j=-1}^{1}\beta^3_{i,j}u^n_{k+i,l+j} + \sum_{i,j=-1}^{1}\gamma^3_{i,j}v^n_{k+i,l+j}. \end{aligned} \tag{8.1}$$

Below the stencil matrices $\underline{\underline{\alpha}}, \underline{\underline{\beta}}, \underline{\underline{\gamma}}$ for each first order EG scheme are given.

EG1 scheme

$$\underline{\underline{\alpha}}^1 := \left\{\begin{array}{ccc} \frac{\nu_x\nu_y}{4\pi} & \frac{\nu_y}{\pi} - \frac{\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{4\pi} \\ \frac{\nu_x}{\pi} - \frac{\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{\pi} - \frac{2\nu_x}{\pi} - \frac{2\nu_y}{\pi} & \frac{\nu_x}{\pi} - \frac{\nu_x\nu_y}{2\pi} \\ \frac{\nu_x\nu_y}{4\pi} & \frac{\nu_y}{\pi} - \frac{\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{4\pi} \end{array}\right\}, \underline{\underline{\beta}}^1 := \left\{\begin{array}{ccc} \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \\ \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_x}{2} & 0 & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_x}{2} \\ \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \end{array}\right\},$$

$$\underline{\underline{\gamma}}^1 := \begin{Bmatrix} \frac{-\nu_x\nu_y}{3\pi} & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_y}{2} & \frac{-\nu_x\nu_y}{3\pi} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{3\pi} & \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_y}{2} & \frac{\nu_x\nu_y}{3\pi} \end{Bmatrix}, \quad \underline{\underline{\alpha}}^2 := \begin{Bmatrix} \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \\ \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_x}{2} & 0 & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_x}{2} \\ \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \end{Bmatrix},$$

$$\underline{\underline{\beta}}^2 := \begin{Bmatrix} \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{4\pi} \\ \frac{-\nu_x\nu_y}{2\pi} + \frac{2\nu_x}{\pi} & \frac{\nu_x\nu_y}{\pi} - \frac{4\nu_x}{\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{2\nu_x}{\pi} \\ \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{4\pi} \end{Bmatrix}, \quad \underline{\underline{\gamma}}^2 := \begin{Bmatrix} \frac{-3\nu_x\nu_y}{16} & 0 & \frac{3\nu_x\nu_y}{16} \\ 0 & 0 & 0 \\ \frac{3\nu_x\nu_y}{16} & 0 & \frac{-3\nu_x\nu_y}{16} \end{Bmatrix},$$

$$\underline{\underline{\alpha}}^3 := \begin{Bmatrix} \frac{-\nu_x\nu_y}{3\pi} & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_y}{2} & -\frac{\nu_x\nu_y}{3\pi} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{3\pi} & \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_y}{2} & \frac{\nu_x\nu_y}{3\pi} \end{Bmatrix}, \quad \underline{\underline{\beta}}^3 := \begin{Bmatrix} \frac{-3\nu_x\nu_y}{16} & 0 & \frac{3\nu_x\nu_y}{16} \\ 0 & 0 & 0 \\ \frac{3\nu_x\nu_y}{16} & 0 & \frac{-3\nu_x\nu_y}{16} \end{Bmatrix},$$

$$\underline{\underline{\gamma}}^3 := \begin{Bmatrix} \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{2\nu_y}{\pi} & \frac{\nu_x\nu_y}{4\pi} \\ \frac{-\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{\pi} - \frac{4\nu_y}{\pi} & \frac{-\nu_x\nu_y}{2\pi} \\ \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{2\nu_y}{\pi} & \frac{\nu_x\nu_y}{4\pi} \end{Bmatrix}.$$

