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EVOLUTION GALERKIN METHODS FOR MULTIDIMENSIONAL HYPERBOLIC SYSTEMS

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Abstract. *This contribution deals with the genuinely multidimensional numerical schemes for hyperbolic equations. After presenting results for the first order evolution Galerkin schemes we will describe their higher order version using finite volumes. These methods couple a finite volume formulation with approximate evolution Galerkin operators constructed using bicharacteristics. Numerical experiments and comparisons with other commonly used schemes are given.*

1 INTRODUCTION

Evolution Galerkin methods (EG-methods) were proposed to approximate evolutionary problems for systems of partial differential equations, see, e.g., [2], [3], [9], [10]. In Ostkamp [11] as well as Lukáčová, Morton and Warnecke [4]-[7] the EG schemes for the wave equation systems and the Euler equations of gas dynamics in two space dimensions were derived and analyzed. In this survey paper we summarize the known results for EG methods and point out some open questions.

It is our belief that the most satisfying methods for approximating evolutionary PDE's are based on approximating the corresponding evolutionary operator. In order to construct a genuinely multidimensional numerical scheme for hyperbolic conservation laws all of the infinitely many directions of propagation of bicharacteristics have to be taken into account. This is the basic idea of the evolution Galerkin methods, which can be described in the following way.

Consider a general hyperbolic conservation law in d space dimensions

$$\underline{U}_t + \sum_{k=1}^{d} \left(\underline{F}_k(\underline{U})\right)_{x_k} = 0, \;\; \underline{x} = (x_1, \ldots, x_d)^T \in I\!R^d, \tag{1}$$

where $\underline{F}_k = \underline{F}_k(\underline{U}), k = 1, \ldots, d$ represent given physical flux functions and the conservative variables are $\underline{U} = (u_1, \ldots, u_m)^T \in I\!R^m$. Let us denote by $E(s) : (H^k(I\!R^d))^m \to (H^k(I\!R^d))^m$ the exact evolution operator associated with a time step s acting on Sobolev spaces for the system (1), i.e.

$$\underline{U}(\cdot, t+s) = E(s)\underline{U}(\cdot, t). \tag{2}$$

We suppose that S_h^p is a finite element space consisting of piecewise polynomials of order $p \geq 0$. Let $\underline{U}^n$ be an approximation in the space S_h^p to the exact solution $\underline{U}(\cdot, t_n)$ at a time $t_n > 0$ and take $E_\tau : S_h^r \to (H^k(I\!R^d))^m$ to be a suitable approximation to the exact evolution operator $E(\tau)$, $r \geq 0$. We denote by $P_h : (H^k(I\!R^d))^m \to S_h^p$ an L^2-projection onto cells, and by $R_h : S_h^p \to S_h^r$ a recovery operator, $r \geq p \geq 0$. In the present paper we shall limit our considerations to cases of constant time step Δt, i.e. $t_n = n\Delta t$, and of a uniform mesh consisting of d-dimensional cubes with a uniform mesh size h.

Definition 1.1 (EG methods) *Starting from some initial value $\underline{U}^0$ at time $t = 0$, the evolution Galerkin scheme (EG) is recursively defined by means of*

$$\underline{U}^{n+1} = P_h E_\Delta R_h \underline{U}^n.$$

For simple linear hyperbolic systems, such as the wave equation system or the Maxwell equations, this approach has been fully exploited. It was shown that using the theoretical background of bicharacteristics for hyperbolic systems accurate approximate evolution operators can be constructed. Using only a projection onto piecewise constant functions

with no recovery step first order schemes are derived. The accuracy of the methods can be increased by increasing the order of the approximation space and the accuracy of the approximate evolution operator. However, this approach is expensive because all integrals that appear in the formulation of the scheme have to be computed over each mesh cell exactly. We use an alternative approach, where the finite volume formulation is coupled with an approximate evolution Galerkin operator. More precisely, the finite volume evolution Galerkin methods are defined as follows.

