Finite volume evolution Galerkin methods for multidimensional hyperbolic problems

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ABSTRACT *The finite volume evolution Galerkin method couples a finite volume formulation with an approximate evolution Galerkin operator, which takes into account all of the infinitely many directions of propagation of bicharacteristics for multidimensional systems. Piecewise linear recovery yields second order accuracy even with the first order approximate evolution operator. Numerical comparisons of the evolution Galerkin schemes with the commonly used finite volume methods for the wave equation system and for the Euler equations are presented.*

Key Words: genuinely multidimensional schemes, hyperbolic systems, wave equation, Euler equations, evolution Galerkin schemes, finite volume methods

1. Introduction

It is our belief that the most satisfying methods for approximating evolutionary PDE's are based on approximating the corresponding evolutionary operator. In order to construct a genuinely multidimensional numerical scheme for hyperbolic conservation laws all of the infinitely many directions of propagation of bicharacteristics have to be taken into account.

This is the main idea of the **evolution Galerkin (EG)** methods, which evolve the initial data using the bicharacteristic cone and then project them onto a finite element space. In the recent paper Lukáčová, Morton and Warnecke [LMW 99] three new first order evolution Galerkin schemes for a system of hyperbolic equations, and particularly for the wave equation system are derived and analysed. It is shown that the evolution Galerkin scheme (denoted in the paper [LMW 99] as the EG3 method), which is based on the general theory of bicharacteristics for hyperbolic conservation laws gives the most accurate numerical scheme. This is a base for the second order scheme, which will be constructed using a piecewise linear recovery.

In the recent years the most commonly used numerical shemes for hyperbolic conservation laws were the finite volume methods, which are based on a type of directional splitting and on an approximate solution of one-dimensional Riemann problems. Their popularity is particularly due to the simplicity of their formulation as well as implementation. However, it is a known fact that these methods can produce structural deficiences in the solution for some special multidimensional problems (see, e.g., [FEY 92], [LEV 97], [LMW 98], [LMW 99]).

The finite volume evolution Galerkin method combines advantages of both approaches: the simplicity of the finite volume formulation and the multidimensionality of the evolution Galerkin schemes.

2. Finite volume evolution Galerkin methods

Consider a general hyperbolic system in d space dimensions

$$\underline{U}_t + \sum_{k=1}^{d} \left(\underline{F}_k(\underline{U})\right)_{x_k} = 0, \;\; \underline{x} = (x_1, \ldots, x_d)^T \in I\!R^d, \tag{1}$$

where $\underline{F}_k = \underline{F}_k(\underline{U}), k = 1, \ldots, d$ represent given flux functions and the unknown functions are $\underline{U} = (u_1, \ldots, u_m)^T \in I\!R^m$. Let us denote by $E(s) : (H^k(I\!R^d))^m \to (H^k(I\!R^d))^m$ the exact evolution operator associated with a time step s for the system (1), i.e.

$$\underline{U}(\cdot, t+s) = E(s)\underline{U}(\cdot, t). \tag{2}$$

We suppose that S_h^p is a finite element space consisting of piecewise polynomials of order $p \geq 0$. Let $\underline{U}^n$ be an approximation in the space S_h^p to the exact solution $\underline{U}(\cdot, t_n)$ at a time $t_n > 0$ and take $E_\tau : S_h^r \to (H^k(I\!R^d))^m$ to be a suitable approximation to the exact evolution operator $E(\tau)$, $r \geq 0$. We denote by $R_h : S_h^p \to S_h^r$ a reconstruction operator, $r > p \geq 0$. In the present paper we

shall limit our consideration to cases of constant time step Δt, i.e. $t_n = n\Delta t$, and of a uniform mesh consisting of d-dimensional cubes with a uniform mesh size h.