EG2 scheme

$$\underline{\underline{\alpha}}^1 := \begin{Bmatrix} \frac{\nu_x\nu_y}{2\pi} & \frac{2\nu_y}{\pi} - \frac{\nu_x\nu_y}{\pi} & \frac{\nu_x\nu_y}{2\pi} \\ \frac{2\nu_x}{\pi} - \frac{\nu_x\nu_y}{\pi} & \frac{2\nu_x\nu_y}{\pi} - \frac{4\nu_x}{\pi} - \frac{4\nu_y}{\pi} & \frac{2\nu_x}{\pi} - \frac{\nu_x\nu_y}{\pi} \\ \frac{\nu_x\nu_y}{2\pi} & \frac{2\nu_y}{\pi} - \frac{\nu_x\nu_y}{\pi} & \frac{\nu_x\nu_y}{2\pi} \end{Bmatrix}, \underline{\underline{\beta}}^1 := \begin{Bmatrix} \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \\ \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_x}{2} & 0 & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_x}{2} \\ \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \end{Bmatrix},$$

$$\underline{\underline{\gamma}}^1 := \begin{Bmatrix} \frac{-\nu_x\nu_y}{3\pi} & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_y}{2} & \frac{-\nu_x\nu_y}{3\pi} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{3\pi} & \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_y}{2} & \frac{\nu_x\nu_y}{3\pi} \end{Bmatrix}, \quad \underline{\underline{\alpha}}^2 := \begin{Bmatrix} \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \\ \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_x}{2} & 0 & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_x}{2} \\ \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \end{Bmatrix},$$

$$\underline{\underline{\beta}}^2 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{\nu_y}{3\pi} & \frac{\nu_x\nu_y}{4\pi} \\ \frac{-\nu_x\nu_y}{2\pi} + \frac{5\nu_x}{3\pi} & \frac{\nu_x\nu_y}{\pi} - \frac{10\nu_x}{3\pi} - \frac{2\nu_y}{3\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{5\nu_x}{3\pi} \\ \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{\nu_y}{3\pi} & \frac{\nu_x\nu_y}{4\pi} \end{array} \right\}, \ \underline{\underline{\gamma}}^2 := \left\{ \begin{array}{ccc} \frac{-\nu_x\nu_y}{8} & 0 & \frac{\nu_x\nu_y}{8} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{8} & 0 & \frac{-\nu_x\nu_y}{8} \end{array} \right\},$$

$$\underline{\underline{\alpha}}^3 := \left\{ \begin{array}{ccc} \frac{-\nu_x\nu_y}{3\pi} & \frac{2\nu_x\nu_y}{3\pi} - \frac{\nu_y}{2} & \frac{-\nu_x\nu_y}{3\pi} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{3\pi} & \frac{-2\nu_x\nu_y}{3\pi} + \frac{\nu_y}{2} & \frac{\nu_x\nu_y}{3\pi} \end{array} \right\}, \ \underline{\underline{\beta}}^3 := \left\{ \begin{array}{ccc} \frac{-\nu_x\nu_y}{8} & 0 & \frac{\nu_x\nu_y}{8} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{8} & 0 & \frac{-\nu_x\nu_y}{8} \end{array} \right\},$$

$$\underline{\underline{\gamma}}^3 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{5\nu_y}{3\pi} & \frac{\nu_x\nu_y}{4\pi} \\ \frac{-\nu_x\nu_y}{2\pi} + \frac{\nu_x}{3\pi} & \frac{\nu_x\nu_y}{\pi} - \frac{2\nu_x}{3\pi} - \frac{10\nu_y}{3\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{\nu_x}{3\pi} \\ \frac{\nu_x\nu_y}{4\pi} & \frac{-\nu_x\nu_y}{2\pi} + \frac{5\nu_y}{3\pi} & \frac{\nu_x\nu_y}{4\pi} \end{array} \right\}.$$

EG3 scheme

$$\underline{\underline{\alpha}}^1 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{4\pi} & \frac{\nu_y}{\pi} - \frac{\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{4\pi} \\ \frac{\nu_x}{\pi} - \frac{\nu_x\nu_y}{2\pi} & -\frac{2\nu}{\pi} - \frac{2\nu_y}{\pi} + \frac{\nu_x\nu_y}{\pi} & \frac{\nu_x}{\pi} - \frac{\nu_x\nu_y}{2\pi} \\ \frac{\nu_x\nu_y}{4\pi} & \frac{\nu_y}{\pi} - \frac{\nu_x\nu_y}{2\pi} & \frac{\nu_x\nu_y}{4\pi} \end{array} \right\}, \underline{\underline{\beta}}^1 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \\ \frac{\nu_x}{2} - \frac{2\nu_x\nu_y}{3\pi} & 0 & -\frac{\nu_x}{2} + \frac{2\nu_x\nu_y}{3\pi} \\ \frac{\nu_x\nu_y}{3\pi} & 0 & -\frac{\nu_x\nu_y}{3\pi} \end{array} \right\},$$