Definition 1.2 (FVEG methods)

$$\underline{U}^{n+1} = \underline{U}^n - \frac{1}{h}\int_0^{\Delta t} \sum_{k=1}^{d} \delta_{x_k} \underline{F}_k(\underline{U}^{n+\tau/\Delta t})\, d\tau, \tag{3}$$

where the central difference $v(x+h/2)-v(x-h/2)$ *is denoted by* $\delta_x v(x)$ *and* $\delta_{x_k}\underline{F}_k(\underline{U}^{n+\tau/\Delta t})$ *represents an approximation to the edge flux difference at intermediate time levels* $t_n + \tau$, $\tau \in (0, \Delta t)$. *The cell boundary flux* $F_k\left(\underline{U}^{n+\tau/\Delta t}\right)$ *is evolved using the approximate evolution operator* E_τ *to* $t_n + \tau$ *and averaged along the cell boundary, i.e. e.g. for* $\underline{U}$ *itself*

$$\underline{U}^{n+\tau/\Delta t} = \sum_{i,j\in\mathbf{Z}} \left(\frac{1}{|\partial\Omega_{ij}|} \int_{\partial\Omega_{ij}} E_\tau R_h \underline{U}^n \, dS\, d\tau \right) \chi_{ij}, \tag{4}$$

where χ_{ij} *is the characteristic function of* $\partial\Omega_{ij}$.

In the finite volume methods typically some one-dimensional Riemann solver is used to compute fluxes through cell edges. Here we will use instead multidimensional approximate evolution operators using bicharacteristics. Combining this with a piecewise linear recovery stage yields very accurate second order schemes even with the first order approximate evolution operators.

In the second section we will describe several approximate evolution operators E_Δ for the wave equation system in two space dimensions. Let us mention that the wave equation system can be considered as a linearized prototype of the nonlinear Euler equations of gas dynamics for the gas at rest. Therefore the approximate evolution operators derived for the wave equation system can be generalized further and used also for the Euler equations. In Section 3 we present results of error analysis and discuss the von Neumann stability. Construction of the second order finite volume evolution Galerkin method is described in Section 4. Some results of numerical experimets and comparisons with other commonly used methods are given in Section 5.

2 APPROXIMATE EVOLUTION OPERATORS FOR THE WAVE EQUATION

Denote by ϕ, u, v the unknown functions of the wave equation system

$$\begin{aligned}
&\phi_t + c(u_x + v_y) = 0,\\
&u_t + c\phi_x = 0,\\
&v_t + c\phi_y = 0.
\end{aligned} \tag{5}$$

Consider a cone with the apex $P = (x, y, t + \Delta t)$ and the base points $Q = Q(\theta) = (x + c\Delta t \cos\theta, y + c\Delta t \sin\theta, t)$ parametrized by the angle $\theta \in [0, 2\pi]$. Denote by $P' = (x, y, t)$ the center of the base of the cone. The lines from $Q(\theta)$ to P generating the mantle of the so-called bicharacteristic cone are called bicharacteristics.

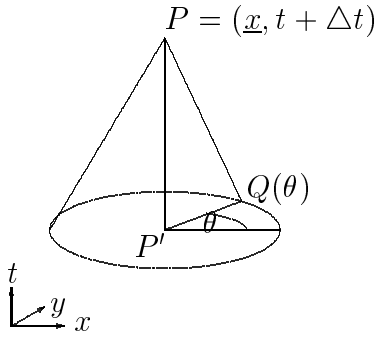

Figure 1: Bicharacteristic along the Mach cone through P and $Q(\theta)$

Using the theory of bicharacteristics it can be shown, that the solution (ϕ, u, v) at the point P is determined by its values on the base as well as on the mantle of the characteristic cone and the exact evolution formulae has been derived; see, e.g. [1], [11], [4]. For the sake of completeness we write down two equivalent formulations of the exact integral equations, which give the basis for further numerical approximations.

2.1 Exact integral equations E1

$$\begin{aligned}
\phi_P = &\frac{1}{2\pi}\int_0^{2\pi}[\phi_Q - u_Q\cos\theta - v_Q\sin\theta]\mathrm{d}\theta\\
&- \frac{1}{2\pi}\int_t^{t+\Delta t}\int_0^{2\pi} S(\tilde{t},\theta)\mathrm{d}\theta\mathrm{d}\tilde{t},
\end{aligned} \tag{6}$$

$$\begin{aligned}
u_P = &\frac{1}{\pi}\int_0^{2\pi}[-\phi_Q\cos\theta + u_Q\cos^2\theta + v_Q\sin\theta\cos\theta]\mathrm{d}\theta\\
&+ \frac{1}{\pi}\int_t^{t+\Delta t}\int_0^{2\pi} S(\tilde{t},\theta)\cos\theta\mathrm{d}\theta\mathrm{d}\tilde{t},
\end{aligned} \tag{7}$$

$$v_P = \frac{1}{\pi}\int_0^{2\pi}[-\phi_Q \sin\theta + u_Q \sin\theta\cos\theta + v_Q \sin^2\theta]\mathrm{d}\theta + \frac{1}{\pi}\int_t^{t+\Delta t}\int_0^{2\pi} S(\tilde{t},\theta)\sin\theta \mathrm{d}\theta\mathrm{d}\tilde{t}, \tag{8}$$