Definition 1. *Starting from some initial value $\underline{U}^0$ at time $t = 0$, the finite volume evolution Galerkin method (FVEG) is recursively defined by means of*

$$\underline{U}^{n+1} = \underline{U}^n - \frac{\Delta t}{h} \sum_{k=1}^{d} \delta_{x_k} \underline{F}_k(\underline{U}^{n+*}), \tag{3}$$

where the central difference $v(x + h/2) - v(x - h/2)$ is denoted by $\delta_x v(x)$ and $\delta_{x_k} \underline{F}_k(\underline{U}^{n+*})$ represents an approximation to the edge flux difference. The cell boundary value $\underline{U}^{n+*}$ is evolved using the approximate evolution operator E_τ to $t_n + \tau$ and averaged over $0 \le \tau \le \Delta t$ and along the cell boundary, i.e.

$$\underline{U}^{n+*} = \sum_{i,j \in \mathsf{Z}} \left(\frac{1}{|\partial\Omega_{ij}|\Delta t} \int_0^{\Delta t} \int_{\partial\Omega_{ij}} E_\tau R_h \underline{U}^n \, dS \, d\tau \right) \chi_{ij}, \tag{4}$$

where χ_{ij} is the characteristic function of $\partial\Omega_{ij}$.

There are several advantages to this formulation. The most important is that the first order accurate approximation E_τ to the evolution operator $E(\tau)$ yields an overall second order update from $\underline{U}^n$ to $\underline{U}^{n+1}$. To obtain this second order approximation in the discrete scheme it is only necessary to carry out a recovery stage at each time level to generate a piecewise linear approximation $\tilde{\underline{U}}^n = R_h \underline{U}^n \in S_h^1$, from the piecewise constant $\underline{U}^n \in S_h^0$, to feed into the calculation of the fluxes. In the next section we illustrate this procedure for the wave equation system in two space dimensions.

3. Wave equation system

The wave equation can be written down as a first order hyperbolic system

$$\begin{aligned} &\phi_t + c(u_x + v_y) = 0, \\ &u_t + c\phi_x = 0, \\ &v_t + c\phi_y = 0 \end{aligned} \tag{5}$$

with the unknown functions ϕ, u, v. Consider a cone with the apex $P = (x, y, t + \Delta t)$ and the base points $Q = Q(\theta) = (x + c\Delta t \cos\theta, y + c\Delta t \sin\theta, t)$ parametrized by the angle $\theta \in [0, 2\pi]$. Denote by $P' = (x, y, t)$ the center of the base of the cone. The lines from $Q(\theta)$ to P generating the mantle of the so-called bicharacteristic cone are called bicharacteristics, see, e.g., [LMW 99] for more details. Using the theory of bicharacteristics it can be shown that the solution (ϕ, u, v) at the point P is determined by its values on the base as well as on the mantle of the characteristic cone and the exact evolution formulae can be derived. In the

recent papers Lukáčová, Morton and Warnecke [LMW 98], [LMW 99] several approximate evolution operator for the wave equation system were analysed. It was shown that the following approximate evolution operator leads to the best first order scheme in terms of accuracy, see [LMW 98], [LMW 99].

3.1. *First order approximate evolution operator*

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi} \phi_Q - 2u_Q\cos\theta - 2v_Q\sin\theta \mathrm{d}\theta + O(\Delta t^2), \tag{6}$$

$$u_P = \frac{1}{2}u_{P'} + \frac{1}{2\pi}\int_0^{2\pi} -2\phi_Q\cos\theta + u_Q(3\cos^2\theta - 1) + 3v_Q\sin\theta\cos\theta \mathrm{d}\theta + O(\Delta t^2), \tag{7}$$

$$v_P = \frac{1}{2}v_{P'} + \frac{1}{2\pi}\int_0^{2\pi} -2\phi_Q\sin\theta + 3u_Q\sin\theta\cos\theta + v_Q(3\sin^2\theta - 1)\mathrm{d}\theta + O(\Delta t^2). \tag{8}$$

Denote by P_h L^2 - projection onto a space of piecewise constant functions in $\mathbb{R}^2$, then we obtain the first order scheme $\underline{U}^{n+1} = P_h E_{\Delta t}\underline{U}^n$, which is in [LMW 99] referred to as the EG3 scheme. Space integrals coming from the projection step are computed exactly, i.e. no numerical quadrature is used. The finite difference formulation can be found in [LMW 99], where the coefficients of the scheme are given explicitely.