$$\underline{\underline{\gamma}}^1 := \left\{ \begin{array}{ccc} -\frac{\nu_x\nu_y}{3\pi} & -\frac{\nu_y}{2} + \frac{2\nu_x\nu_y}{3\pi} & -\frac{\nu_x\nu_y}{3\pi} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{3\pi} & \frac{\nu_y}{2} - \frac{2\nu_x\nu_y}{3\pi} & \frac{\nu_x\nu_y}{3\pi} \end{array} \right\}, \ \underline{\underline{\alpha}}^2 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{3\pi} & 0 & -\frac{\nu_x\nu_y}{3\pi} \\ \frac{\nu_x}{2} - \frac{2\nu_x\nu_y}{3\pi} & 0 & -\frac{\nu_x}{2} + \frac{2\nu_x\nu_y}{3\pi} \\ \frac{\nu_x\nu_y}{3\pi} & 0 & \frac{-\nu_x\nu_y}{3\pi} \end{array} \right\},$$

$$\underline{\underline{\beta}}^2 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{8\pi} & -\frac{\nu_x\nu_y}{4\pi} & \frac{\nu_x\nu_y}{8\pi} \\ \frac{\nu_x}{\pi} - \frac{\nu_x\nu_y}{4\pi} & -\frac{2\nu_x}{\pi} + \frac{\nu_x\nu_y}{2\pi} & \frac{\nu_x}{\pi} - \frac{\nu_x\nu_y}{4\pi} \\ \frac{\nu_x\nu_y}{8\pi} & -\frac{\nu_x\nu_y}{4\pi} & \frac{\nu_x\nu_y}{8\pi} \end{array} \right\}, \ \underline{\underline{\gamma}}^2 := \left\{ \begin{array}{ccc} -\frac{3\nu_x\nu_y}{32} & 0 & \frac{3\nu_x\nu_y}{32} \\ 0 & 0 & 0 \\ \frac{3\nu_x\nu_y}{32} & 0 & -\frac{3\nu_x\nu_y}{32} \end{array} \right\},$$

$$\underline{\underline{\alpha}}^3 := \left\{ \begin{array}{ccc} -\frac{\nu_x\nu_y}{3\pi} & -\frac{\nu_y}{2} + \frac{2\nu_x\nu_y}{3\pi} & -\frac{\nu_x\nu_y}{3\pi} \\ 0 & 0 & 0 \\ \frac{\nu_x\nu_y}{3\pi} & +\frac{\nu_y}{2} - \frac{2\nu_x\nu_y}{3\pi} & \frac{\nu_x\nu_y}{3\pi} \end{array} \right\}, \quad \underline{\underline{\beta}}^3 := \left\{ \begin{array}{ccc} -\frac{3\nu_x\nu_y}{32} & 0 & \frac{3\nu_x\nu_y}{32} \\ 0 & 0 & 0 \\ \frac{3\nu_x\nu_y}{32} & 0 & -\frac{3\nu_x\nu_y}{32} \end{array} \right\},$$

$$\underline{\underline{\gamma}}^3 := \left\{ \begin{array}{ccc} \frac{\nu_x\nu_y}{8\pi} & \frac{\nu_y}{\pi} - \frac{\nu_x\nu_y}{4\pi} & \frac{\nu_x\nu_y}{8\pi} \\ -\frac{\nu_x\nu_y}{4\pi} & -\frac{2\nu_y}{\pi} + \frac{\nu_x\nu_y}{2\pi} & -\frac{\nu_x\nu_y}{4\pi} \\ \frac{\nu_x\nu_y}{8\pi} & \frac{\nu_y}{\pi} - \frac{\nu_x\nu_y}{4\pi} & \frac{\nu_x\nu_y}{8\pi} \end{array} \right\}.$$