with the source term S given as

$$S(\tilde{t},\theta) = c[u_x(\tilde{x},\tilde{y},\tilde{t})\sin^2\theta - (u_y(\tilde{x},\tilde{y},\tilde{t}) + v_x(\tilde{x},\tilde{y},\tilde{t}))\sin\theta\cos\theta + v_y(\tilde{x},\tilde{y},\tilde{t})\cos^2\theta], \tag{9}$$

where $(\tilde{x},\tilde{y}) = (x + c(t+\Delta t - \tilde{t})\cos\theta, y + c(t+\Delta t - \tilde{t})\sin\theta)$ and $\tilde{t} \in [t, t+\Delta t]$.

2.2 Exact integral equations E2

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi}[\phi_Q - u_Q\cos\theta - v_Q\sin\theta]\,\mathrm{d}\theta - \frac{1}{2\pi}\int_t^{t+\Delta t}\int_0^{2\pi} S(\tilde{t},\theta)\,\mathrm{d}\theta\,\mathrm{d}\tilde{t}, \tag{10}$$

$$u_P = \frac{1}{2\pi}\int_0^{2\pi}[-\phi_Q\cos\theta + u_Q\cos^2\theta + v_Q\sin\theta\cos\theta]\,\mathrm{d}\theta + \frac{1}{2}u_{P'} + \frac{1}{2\pi}\int_t^{t+\Delta t}\int_0^{2\pi} S(\tilde{t},\theta)\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\tilde{t} - \frac{1}{2}c\int_t^{t+\Delta t}\phi_x(x,y,\tilde{t})\,\mathrm{d}\tilde{t}, \tag{11}$$

$$v_P = \frac{1}{2\pi}\int_0^{2\pi}[-\phi_Q\ \sin\theta + u_Q\ \cos\theta\ \sin\theta + v_Q\ \sin^2\theta]\,\mathrm{d}\theta + \frac{1}{2}v(Q_2) + \frac{1}{2\pi}\int_t^{t+\Delta t}\int_0^{2\pi} S(\tilde{t},\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\tilde{t} - \frac{1}{2}c\int_t^{t+\Delta t}\phi_y(x,y,\tilde{t})\,\mathrm{d}\tilde{t}, \tag{12}$$

where $S(\tilde{t},\theta)$ is given by (9). The fact, that the above two integral equations are equivalent can be easily verified by integrating the second and third equations of (5) from P' to P.

Different time integral approximations in the E1 as well as in the E2 lead to various approximate evolution operators. Thus, using for example the rectangle rule in the E1 and the E2 gives the approximate evolution operators for the EG1 and the EG3 scheme, respectively; whereas applying the trapezoidal rule for time integration in the E1 gives the approximate evolution operator for the EG2 scheme. In [8] the authors have derived integral equations for the wave equation system in three space dimensions, which can be considered as a generalization of the E2 operator. Taking a projection onto two space dimensions a new approximate evolution operator for the so-called EG4 scheme is obtained. In what follows we give a list of approximate evolution operators for the wave equation system; for more details on their derivations see [4], [8].

2.3 Approximate evolution operator for the EG1 scheme

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi} \phi_Q - 2u_Q\cos\theta - 2v_Q\sin\theta \mathrm{d}\theta + O(\Delta t^2) \tag{13}$$

$$u_P = \frac{1}{\pi}\int_0^{2\pi} -\phi_Q\cos\theta + u_Q(3\cos^2\theta - 1) + 3v_Q\sin\theta\cos\theta \mathrm{d}\theta + O(\Delta t^2) \tag{14}$$

$$v_P = \frac{1}{\pi}\int_0^{2\pi} -\phi_Q\sin\theta + 3u_Q\sin\theta\cos\theta + v_Q(3\sin^2\theta - 1)\mathrm{d}\theta + O(\Delta t^2) \tag{15}$$

2.4 Approximate evolution operator for the EG2 scheme

$$\phi_P = \frac{1}{\pi}\int_0^{2\pi} \phi_Q - u_Q\cos\theta - v_Q\sin\theta \mathrm{d}\theta - \phi_{P'} + O(\Delta t^3) \tag{16}$$

$$u_P = \frac{1}{\pi}\int_0^{2\pi} -\phi_Q\cos\theta + u_Q(2\cos^2\theta - \tfrac{1}{2}) + 2v_Q\sin\theta\cos\theta \mathrm{d}\theta + O(\Delta t^3) \tag{17}$$

$$v_P = \frac{1}{\pi}\int_0^{2\pi} -\phi_Q\sin\theta + 2u_Q\sin\theta\cos\theta + v_Q(2\sin^2\theta - \tfrac{1}{2})\mathrm{d}\theta + O(\Delta t^3) \tag{18}$$

