A Genuinely Multi-dimensional Relaxation Scheme for Hyperbolic Conservation Laws

K. R. Arun. [*], S. V. Raghurama Rao. [†], M. Lukáčová - Medviďová. [‡]and Phoolan Prasad[§]

Abstract

A novel genuinely multi-dimensional relaxation scheme is proposed. Based on a new discrete velocity Boltzmann equation, which is an improvement over previously introduced relaxation systems in terms of isotropic coverage of the multi-dimensional domain by the foot of the characteristic, a finite volume method is developed in which the fluxes at the cell interfaces are evaluated in a genuinely multi-dimensional way, in contrast to the traditional dimension-by-dimension treatment. This algorithm is tested on some bench-mark test problems for hyperbolic conservation laws.

Keywords: genuinely multi-dimensional schemes, relaxation systems, isotropy, hyperbolic conservation laws, discrete velocity Boltzmann equation.

1 Introduction

Finite volume methods have been popular for the numerical solution of hyperbolic conservation laws in the last three decades. Numerical schemes based on kinetic theory represent interesting alternatives to the classical Riemann solver based schemes. A review of *upwind* methods based on *kinetic theory* is given in [6]. Jin and Xin [7] have introduced a new category of upwind method called relaxation schemes. A relaxation system converts a nonlinear convection

[*]Ph.D student, Department of Mathematics, Indian Institute of Science Bangalore, arunkr@math.iisc.ernet.in

[†]Assistant Professor, Department of Aerospace Engineering, Indian Institute of Science Bangalore, raghu@aero.iisc.ernet.in

[‡]Professor, Institute of Numerical Simulation, Hamburg University of Technology Harburg, Germany, lukacova@tu-harburg.de

[§]Honorary Professor and DAE Raja Ramanna Fellow, Department of Mathematics, Indian Institute of Science Bangalore, prasad@math.iisc.ernet.in

equation into linear convection equations with nonlinear source terms. The numerical methods based on a *relaxation system* are called *relaxation schemes*. These schemes avoid solution of Riemann problems. The diagonal form of the relaxation system can be viewed as a *discrete velocity Boltzmann equation* [1]. The so called *discrete kinetic schemes* are based on the discrete velocity Boltzmann equation. Some of the numerical investigations using the discrete Boltzmann equation can be found in [1], [2], [14]. For multi-dimensional flows, however, the traditional finite volume methods are typically based on a dimension-by-dimension treatment using one-dimensional approximate Riemann solver. As a result of this inherently one-dimensional treatment, the discontinuities which are oblique to the coordinate directions are not be resolved accurately. Developing genuinely multi-dimensional algorithms has been a topic of intense research in the last decade and a half. The reader is referred to [5], [8], [9], [11], [15], [17] for some multi-dimensional schemes. In the present work, we have developed a genuinely multi-dimensional relaxation scheme using a discrete velocity Boltzmann equation. In the relaxation system of Aregba-Driollet and Natalini [1], the foot of the bicharacteristic curves are not distributed in an isotropic way. To overcome this deficiency Manisha *et. al* [12] have given an *isotropic relaxation system* in which the foot of the characteristic traverses all quadrants in an isotropic way. The goal of this paper is to derive a genuinely multi-dimensional finite volume scheme based on the relaxation system in [12], that follows the work of Lukáčová; see [10], [11] and the references therein.

2 Relaxation systems for hyperbolic conservation laws

In this section we introduce relaxation systems for conservation laws. For the sake of convenience we present the details only for two space dimensions. Extension to arbitrary space dimensions is straight forward. Consider the following Cauchy problem

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{g}_1(\boldsymbol{u}) + \partial_y \boldsymbol{g}_2(\boldsymbol{u}) = 0, \tag{1}$$

$$\boldsymbol{u}(x, y, 0) = \boldsymbol{u}_0(x, y), \tag{2}$$

where $\boldsymbol{u} \in \mathbb{R}^n$ is the unknown vector, $\boldsymbol{g}_1 \in \mathbb{R}^n, \boldsymbol{g}_2 \in \mathbb{R}^n$ are locally Lipschitz continuous flux functions. We assume that the system (1) is hyperbolic.