9 Appendix 2

N	$\Vert\underline{U}(T) - \underline{U}^n\Vert$	EOC	$\Vert\phi(T) - \phi^n\Vert$	EOC	$\Vert u(T) - u^n\Vert$	EOC
20	0.1278012		0.1226455		0.0254091	
40	0.0330507	1.9511493	0.0325366	1.9143586	0.0041060	2.6295490
80	0.0084099	1.9745238	0.0083474	1.9626589	0.0007233	2.5050121
160	0.0021160	1.9907152	0.0021065	1.9865045	0.0001422	2.3470948
320	0.0005312	1.9940699	0.0005295	1.9920389	0.0000297	2.2612327
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 1. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.12) and $\underline{\underline{\hat{\beta}}}^2$ (5.16) ...EG1-A1

N	$\Vert\underline{U}(T) - \underline{U}^n\Vert$	EOC	$\Vert\phi(T) - \phi^n\Vert$	EOC	$\Vert u(T) - u^n\Vert$	EOC
20	0.1278009		0.1226451		0.0254092	
40	0.0330501	1.9511714	0.0325359	1.9143847	0.0041063	2.6294249
80	0.0084086	1.9747103	0.0083461	1.9628674	0.0007241	2.5035540
160	0.0021136	1.9921658	0.0021038	1.9880946	0.0001438	2.3322819
320	0.0005263	2.0057572	0.0005242	2.0047193	0.0000329	2.1264974
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 2. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.13) and $\underline{\underline{\hat{\beta}}}^2$ (5.16) ...EG1-B1

N	$\Vert\underline{U}(T) - \underline{U}^n\Vert$	EOC	$\Vert\phi(T) - \phi^n\Vert$	EOC	$\Vert u(T) - u^n\Vert$	EOC
20	0.1278011		0.1226454		0.0254091	
40	0.0330506	1.9511542	0.0325364	1.9143645	0.0041061	2.6295185
80	0.0084096	1.9745698	0.0083471	1.9627103	0.0007235	2.5046565
160	0.0021154	1.9910690	0.0021058	1.9868920	0.0001426	2.3434728
320	0.0005300	1.9969112	0.0005282	1.9951079	0.0000305	2.2269886
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 3. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.14) and $\underline{\underline{\hat{\beta}}}^2$ (5.16) ...EG1-C1

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226453		0.0254092	
40	0.0330504	1.9511610	0.0325362	1.9143725	0.0041062	2.6294787
80	0.0084092	1.9746301	0.0083467	1.9627776	0.0007238	2.5041849
160	0.0021147	1.9915371	0.0021050	1.9874053	0.0001431	2.3386662
320	0.0005284	2.0007061	0.0005265	1.9992201	0.0000315	2.1828538
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 4. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.15) and $\underline{\underline{\hat{\beta}}}^2$ (5.16) …EG1-D1

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278012		0.1226455		0.0254091	
40	0.0330507	1.9511493	0.0325366	1.9143586	0.0041060	2.6295490
80	0.0084099	1.9745237	0.0083474	1.9626588	0.0007233	2.5050097
160	0.0021160	1.9907151	0.0021065	1.9865043	0.0001422	2.3470921
320	0.0005312	1.9940697	0.0005295	1.9920387	0.0000297	2.2612283
640	0.0001361	1.9646285	0.0001359	1.9619296	0.0000048	2.6213262

Tab 5. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.12) and $\underline{\underline{\hat{\beta}}}^2$ (5.17) …EG1-A2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278009		0.1226451		0.0254092	
40	0.0330501	1.9511714	0.0325359	1.9143847	0.0041063	2.6294249
80	0.0084086	1.9747102	0.0083461	1.9628673	0.0007241	2.5035516
160	0.0021136	1.9921657	0.0021038	1.9880945	0.0001438	2.3322793
320	0.0005263	2.0057569	0.0005242	2.0047190	0.0000329	2.1264934
640	0.0001264	2.0575519	0.0001254	2.0636897	0.0000114	1.5314628

Tab 6. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.13) and $\underline{\underline{\hat{\beta}}}^2$ (5.17) …EG1-B2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226454		0.0254091	
40	0.0330506	1.9511542	0.0325364	1.9143644	0.0041061	2.6295185
80	0.0084096	1.9745698	0.0083471	1.9627102	0.0007235	2.5046541
160	0.0021154	1.9910689	0.0021058	1.9868919	0.0001426	2.3434701
320	0.0005300	1.9969110	0.0005282	1.9951077	0.0000305	2.2269848
640	0.0001337	1.9872785	0.0001334	1.9858337	0.0000064	2.2463000