2.5 Approximate evolution operator for the EG3 scheme

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi} \phi_Q - 2u_Q\cos\theta - 2v_Q\sin\theta \mathrm{d}\theta + O(\Delta t^2) \tag{19}$$

$$u_P = \frac{1}{2}u_{P'} + \frac{1}{2\pi}\int_0^{2\pi} -2\phi_Q\cos\theta + u_Q(3\cos^2\theta - 1) + 3v_Q\sin\theta\cos\theta \mathrm{d}\theta + O(\Delta t^2) \tag{20}$$

$$v_P = \frac{1}{2}v_{P'} + \frac{1}{2\pi}\int_0^{2\pi} -2\phi_Q\sin\theta + 3u_Q\sin\theta\cos\theta + v_Q(3\sin^2\theta - 1)\mathrm{d}\theta + O(\Delta t^2) \tag{21}$$

2.6 Approximate evolution operator for the EG4 scheme

$$\phi_P = \frac{1}{2\pi}\int_0^{2\pi} \phi_Q - 2u_Q\cos\theta - 2v_Q\sin\theta \mathrm{d}\theta + O(\Delta t^2), \tag{22}$$

$$u_P = \frac{1}{2\pi}\int_0^{2\pi} -2\phi_Q\cos\theta + 2u_Q\cos^2\theta + 2v_Q\sin\theta\cos\theta \mathrm{d}\theta + O(\Delta t^2), \tag{23}$$

$$v_P = \frac{1}{2\pi}\int_0^{2\pi} -2\phi_Q\sin\theta + 2u_Q\sin\theta\cos\theta + 2v_Q\sin^2\theta \mathrm{d}\theta + O(\Delta t^2). \tag{24}$$

Denote by P_h an L^2 - projection onto a space of piecewise constant functions S_h^0. Applying P_h onto the approximate evolution operators (13)-(15), (16)-(18), (19)-(21), (22)-(24) yields first order schemes $\underline{U}^{n+1} = P_h E_\Delta \underline{U}^n$, which are in [4] and [8] referred to as the EG1, EG2, EG3 and EG4 schemes. Space integrals coming from the projection step are computed exactly, i.e. no numerical quadrature is used. The finite difference formulation can be found in [4], [8], where the coefficients of the EG schemes in finite difference formulation are given explicitly.

3 STABILITY AND ERROR ANALYSIS

The aim of this section is to sketch the error analysis and discuss the strong stability of the EG1 – EG4 methods. The global error between the exact solution and the approximate solution $\underline{U}^n$ is defined as

$$\underline{e}^n := \underline{U}(t_n) - \underline{U}^n.$$

The error can be decomposed into a projection error $\underline{\eta}$ and an evolutionary error $\underline{\xi}$:

$$\underline{e}^n = (\underline{U}(t_n) - Q\underline{U}(t_n)) + (Q\underline{U}(t_n) - \underline{U}^n) \equiv \underline{\eta}^n + \underline{\xi}^n, \tag{25}$$

where $Q : L^2 \to S_h^0$ is a suitable projection onto S_h^0, possibly different from P_h. In our case we work with a projection $Q : (L^2(I\!R^d))^m \to S_h^0$ given by point evaluation, i.e.

$$Q\underline{U}(t_n) := \sum_{k,l\in\mathbf{Z}} \underline{U}(x_k, y_l, t_n)\ \chi_{kl} \qquad \underline{U} \in (L^2(I\!R^d))^m,$$

where $x_k = kh,\ y_l = lh$ is the midpoint of the square cell Ω_{kl}. Clearly, $\|\eta^n\| = O(h)$.

We derive an evolution estimate for ξ by introducing the term $P_h E_\Delta Q\underline{U}$ as follows,

$$\underline{\xi}^n = (Q\underline{U}(t_n) - P_h E_\Delta Q\underline{U}(t_{n-1})) + (P_h E_\Delta Q\underline{U}(t_{n-1}) - P_h E_\Delta \underline{U}^{n-1}). \tag{26}$$

If the operator $P_h E_\Delta$ is strongly stable, i.e.

$$\|P_h E_\Delta\| \leq 1, \tag{27}$$

the last term is bounded by $\|\underline{\xi}^{n-1}\|$, so that the evolutionary error $\underline{\xi}^n$ is determined by the truncation error

$$T^n := \frac{1}{\Delta t}(Q\underline{U}(t_n) - P_h E_\Delta Q\underline{U}(t_{n-1})), \tag{28}$$

through the recurrence relation $\|\underline{\xi}^n\| \leq \Delta t\|T^n\| + \|\underline{\xi}^{n-1}\|$.