2.1 Relaxation system of Jin and Xin

Jin and Xin [7] has proposed the following relaxation system for (1).

$$\begin{aligned}
\partial_t \boldsymbol{u} + \partial_x \boldsymbol{v} + \partial_y \boldsymbol{w} &= \boldsymbol{0}, \\
\partial_t \boldsymbol{v} + \lambda_1^2 \partial_x \boldsymbol{u} &= \frac{1}{\epsilon}\left(\boldsymbol{g}_1(\boldsymbol{u}) - \boldsymbol{v}\right), \\
\partial_t \boldsymbol{w} + \lambda_2^2 \partial_y \boldsymbol{u} &= \frac{1}{\epsilon}\left(\boldsymbol{g}_2(\boldsymbol{u}) - \boldsymbol{w}\right)
\end{aligned} \tag{3}$$

with the initial conditions

$$\boldsymbol{v}(x,y,0) = \boldsymbol{g}_1\left(\boldsymbol{u}_0(x,y)\right), \quad \boldsymbol{w}(x,y,0) = \boldsymbol{g}_2\left(\boldsymbol{u}_0(x,y)\right). \tag{4}$$

Here $\boldsymbol{v}, \boldsymbol{w}$ are new variables, λ_1 and λ_2 are positive constants and ϵ is a small positive constant called the relaxation parameter. In the relaxation limit as $\epsilon \to 0$, from the last two equations in (3) it can be seen that,

$$\boldsymbol{v} = \boldsymbol{g}_1(\boldsymbol{u}), \quad \boldsymbol{w} = \boldsymbol{g}_2(\boldsymbol{u}), \tag{5}$$

up to first order terms. Then from the first equation in (3) one recovers the original conservation law (1). The state satisfying (5) is called a *local equilibrium*. Solution of (3) with the initial conditions (4) in the limit $\epsilon \to 0$ formally satisfies the Cauchy problem (1)-(2). The main advantage of using (3) is that the convection terms are linear. The nonlinear source terms on the right hand side of (3) can be separated by a splitting method, see [7] for more details.

2.2 Discrete velocity Boltzmann equation

We should point out that the relaxation system (3) is not diagonalizable. As it is preferable to work with a system in diagonal form, Aregba-Driollet and Natalini [1] has given the following diagonal relaxation system.

$$\partial_t \boldsymbol{f} + \boldsymbol{\Lambda}_1 \partial_x \boldsymbol{f} + \boldsymbol{\Lambda}_2 \partial_y \boldsymbol{f} = \frac{1}{\epsilon}(\boldsymbol{F} - \boldsymbol{f}). \tag{6}$$

The system (6) is termed as a discrete velocity Boltzmann equation. The initial condition for (6) is given by

$$\boldsymbol{f}(x,y,0) = \boldsymbol{F}\left(\boldsymbol{u}_0(x,y)\right). \tag{7}$$

For the sake of isotropy (see comments below), we choose

$$\boldsymbol{f} = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_4)^{\mathrm{t}} \quad \boldsymbol{F} = (\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \boldsymbol{F}_4)^{\mathrm{t}}.$$

Each of the $\boldsymbol{f}_i$'s and $\boldsymbol{F}_i$'s are from $\mathbb{R}^n$. The system (2.6) consists of $4n$ equations but in very simple form for numerical integration. The matrices $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are block diagonal matrices. The quantities $\boldsymbol{F}_i$'s are called local Maxwellians. The relaxation system (6) is endowed with a matrix operator $\boldsymbol{P}$ such that

$$\boldsymbol{P}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{u}, \quad \boldsymbol{P}\boldsymbol{\Lambda}_j\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{g}_j(\boldsymbol{u}), \quad j = 1, 2. \tag{8}$$