3.2. *Second order reconstruction*

In order to construct the second order FVEG scheme we take the first order accurate approximate evolution operator (6) - (8) and define a bilinear reconstruction R_h. There are many possible recovery schemes, which could be used. For our computation we choose a discontinous bilinear recovery using a four point averages at each vertex, but others can be used as well. It is taken to be conservative and given as

$$R_h\underline{U}|_{\Omega_{ij}} = \underline{U}_{ij} + \frac{(x-x_i)}{4h}\left(\Delta_{0x}\underline{U}_{ij+1} + 2\Delta_{0x}\underline{U}_{ij} + \Delta_{0x}\underline{U}_{ij-1}\right)$$
$$+\frac{(y-y_j)}{4h}\left(\Delta_{0y}\underline{U}_{i+1j} + 2\Delta_{0y}\underline{U}_{ij} + \Delta_{0y}\underline{U}_{i-1j}\right) + \frac{(x-x_i)(y-y_j)}{h^2}\Delta_{0y}\Delta_{0x}\underline{U}_{ij},$$

where $\Delta_{0x}v(x) = \frac{1}{2}\left(v(x+h) - v(x-h)\right) = \frac{1}{2}\left(v_{i+1} - v_{i-1}\right)$, an analogous notation is used for Δ_{0y}. For the computation of fluxes through cell edges the time averaged value $\underline{U}^{n+*}$ has to be known, see (4). Instead of exact time integration

the second order midpoint rule is used, i.e.

$$\underline{U}^{n+*} = \int_{\partial\Omega_{ij}} E_{\Delta t/2} R_h \underline{U}^n \mathrm{d}S + O(\Delta t^2). \tag{9}$$

Two dimensional space integrals of bilinear functions $R_h \underline{U}^n$ with respect to θ and S which occur in (9) are computed exactly without any numerical quadrature and thus all of the infinitely many directions of propagation of flow information are taken explicitely into account. The above construction leads to the overall second order scheme, which gives in regions of smooth solution very accurate results even on coarse grids, see Table 1 below.

3.3. *Numerical results*

We consider the space periodic problem for the wave equation system (5) with the initial data

$$\phi(\underline{x}, 0) = \frac{1}{c}(\sin 2\pi x + \sin 2\pi y), \;\; u(\underline{x}, 0) = 0 = v(\underline{x}, 0).$$

In this case the exact solution can be easily computed [LMW 99]. In Table 1 we compare errors of several second order schemes, namely the second order FVEG method, the Lax-Wendroff method (LW) and the standart finite volume flux-vector directional splitting method (FV-FVS), which uses a MUSCL technique for the flux computation and a second order Runge-Kutta time approximation. For the details on the latter one see, e.g., [KRO 97].

We use meshes of 20×20, $40 \times 40, \ldots, 640 \times 640$ cells and compute also the experimental order of convergence (EOC) from two meshes of sizes N_1 and N_2 as

$$EOC = \ln \frac{\|\underline{U}_{N_1}(T) - \underline{U}^n_{N_1}\|}{\|\underline{U}_{N_2}(T) - \underline{U}^n_{N_2}\|} / \ln\left(\frac{N_2}{N_1}\right).$$

In all cases the results are for a CFL-number ν of 0.45 and an end time of $T = 0.2$. Experiments for several other values of ν and T confirm the second order accuracy of the FVEG scheme.