Generally, to obtain an order of accuracy p, we need both the projection error $\underline{\eta}^n$ and the truncation error T^n to be of this order. But we have a free choice of the projection Q to ensure that this holds. Therefore, it remains to establish the strong stability of $P_h E_\Delta$ and compute the order of the truncation error.

In [4] the following results for the EG1, EG2 and EG3 schemes were proved. Moreover, it can be shown analogously that the same results hold also for the EG4 scheme.

Lemma 3.1 (strong stability) *Each of the difference operators representing EG1, EG2, EG3 and EG4 is strongly stable in some positive interval* $0 \leq \nu < \nu_{max}$ *for the CFL number* ν.

Lemma 3.2 (truncation error) *The truncation error* T^n *of the evolution Galerkin schemes EG1, EG2, EG3 and EG4 is of first order, i.e.* $\|T^n\| = O(h)$ *for* $\Delta t/h = \lambda$ *fixed.*

Theorem 3.1 (error estimate) *The evolution Galerkin schemes EG1, EG2, EG3 and EG4 are of first order for* $0 \leq \nu \leq \nu_{max}$, *where* $\nu_{max} \in (0, 1]$. *Suppose we are interested in the approximation at time* $t = T$ *and* $n = T/\Delta t$. *Then there exists a constant* $C > 0$, *such that*

$$\|e^n\| \leq Ch. \tag{29}$$

Due to the linearity of the problem it is possible, and most convenient, to use Fourier analysis to establish the stability of the EG schemes. It should be pointed out that this is nontrivial because of the approximations to the evolution operator that have been made: if it were possible to use the exact evolution operator unconditional stability would follow immediately. As stated in Lemma 3.1 we are able to prove that the interval of stability $[0, \nu_{max})$ is nonempty, however analytical formulae for ν_{max} are still unknown for the general 2D case.

Neverthless, for practical purposes we are able to estimate the interval of stability with the aid of numerical computations. The methods are strongly stable on a particular grid if $\rho(T(\xi, \eta)) \leq 1$ for all ξ and η, where ρ is the spectral radius and $T(\xi, \eta)$ is the amplification matrix. We can test the stability of methods by computing $\max_{\xi,\eta} \rho(T(\xi, \eta))$ over a discrete set of points ξ,η in $[0, 2\pi] \times [0, 2\pi]$. Doing this for different values of ν and

ν	$\max_{\xi,\eta} \rho(T(\xi,\eta))$			
	EG1	EG2	EG3	EG4
0.10	1.00000	1.00000	1.00000	1.00000
0.20	1.00000	1.00000	1.00000	1.00000
0.30	1.00000	1.00000	1.00000	1.00000
0.40	1.00000	1.00000	1.00000	1.00000
0.50	1.00000	1.00000	1.00000	1.00000
0.51	1.00000	1.00000	1.00000	1.00000
0.53	1.00000	1.00000	1.00000	1.00000
0.54	1.00000	**1.00764**	1.00000	1.00000
0.59	1.00000	1.11842	1.00000	1.00000
0.60	1.00000	1.13904	**1.00011**	1.00000
0.70	1.00000	1.31730	1.00856	1.00000
0.72	1.00000	1.34684	1.01145	1.00000
0.73	1.00000	1.36084	1.01315	**1.00001**
0.78	1.00000	1.42323	1.02335	1.00260
0.79	**1.01172**	1.43418	1.02551	1.00372
0.80	1.03718	1.44462	1.02787	1.00486

Table 1: Amplification factors for the first order EG-schemes. Calculation by Y. Zahaykah.

observing at which point the maximum exceeds 1, it is possible to estimate the stability limit ν_{max}.

Note that the EG1 scheme has the best stability, namely it is stable up to the CFL number $\nu = 0.78$. On the other hand, this scheme still has a significant numerical dissipation compared to the EG3 scheme, which gives the best results in term of accuracy, see [4], [5]. Therefore, to achieve better approximation of the solution it is necessary to construct higher order schemes. In the next section we describe a construction of the second order schemes using the finite volume formulation.