The conditions (8) are necessary for any solution of the relaxation system (6) to converge to the correct entropy weak solution of the hyperbolic conservation law (1), cf [4], [13]. Under the assumption that the Maxwellians are monotone, Natalini [13] has proved the convergence of the solution of a relaxation system (6) to the Kruzkov entropy solution of a scalar conservation law. The relaxation system (6) is completely determined once the matrices $\boldsymbol{\Lambda}_i, i = 1, 2$ and the Maxwellians $\boldsymbol{F}_i, i = 1, 2, 3, 4$ are fixed. Following [1] and [12], we adopt a block structure for the matrices $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ as given below.

$$\boldsymbol{\Lambda}_1 = \text{diag}\left(-\lambda\boldsymbol{I}_n, \lambda\boldsymbol{I}_n, \lambda\boldsymbol{I}_n, -\lambda\boldsymbol{I}_n\right), \quad \boldsymbol{\Lambda}_2 = \text{diag}\left(-\lambda\boldsymbol{I}_n, -\lambda\boldsymbol{I}_n, \lambda\boldsymbol{I}_n, \lambda\boldsymbol{I}_n\right).$$

Here, $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. Note that this choice of the discrete velocities admits the following nice feature, which we call *isotropy*. The bicharacteristics curves of (6) through any point $P(x, y, t + \Delta t)$ falls evenly in all the four quadrants around the point $Q(x, y, t)$; see Figure 1.

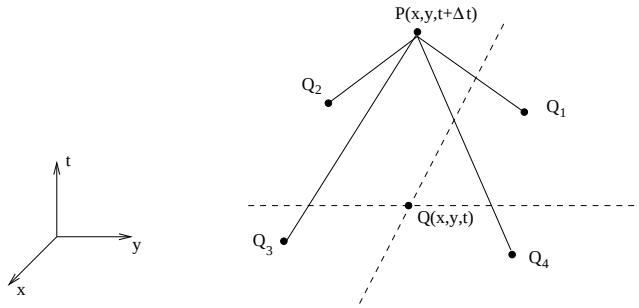

Figure 1: Feet of the bicharacteristics through P

The isotropy is a special property of our relaxation system. This feature enables us to design a genuinely multi-dimensional finite volume scheme for (6). Following [1], [3] we choose the Maxwellians $\boldsymbol{F}_i, i = 1, 2, 3, 4$, in the following way.

$$\boldsymbol{F}_i(\boldsymbol{u}) = \alpha_{i0}\boldsymbol{u} + \alpha_{i1}\boldsymbol{g}_1(\boldsymbol{u}) + \alpha_{i2}\boldsymbol{g}_2(\boldsymbol{u}), \tag{9}$$

where the coefficients α_{ij} are to be fixed so that the conditions (8) are satisfied. A simple calculation yields

$$\begin{aligned} \boldsymbol{F}_1(\boldsymbol{u}) &= \frac{\boldsymbol{u}}{4} - \frac{\boldsymbol{g}_1(\boldsymbol{u})}{4\lambda} - \frac{\boldsymbol{g}_2(\boldsymbol{u})}{4\lambda}, \quad \boldsymbol{F}_2(\boldsymbol{u}) = \frac{\boldsymbol{u}}{4} + \frac{\boldsymbol{g}_1(\boldsymbol{u})}{4\lambda} - \frac{\boldsymbol{g}_2(\boldsymbol{u})}{4\lambda}, \\ \boldsymbol{F}_3(\boldsymbol{u}) &= \frac{\boldsymbol{u}}{4} + \frac{\boldsymbol{g}_1(\boldsymbol{u})}{4\lambda} + \frac{\boldsymbol{g}_2(\boldsymbol{u})}{4\lambda}, \quad \boldsymbol{F}_4(\boldsymbol{u}) = \frac{\boldsymbol{u}}{4} - \frac{\boldsymbol{g}_1(\boldsymbol{u})}{4\lambda} + \frac{\boldsymbol{g}_2(\boldsymbol{u})}{4\lambda}. \end{aligned} \tag{10}$$