Tab 7. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.14) and $\underline{\underline{\hat{\beta}}}^2$ (5.17) …EG1-C2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226453		0.0254092	
40	0.0330504	1.9511609	0.0325362	1.9143725	0.0041062	2.6294787
80	0.0084092	1.9746300	0.0083467	1.9627776	0.0007238	2.5041825
160	0.0021147	1.9915370	0.0021050	1.9874052	0.0001431	2.3386635
320	0.0005284	2.0007059	0.0005265	1.9992199	0.0000315	2.1828499
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 8. EG1 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.15) and $\underline{\underline{\hat{\beta}}}^2$ (5.17) …EG1-D2

N	$\Vert\underline{U}(T)-\underline{U}^n\Vert$	EOC	$\Vert\phi(T)-\phi^n\Vert$	EOC	$\Vert u(T)-u^n\Vert$	EOC
20	0.1278013		0.1226456		0.0254090	
40	0.0330509	1.9511425	0.0325368	1.9143506	0.0041059	2.6295864
80	0.0084102	1.9744742	0.0083478	1.9626028	0.0007231	2.5054451
160	0.0021168	1.9902785	0.0021073	1.9860267	0.0001417	2.3515361
320	0.0005327	1.9905951	0.0005311	1.9882978	0.0000287	2.3044914
640	0.0001391	1.9369768	0.0001391	1.9333942	0.0000029	3.3232421

Tab 9. EG2 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.18) and $\underline{\underline{\hat{\beta}}}^2$ (5.22) ...EG2-A1

N	$\Vert\underline{U}(T)-\underline{U}^n\Vert$	EOC	$\Vert\phi(T)-\phi^n\Vert$	EOC	$\Vert u(T)-u^n\Vert$	EOC
20	0.1278010		0.1226452		0.0254092	
40	0.0330502	1.9511666	0.0325361	1.9143790	0.0041063	2.6294535
80	0.0084090	1.9746592	0.0083464	1.9628113	0.0007239	2.5038717
160	0.0021141	1.9918613	0.0021044	1.9877598	0.0001434	2.3354790
320	0.0009876	1.0981349	0.0009864	1.0931307	0.0000335	2.0973744
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 10. EG2 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.19) and $\underline{\underline{\hat{\beta}}}^2$ (5.22) ...EG2-B1

N	$\Vert\underline{U}(T)-\underline{U}^n\Vert$	EOC	$\Vert\phi(T)-\phi^n\Vert$	EOC	$\Vert u(T)-u^n\Vert$	EOC
20	0.1278011		0.1226453		0.0254091	
40	0.0330505	1.9511566	0.0325364	1.9143673	0.0041061	2.6295053
80	0.0084094	1.9745899	0.0083469	1.9627327	0.0007236	2.5044972
160	0.0021152	1.9912252	0.0021055	1.9870633	0.0001427	2.3418670
320	0.0005295	1.9981770	0.0005277	1.9964777	0.0000308	2.2120888
640	0.0001326	1.9973480	0.0001322	1.9966285	0.0000071	2.1115203

Tab 11. EG2 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.20) and $\underline{\underline{\hat{\beta}}}^2$ (5.22) ...EG2-C1

N	$\Vert\underline{U}(T)-\underline{U}^n\Vert$	EOC	$\Vert\phi(T)-\phi^n\Vert$	EOC	$\Vert u(T)-u^n\Vert$	EOC
20	0.1278010		0.1226452		0.0254092	
40	0.0330503	1.9511638	0.0325361	1.9143758	0.0041062	2.6294661
80	0.0084090	1.9746494	0.0083465	1.9627994	0.0007239	2.5040270
160	0.0021144	1.9916931	0.0021047	1.9875763	0.0001433	2.3370703
320	0.0005299	1.9963633	0.0005280	1.9949611	0.0000319	2.1664895
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 12. EG2 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.21) and $\underline{\underline{\hat{\beta}}}^2$ (5.22) ...EG1-D1