4 SECOND ORDER FINITE VOLUME EVOLUTION GALERKIN SCHEMES

There are many possible recovery schemes, which could be used. We only prescribe that the following conservativity property holds

$$P_h R_h \underline{V} = \underline{V} \qquad \text{for all } \underline{V} \in S_h^p. \tag{30}$$

For our computation we choose a discontinous bilinear recovery using finite difference approximation to derivatives, but others can be used and were tested as well.

$$R_h \underline{U}|_{\Omega_{ij}} = \underline{U}_{ij} + \frac{(x - x_i)}{4h} \left(\Delta_{0x} \underline{U}_{ij+1} + 2\Delta_{0x} \underline{U}_{ij} + \Delta_{0x} \underline{U}_{ij-1} \right)$$

$$+\frac{(y-y_j)}{4h}\left(\Delta_{0y}\underline{U}_{i+1j}+2\Delta_{0y}\underline{U}_{ij}+\Delta_{0y}\underline{U}_{i-1j}\right)+\frac{(x-x_i)(y-y_j)}{h^2}\Delta_{0y}\Delta_{0x}\underline{U}_{ij},$$

where $\Delta_{0x}v(x)=\frac{1}{2}\left(v(x+h)-v(x-h)\right)=\frac{1}{2}\left(v_{i+1}-v_{i-1}\right)$, an analogous notation is used for Δ_{0y}.

For the computation of fluxes through cell edges the boundary value of $\underline{U}$ has to be determined. Instead of the exact time integration the second order midpoint rule is used. Thus, the finite volume evolution Galerkin scheme (3) gives

$$\underline{U}^{n+1}=\underline{U}^n-\frac{\Delta t}{h}\sum_{k=1}^{d}\delta_{x_k}\underline{F}_k(\underline{U}^{n+*}), \tag{31}$$

where $\underline{U}^{n+*}=\frac{1}{|\partial\Omega_{ij}|}\int_{\partial\Omega_{ij}}E_{\Delta t/2}R_h\underline{U}^n\mathrm{d}S$.

Two dimensional space integrals of bilinear function $R_h\underline{U}^n$ with respect to θ and cell edges are computed exactly without any numerical quadrature and thus all of the infinitely many directions of propagation of flow information are taken explicitly into account. Example of stencils can be found in [7]. The above construction leads for every approximate evolution operator given in sections 2.3 – 2.6 to the overall second order schemes. Numerical experiments show that these schemes give very accurate results in regions were the solution is smooth.

5 NUMERICAL RESULTS

Consider the initial value problem for the wave equation with the initial data

$$\phi(\underline{x},0)=-c\exp(-15*x^2-15*y^2),\ \ u(\underline{x},0)=0=v(\underline{x},0).$$

The computational domain $[-3,3]\times[-3,3]$ is divided into 100×100 cells. The solution obtained by the second order FVEG3 scheme at different times is depicted in Figures 2-3. We can see well resolved symmetric circular propagation of waves as time increases. This problem can be considered as a model for a pointwise disturbance of a still water surface, e.g. as it occurs when a stone is thrown into a lake.

Although an exact solution is not known for the initial-value problem considered, we can study the experimental order of convergence (EOC). This is computed in the following way using three meshes of sizes $N_1,\ N_2:=N_1/2,\ N_3:=N_2/2$, respectively

$$\mathrm{EOC}=\log_2\frac{\|\underline{U}^n_{N_1}-\underline{U}^n_{N_2}\|_{L^2}}{\|\underline{U}^n_{N_2}-\underline{U}^n_{N_3}\|_{L^2}}.$$

Here $\underline{U}^n_N$ is the approximate solution on the mesh with $N\times N$ cells. The computational domain $[-3,3]\times[-3,3]$ was consecutively divided into 40×40, $80\times80,\ldots,640\times640$ cells in order to obtain the experimental order of the convergence. The CFL-number was set to 0.45 and the final time was taken to be $T=1.0$.