Let $\sigma\left(\boldsymbol{F}_i(\boldsymbol{u})\right)$ denote the eigenvalues of $\boldsymbol{F}_i(\boldsymbol{u})$. Then, under some technical assumptions, it is shown in [3] that the Jacobians $\partial_u \boldsymbol{F}_i(\boldsymbol{u})$ are diagonalizable. Moreover if $\sigma\left(\boldsymbol{F}_i(\boldsymbol{u})\right) \subset [0, +\infty[$, then (6) admits a kinetic entropy and in the hydrodynamic limit as $\epsilon \to 0$ the Lax entropy inequality is satisfied; see also [4].

2.3 Chapman-Enskog Analysis

The Chapman-Enskog expansion gives the stability condition for the relaxation system. Since the local equilibrium of the relaxation system (6) is the hyperbolic conservation law (1), we derive a viscous first order approximation to (1), analogous to the compressible Navier-Stokes equations in classical kinetic theory. Consider the asymptotic expansion,

$$\boldsymbol{f}_i = \boldsymbol{F}_i + \epsilon \boldsymbol{f}_i^{(1)} + \epsilon^2 \boldsymbol{f}_i^{(2)} + \cdots . \tag{11}$$

Substituting this expression in the discrete velocity Boltzmann equation (6) and using the consistency conditions (8) the following system will be obtained.

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{g}_1(\boldsymbol{u}) + \partial_y \boldsymbol{g}_2(\boldsymbol{u}) = \epsilon \left(\partial_x \left(\boldsymbol{Q}_{11}\partial_x \boldsymbol{u} + \boldsymbol{Q}_{12}\partial_y \boldsymbol{u}\right) + \partial_y \left(\boldsymbol{Q}_{21}\partial_x \boldsymbol{u} + \boldsymbol{Q}_{22}\partial_y \boldsymbol{u}\right)\right). \tag{12}$$

In (12) the matrices $\boldsymbol{Q}_{ij}$ are dissipation matrices. For (12) to be parabolic all the $\boldsymbol{Q}_{ij}$ should be positive definite. This yields a condition on λ; see [1], [4] for more details. For a scalar conservation law in two dimensions the stability condition turns out to be

$$\lambda^2 \geq \left(\partial_u g_1(u)\right)^2 + \left(\partial_u g_2(u)\right)^2 . \tag{13}$$

3 A numerical scheme based on bicharacteristics

In the literature numerous approaches based on the relaxation systems can be found. For example, in [7] Jin and Xin classify their schemes into two categories, namely, relaxing schemes and relaxed schemes. *Relaxing schemes* depend on ϵ and the artificial variables $\boldsymbol{v}$ and $\boldsymbol{w}$. The zero relaxation limit of the relaxing schemes are called relaxed schemes. The *relaxed schemes* are stable discretizations of the original conservation law and thus are independent of ϵ and the

artificial variables $\boldsymbol{v}$ and $\boldsymbol{w}$. Jin and Xin construct upwinding based on characteristic variables. To achieve second order accuracy they use van Leer's MUSCL discretization. Time discretization is realized by means of a second order TVD Runge-Kutta method. New discrete kinetic schemes have been introduced by Aregba-Driollet and Natalini in [1]. The authors start with a set of decoupled equations, in which the dependent variables are the characteristic variables. The main advantage in this approach is the fact that upwinding becomes very simple. In order to treat multi-dimensional systems the dimensional splitting has been used.
Our goal is to use the discrete Boltzmann equation (6) and approximate it in a genuinely multi-dimensional way using the isotropy of the system. The discrete Boltzmann equation (6) is solved by splitting method as

$$\partial_t \boldsymbol{f} + \boldsymbol{\Lambda}_1 \partial_x \boldsymbol{f} + \boldsymbol{\Lambda}_2 \partial_y \boldsymbol{f} = 0 \quad \text{(convection step)}, \tag{14}$$

$$\frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t} = \frac{1}{\epsilon}(\boldsymbol{F} - \boldsymbol{f}) \quad \text{(relaxation step)}. \tag{15}$$