N / $\|\underline{U}(T) - \underline{U}^n\|$	**FVEG**	**LW**	**FV-FVS**
20	0.008647	0.065500	0.057748
40	0.001789	0.016472	0.014502
80	0.000925	0.004133	0.003617
160	0.000290	0.001033	0.000905
320	0.000080	0.000258	0.000227
640	0.000021	0.000064	0.000057
EOC	1.94	1.99	2.00

Table 1. *Comparison of accuracy of the FVEG method, the Lax-Wendroff method and the FV flux-vector splitting method*

4. Euler equations

The system of Euler equations describing the motion of compressible flow in two space dimensions can be written in the form of hyperbolic system (1) with $d = 2$; the definition of $\underline{U}$ and $\underline{F}_i(\underline{U})$ is well-known and can be found e.g. in [KRO 97]. If $\underline{\underline{R}} = (\underline{r}_{1,\underline{n}}, \underline{r}_{2,\underline{n}}, \underline{r}_{3,\underline{n}}, \underline{r}_{4,\underline{n}})$ denotes the matrix of right eigenvectors we can decompose the vector of conservative variables $\underline{U}$ for any direction $\underline{n}$ in the following way

$$\underline{U} = \underline{\underline{R}}\,\underline{\underline{R}}^{-1}\underline{U} = \sum_{k=1}^{4} \alpha_k \underline{r}_{k,\underline{n}}. \tag{10}$$

A general procedure for derivation of the exact evolution operator for hyperbolic problems can also be applied to the Euler equations. For linear problems this procedure works with the characteristic variables $\underline{W} = \underline{\underline{R}}^{-1}\underline{U}$. Now, instead of $\underline{W}$ the vector $\underline{\alpha}$ is used. The following **approximate evolution operator for the Euler equation** was derived in [OST 97]

$$\begin{aligned} \underline{U}(\underline{x}, t+\Delta t) = \frac{1}{|O|}\int_O \sum_{k=1}^{4} \alpha_k \underline{r}_k(\underline{U}(Q_k(\underline{x},\underline{n})))\mathrm{d}O \\ + \frac{1}{|O|}\int_O \left[\alpha_4 \underline{r}_4(\underline{U}(Q_4(\underline{x},\underline{n}))) - \alpha_1 \underline{r}_1(\underline{U}(Q_4(\underline{x},\underline{n})))\right]\mathrm{d}O, \end{aligned} \tag{11}$$

where O denotes the unit sphere, $Q_1(\underline{x},\underline{n}) = (\underline{x} - \Delta t(\underline{u} - c\underline{n}), t)$, $Q_2(\underline{x},\underline{n}) = Q_3(\underline{x},\underline{n}) = (\underline{x} - \Delta t\underline{u}, t)$, $Q_4(\underline{x},\underline{n}) = (\underline{x} - \Delta t(\underline{u} + c\underline{n}), t)$, c is a local speed of sound and $\underline{u}$ represents the fluid velocity.

After L^2 - projection onto a space of piecewise constants we obtain the EG scheme for the Euler equations. It was shown by Ostkamp [OST 97] that Fey's method of transport [FEY 92] for the Euler equations can be reinterpreted as the above evolution Galerkin scheme.

4.1 *Numerical results*

We take the well-known test problem, namely the two-dimensional Sod's problem and compare the behaviour of the evolution Galerkin method (11) and the LeVeque wave propagation algorithm [LEV 97], which is available as a public domain software package called CLAWPACK. In order to avoid a discussion of the limiters we compare here only *first order schemes*. Note however that in LeVeque's wave propagation algorithm also first order correction terms for xy-cross derivative are included.

The computational domain is the square [-1,1] × [-1,1]. To ensure the CFL stability condition, the CFL number is taken 0.8. We choose periodic boundary conditions and the following initial data

$$\begin{aligned} &\rho = 1, u = 0, v = 0, p = 1 && \text{if } |x| \leq 0.4, \\ &\rho = 0.125, u = 0, v = 0, p = 0.1 && \text{otherwise.} \end{aligned} \tag{12}$$

Solutions at time $T = 0.2$ are computed on a quadrilateral grid with 200×200 grid cells. In Figures 1 and 2 the isolines of solution obtained by the evolution Galerkin method (11) and by the wave propagation algorithm, respectively are drawn. The y-velocity is not depicted since it is symmetric to the x-velocity. For the evolution Galerkin scheme the resolution of the flow phenomena is the same in all directions and information is moving in infinitely many directions in a circular manner. However, we can notice that the wave propagation algorithm, which makes use of an improved directional splitting, does not preserve circular symmetry in such a good manner as our scheme and some dependence of the solution on the grid can well be seen.