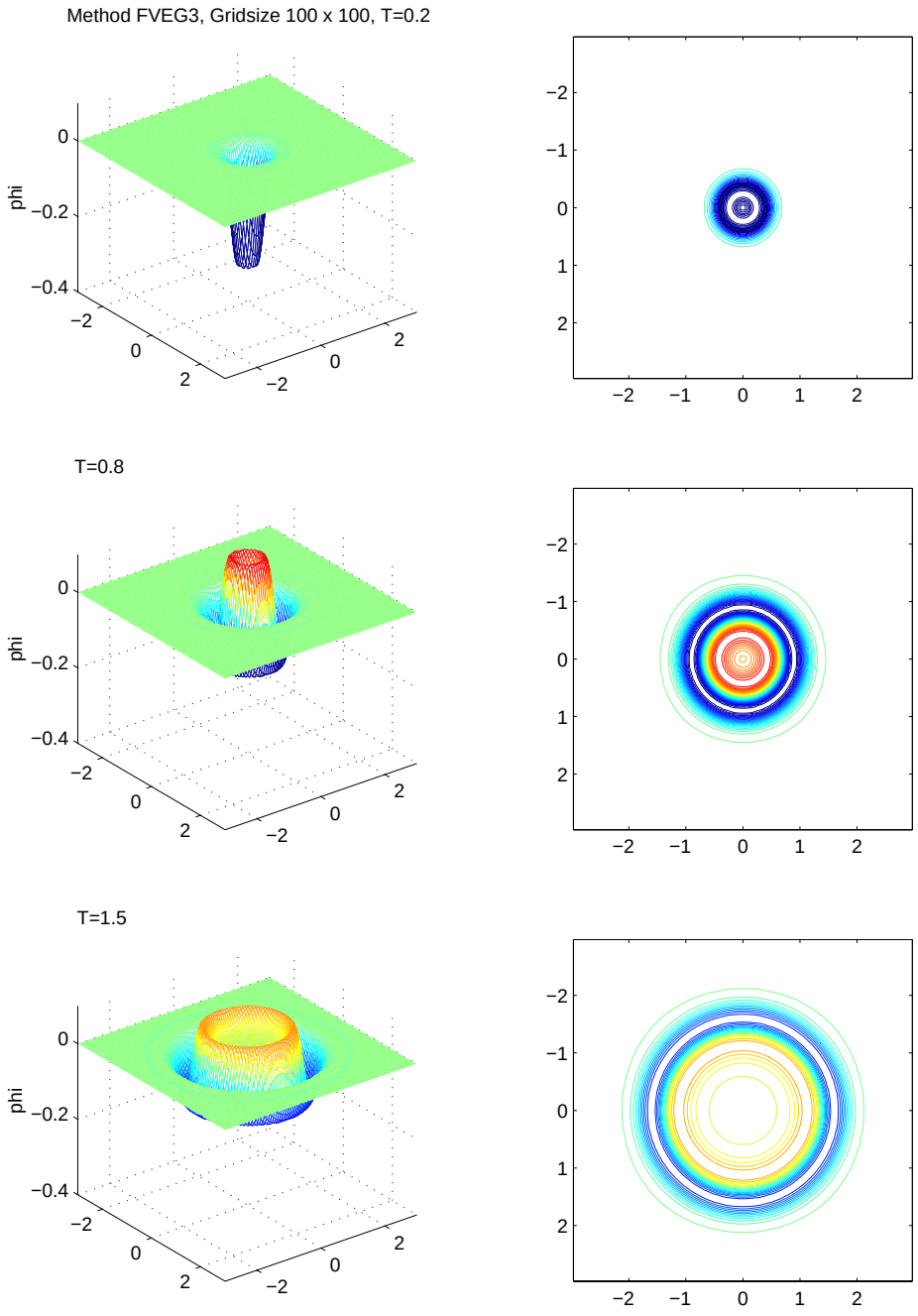

Figure 2: Water wave propagation computed by the FVEG3 scheme at time $T = 0.2, 0.8$ and 1.5.