N	$\Vert\underline{U}(T)-\underline{U}^n\Vert$	EOC	$\Vert\phi(T)-\phi^n\Vert$	EOC	$\Vert u(T)-u^n\Vert$	EOC
20	0.1278013		0.1226456		0.0254090	
40	0.0330509	1.9511424	0.0325368	1.9143506	0.0041059	2.6295860
80	0.0084102	1.9744741	0.0083478	1.9626028	0.0007231	2.5054438
160	0.0021168	1.9902784	0.0021073	1.9860265	0.0001417	2.3515324
320	0.0005327	1.9905949	0.0005311	1.9882975	0.0000287	2.3044829
640	0.0001391	1.9369763	0.0001391	1.9333938	0.0000029	3.3231907

Tab 13. EG2 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.18) and $\underline{\underline{\hat{\beta}}}^2$ (5.23) ...EG2-A2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278010		0.1226452		0.0254092	
40	0.0330502	1.9511666	0.0325361	1.9143790	0.0041063	2.6294531
80	0.0084090	1.9746591	0.0083464	1.9628113	0.0007239	2.5038705
160	0.0021141	1.9918611	0.0021044	1.9877597	0.0001434	2.3354754
320	0.0010078	1.0689171	0.0010067	1.0638058	0.0000331	2.1138504
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 14. EG2 scheme with $\hat{\underline{\underline{\alpha}}}^1$ (5.19) and $\hat{\underline{\underline{\beta}}}^2$ (5.23) . . . EG2-B2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226453		0.0254091	
40	0.0330505	1.9511565	0.0325364	1.9143673	0.0041061	2.6295049
80	0.0084094	1.9745898	0.0083469	1.9627327	0.0007236	2.5044959
160	0.0021152	1.9912251	0.0021055	1.9870631	0.0001427	2.3418635
320	0.0005295	1.9981767	0.0005277	1.9964774	0.0000308	2.2120810
640	0.0001326	1.9973474	0.0001322	1.9966281	0.0000071	2.1115071

Tab 15. EG2 scheme with $\hat{\underline{\underline{\alpha}}}^1$ (5.20) and $\hat{\underline{\underline{\beta}}}^2$ (5.23) . . . EG2-C2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278010		0.1226452		0.0254092	
40	0.0330503	1.9511638	0.0325361	1.9143758	0.0041062	2.6294657
80	0.0084090	1.9746493	0.0083465	1.9627993	0.0007239	2.5040258
160	0.0021144	1.9916930	0.0021047	1.9875762	0.0001433	2.3370667
320	0.0005297	1.9969270	0.0005278	1.9955418	0.0000320	2.1646132
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 16. EG2 scheme with $\hat{\underline{\underline{\alpha}}}^1$ (5.21)and $\hat{\underline{\underline{\beta}}}^2$ (5.23) . . . EG2-D2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278012		0.1226455		0.0254091	
40	0.0330507	1.9511493	0.0325366	1.9143586	0.0041060	2.6295495
80	0.0084099	1.9745238	0.0083474	1.9626589	0.0007233	2.5050110
160	0.0021160	1.9907152	0.0021065	1.9865045	0.0001422	2.3470953
320	0.0005312	1.9940700	0.0005295	1.9920389	0.0000297	2.2612356
640	0.0001361	1.9646290	0.0001359	1.9619300	0.0000048	2.6213524

Tab 17. EG3 scheme with $\hat{\underline{\underline{\alpha}}}^1$ (5.12) and $\hat{\underline{\underline{\beta}}}^2$ (5.24) . . . EG3-A1

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278009		0.1226451		0.0254092	
40	0.0330501	1.9511714	0.0325359	1.9143847	0.0041063	2.6294254
80	0.0084086	1.9747102	0.0083461	1.9628674	0.0007241	2.5035529
160	0.0021136	1.9921658	0.0021038	1.9880946	0.0001438	2.3322824
320	0.0005263	2.0057572	0.0005242	2.0047193	0.0000329	2.1265000
640	0.0001264	2.0575526	0.0001254	2.0636902	0.0000114	1.5314687

Tab 18. EG3 scheme with $\hat{\underline{\underline{\alpha}}}^1$ (5.13) and $\hat{\underline{\underline{\beta}}}^2$ (5.24) . . . EG3-B1

