1) $\int \frac{2x}{x^2+x+1} \, dx$

Partielle Bruchzerlegung:

$$\frac{2x}{x^2+x+1} = \frac{2x+1}{x^2+x+1} - \frac{1}{x^2+x+1}$$

• $\int \frac{2x+1}{x^2+x+1} \, dx = \int \frac{(x^2+x+1)'}{x^2+x+1} \, dx =$

(*)

$= \log|x^2+x+1| + C_1$

• $\int \frac{1}{x^2+x+1} \, dx = \int \frac{dx}{(x+\frac{1}{2})^2 - \frac{1}{4} + 1} =$

$= \frac{4}{3} \int \frac{dx}{\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right)^2 + 1} = \frac{4}{3} \frac{\sqrt{3}}{2} \int \frac{dy}{y^2+1}$

$\frac{2}{\sqrt{3}}(x+\frac{1}{2}) = y$

$\frac{2}{\sqrt{3}} \, dx = dy$

(**)

$= \frac{2}{\sqrt{3}} \operatorname{arctg}(y) + C_2 = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C_2$

Zusammen $\Rightarrow$ $\int \frac{2x}{x^2+x+1} \, dx = \underline{\log|x^2+x+1| - \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$

(2.) $\int \frac{1}{(x+1)^2(x+2)}\,dx$

$$\frac{a}{(x+1)^2} + \frac{b}{(x+1)} + \frac{c}{(x+2)} = \frac{a(x+2) + b(x+1)(x+2) + c(x+1)^2}{(x+1)^2(x+2)}$$

$$\to \quad a(x+2) + b(x+1)(x+2) + c(x+1)^2 \overset{!}{=} 1$$

$$\Leftrightarrow \quad x^2(b+c) + x(a+3b+2c) + (2a+2b+c) = 1$$

$$\Rightarrow \quad b+c=0, \quad a+3b+2c=0, \quad 2a+2b+c=1$$

$$\Rightarrow \quad a=1, \quad b=-1, \quad c=1$$

$$\int \frac{1}{(x+1)^2(x+2)}\,dx = \int \frac{1}{(x+1)^2}\,dx + \int \frac{-1}{(x+1)}\,dx + \int \frac{1}{(x+2)}\,dx$$

$$= -\frac{1}{(x+1)} - \log|x+1| + \log|x+2| + C$$

(3.) $\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3} x^3 \frac{1}{x} dx$ (3)

$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \cdot \frac{1}{3} x^3 + C =$

$= \frac{1}{3} x^3 \ln x - \frac{x^3}{9} + C$ ✓

(4.) $\int (x^2 + x + 2) \sin x \, dx =$

$= (x^2 + x + 2)(-\cos x) - \int (2x+1)(-\cos x) dx$

$= -(x^2 + x + 2) \cos x + \int (2x+1) \cos x \, dx =$

$= -(x^2 + x + 2) \cos x + (2x+1) \sin x - \int 2 \sin x \, dx$

$= -(x^2 + x + 2) \cos x + (2x+1) \sin x + 2 \cos x + C$

(5.) $\int \frac{1}{1+\sqrt{x}} dx = 2\int \frac{y}{1+y} dy =$

$\sqrt{x} = y \quad (x \geq 0)$

$x = y^2$

$dx = 2y\,dy$

$= 2 \int \frac{1+y}{1+y} dy - 2\int \frac{1}{1+y} dy$

$= 2y - 2 \log |1+y| + C$

$= 2\sqrt{x} - 2 \log |1+\sqrt{x}| + C$

(6.) (4)

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{y}\,dy =$$

$1-x^2 = y$

$-2x\,dx = dy$

$x\,dx = -\frac{1}{2}dy$

$$= -\frac{1}{2}\frac{y^{1/2+1}}{1/2+1} + C =$$

$$= -\frac{1}{3}y^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$$

(7.)

$$\int \cos^7 dx = \int \cos^6 x \cos x\,dx =$$

$$= \int (\underbrace{1-\sin^2 x}_{\cos^2 x})^3 \cos x\,dx = \int (1-y^2)^3\,dy$$

$y = \sin x$

$dy = \cos x\,dx$

$$= \int 1 - 3y^2 + 3y^4 - y^6\,dy =$$

$$= y - y^3 + \frac{3}{5}y^5 - \frac{y^7}{7} + C =$$

$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$