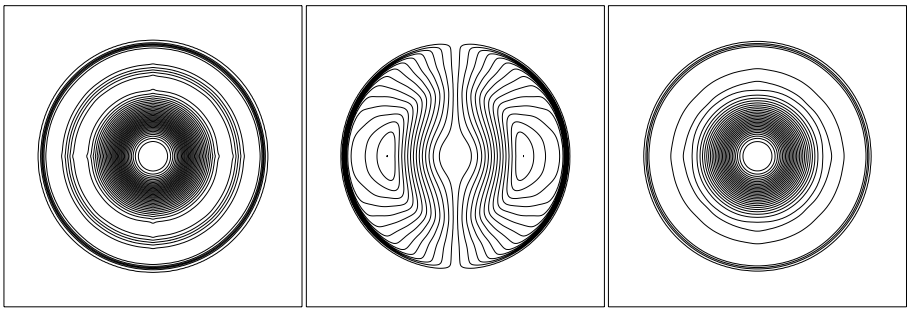

Figure 1. *Evolution Galerkin scheme: density (left), x-velocity (middle), pressure (right)*

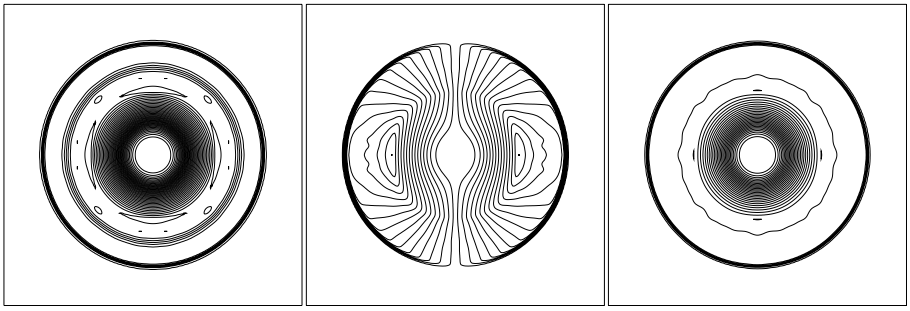

Figure 2. *Wave propagation algorithm: density (left), x-velocity (middle), pressure (right)*

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6. Bibliography

[FEY 92] FEY M., JELTSCH R., «A simple multidimensional Euler scheme», *Proceedings of ECCOMAS'92*, Elsevier Science Publishers, Amsterdam, 1992, p. 667-671.

[KRO 97] KRÖNER D., *Numerical Schemes for Conservation Laws*, Wiley - Teubner, Stuttgart, 1997.

[LEV 97] LEVEQUE R.J., «Wave propagation algorithms for multidimensional hyperbolic systems», *J. Comp. Phys., 131*, 1997, p. 327-353.

[LMW 98] LUKÁČOVÁ - MEDVIĎOVÁ M., MORTON K.W., WARNECKE G., «On the evolution Galerkin method for solving multidimensional hyperbolic systems», *Proceedings of ENUMATH'98*, World Scientific Publishing Company, Singapore, 1998, p. 445-452.

[LMW 99] LUKÁČOVÁ - MEDVIĎOVÁ M., MORTON K.W., WARNECKE G., «Evolution Galerkin methods for hyperbolic systems in two space dimensions», Preprint University of Bath, England, 1999, submitted to *MathComp.*

[OST 97] OSTKAMP S., «Multidimensional characterisitic Galerkin schemes and evolution operators for hyperbolic systems», *Math. Meth. Appl. Sci., 20*, 1997, p. 1111-1125.