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226454		0.0254091	
40	0.0330506	1.9511542	0.0325364	1.9143645	0.0041061	2.6295191
80	0.0084096	1.9745698	0.0083471	1.9627103	0.0007235	2.5046554
160	0.0021154	1.9910690	0.0021058	1.9868920	0.0001426	2.3434733
320	0.0005300	1.9969113	0.0005282	1.9951079	0.0000305	2.2269923
640	0.0001337	1.9872791	0.0001334	1.9858341	0.0000064	2.2463186

Tab 19. EG3 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.14) and $\underline{\underline{\hat{\beta}}}^2$ (5.24) ...EG3-C1

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226453		0.0254092	
40	0.0330504	1.9511610	0.0325362	1.9143725	0.0041062	2.6294793
80	0.0084092	1.9746300	0.0083467	1.9627776	0.0007238	2.5041838
160	0.0021147	1.9915371	0.0021050	1.9874053	0.0001431	2.3386667
320	0.0005284	2.0007062	0.0005265	1.9992201	0.0000315	2.1828570
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 20. EG3 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.15) and $\underline{\underline{\hat{\beta}}}^2$ (5.24) ...EG3-D1

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278012		0.1226455		0.0254091	
40	0.0330507	1.9511493	0.0325366	1.9143586	0.0041060	2.6295492
80	0.0084099	1.9745238	0.0083474	1.9626588	0.0007233	2.5050104
160	0.0021160	1.9907151	0.0021065	1.9865044	0.0001422	2.3470937
320	0.0005312	1.9940698	0.0005295	1.9920388	0.0000297	2.2612319
640	0.0001361	1.9646287	0.0001359	1.9619299	0.0000048	2.6213408

Tab 21. EG3 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.12) and $\underline{\underline{\hat{\beta}}}^2$ (5.25) ...EG3-A2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278009		0.1226451		0.0254092	
40	0.0330501	1.9511714	0.0325359	1.9143847	0.0041063	2.6294251
80	0.0084086	1.9747102	0.0083461	1.9628674	0.0007241	2.5035523
160	0.0021136	1.9921657	0.0021038	1.9880946	0.0001438	2.3322808
320	0.0005263	2.0057571	0.0005242	2.0047192	0.0000329	2.1264969
640	0.0001264	2.0575523	0.0001254	2.0636899	0.0000114	1.5314655

Tab 22. EG3 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.13) and $\underline{\underline{\hat{\beta}}}^2$ (5.25) ...EG3-B2

N	$\|\underline{U}(T) - \underline{U}^n\|$	EOC	$\|\phi(T) - \phi^n\|$	EOC	$\|u(T) - u^n\|$	EOC
20	0.1278011		0.1226454		0.0254091	
40	0.0330506	1.9511542	0.0325364	1.9143645	0.0041061	2.6295187
80	0.0084096	1.9745698	0.0083471	1.9627102	0.0007235	2.5046548
160	0.0021154	1.9910689	0.0021058	1.9868919	0.0001426	2.3434717
320	0.0005300	1.9969112	0.0005282	1.9951078	0.0000305	2.2269883
640	0.0001337	1.9872788	0.0001334	1.9858339	0.0000064	2.2463095

Tab 23. EG3 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.14) and $\underline{\underline{\hat{\beta}}}^2$ (5.25) ...EG3-C2

N	$\|\underline{U}(T)-\underline{U}^n\|$	EOC	$\|\phi(T)-\phi^n\|$	EOC	$\|u(T)-u^n\|$	EOC
20	0.1278011		0.1226453		0.0254092	
40	0.0330504	1.9511610	0.0325362	1.9143725	0.0041062	2.6294790
80	0.0084092	1.9746300	0.0083467	1.9627776	0.0007238	2.5041832
160	0.0021147	1.9915371	0.0021050	1.9874052	0.0001431	2.3386651
320	0.0005284	2.0007060	0.0005265	1.9992200	0.0000315	2.1828532
640	unstable	unstable	unstable	unstable	unstable	unstable

Tab 24. EG3 scheme with $\underline{\underline{\hat{\alpha}}}^1$ (5.15) and $\underline{\underline{\hat{\beta}}}^2$ (5.25) ...EG3-D2

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