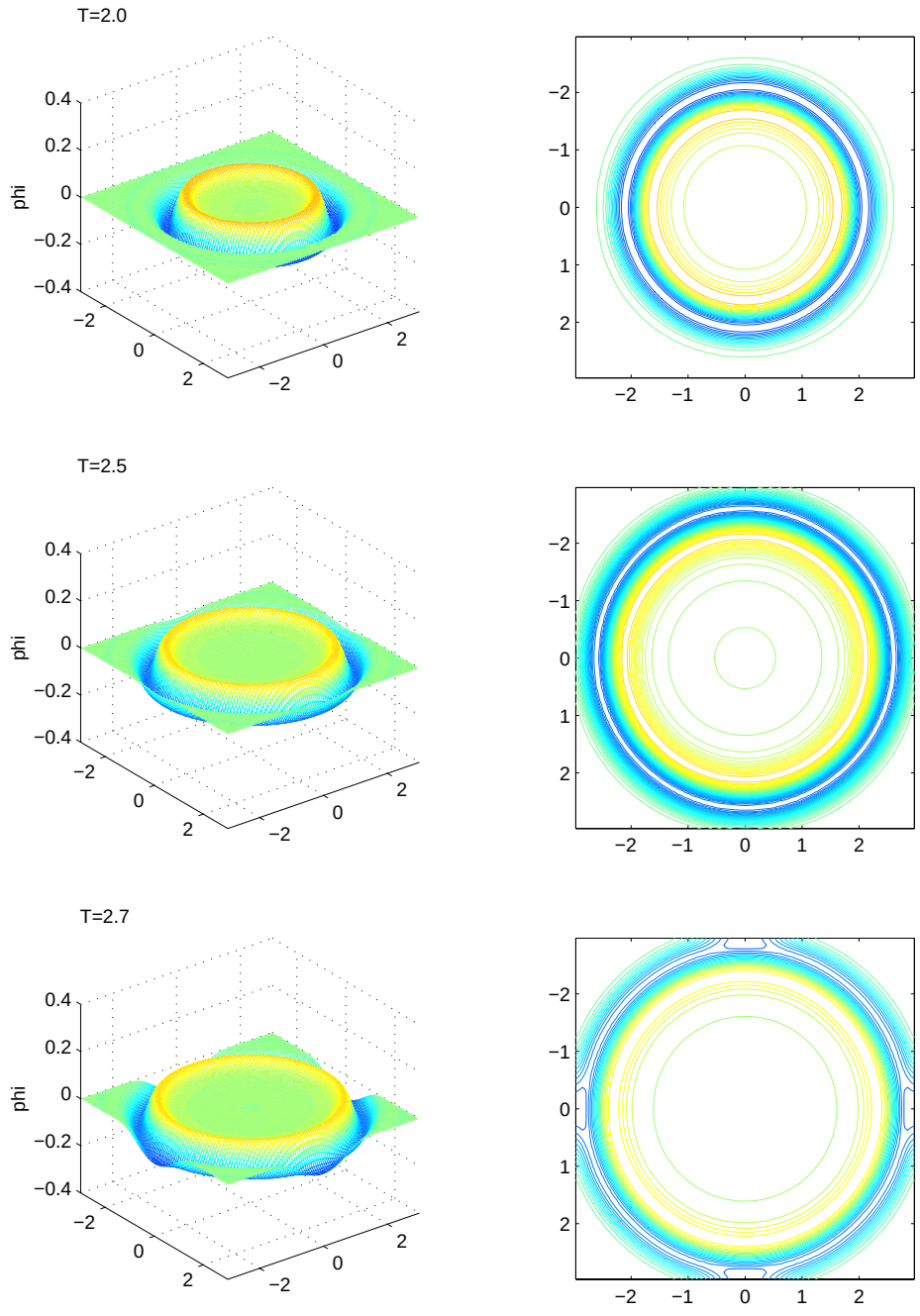

Figure 3: Water wave propagation computed by the FVEG3 scheme at time $T = 2.0, 2.5$ and 2.7.

For a comparison we also give results for other commonly used second order schemes, namely the rotated-Richtmyer Lax-Wendroff method (LW) and the standard finite volume flux-vector directional splitting method (FV-FVS). The finite volume flux splitting scheme computes only fluxes in the normal directions to the cell boundary. To get the second order a MUSCL technique is applied for the flux computation and a second order Runge-Kutta for the time approximation.

$N/\|\underline{U}_N^n - \underline{U}_{N/2}^n\|$	**FVEG1**	**FVEG2**	**FVEG3**	**FVEG4**	**LW**	**FV-FVS**
40	0.106693	0.130002	0.105862	0.108216	0.189925	0.112468
80	0.074452	0.103183	0.077221	0.075361	0.148375	0.091892
160	0.024858	0.048649	0.027619	0.026062	0.067839	0.042943
320	0.004290	0.015035	0.005349	0.005415	0.020317	0.012923
640	0.000593	0.003958	0.000953	0.001110	0.005264	0.003322
EOC	2.85	1.92	2.48	2.28	1.95	1.96

Table 2: L^2 differences and the order of convergence.

The best accuracy is achieved with the first order approximate operator (13)-(15), i.e. for the FVEG1 scheme. It is actually 5 times more accurate than the finite volume directional splitting scheme and 8 times better than the Lax-Wendroff scheme. However, we should note that we have a stricter CFL condition than that of the Lax-Wendroff method, rotated Richtmyer version, for which $\nu_{max} = 1$.

The FVEG3 and FVEG4 have the second best behaviour in terms of accurcay. The FVEG2 scheme, which is obtained from the second order approximate evolution operator, has the third best accuracy. In fact, it is comparable with the accuracy of the FV directional splitting scheme. However, the computational efficiency of the FVEG algorithms is twice as good as that of the FV-FVS since no intermediate state as in the Runge-Kutta method is needed. Fluxes for the FVEG methods are computed only once at the point $\underline{U}^{n+\Delta t/2}$.

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