The relaxation step is further simplified by taking $\epsilon = 0$, leading to $\boldsymbol{f} = \boldsymbol{F}$. Hence at any stage we need to solve only the set of linear convection equations in (14). If we integrate (14) over a mesh cell and over the time interval from $n\Delta t$ to $(n+1)\Delta t$, application of Gauss formula gives,

$$\boldsymbol{f}^{n+1} = \boldsymbol{f}^n - \Delta t \int_0^{\Delta t} \left[\boldsymbol{\Lambda}_1 \delta_x \boldsymbol{f}^{n+\tilde{t}/\Delta t} + \boldsymbol{\Lambda}_2 \delta_y \boldsymbol{f}^{n+\tilde{t}/\Delta t} \right] \mathrm{d}\tilde{t}. \tag{16}$$

In this formula, $\boldsymbol{f}^{n+1}$ and $\boldsymbol{f}^n$ represents the cell averages, while $\delta_x \boldsymbol{f}^{n+\tilde{t}/\Delta t}$ involves averages along the cell edges to the right and left and $\delta_y \boldsymbol{f}^{n+\tilde{t}/\Delta t}$ along the edges to the top and bottom, in all cases at an intermediate time step $n + \tilde{t}/\Delta t$. We approximate the time integrals in (16) by midpoint rule by $\tilde{t} = \Delta t/2$. Then the cell boundary flux $\boldsymbol{f}^{n+1/2}$ is evolved using the exact evolution operator $E_{\Delta t/2}$ for (14) and suitable recovery operators R_h. For example on vertical edges,

$$\boldsymbol{f}^{n+1/2} = \frac{1}{h} \int_0^h E_{\Delta t/2} R_h \boldsymbol{f}^n \mathrm{d}s_y, \tag{17}$$

where $\mathrm{d}s_y$ is an element of arc length along the vertical edges. A similar expression holds for horizontal edges also. We have used the Simpson rule for the cell interface integrals in (17) and evaluated the fluxes at the midpoints and corners of the edges. Note that the Simpson quadrature allows us to take into account the multi-dimensional effects.

4 Numerical results

The new multi-dimensional relaxation scheme is tested on some standard test problems for inviscid Burgers equation and the Euler equations in two space dimensions. In all the problems

the computations were carried out on uniform Cartesian grids.

Burgers equation test cases: These test cases are taken from Spekreijse [16]. It models a shock and a smooth variation representing an expansion fan in 2D domain. The inviscid Burgers equation considered here is given by

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_y u = 0.$$

The computational domain is $[0, 1] \times [0, 1]$. The following two test cases has been considered.

Test case 1: The boundary conditions are given by

$$u(0, y) = 1 \quad 0 < y < 1, \; u(1, y) = -1 \quad 0 < y < 1, \; u(x, 0) = 1 - 2x \quad 0 < x < 1.$$

The solution is computed for time $T = 1.5$ on a 128×128 grid with a CFL number 0.7. The solution contains a shock originating at the point $(0.5, 0.5)$. The isolines of the solution is given in Figure 2.

Test case 2: The boundary conditions are

$$u(0, y) = 1.5 \quad 0 < y < 1, \; u(1, y) = -0.5 \quad 0 < y < 1, \; u(x, 0) = 1.5 - 2x \quad 0 < x < 1.$$

The solution is computed for time $T = 1.5$ on a 128×128 grid with a CFL number 0.7. The solution contains an oblique shock originating at the point $(0.75, 0.5)$. See Figure 2.

Euler equations test cases:

Test case 3: The first test case is the cylindrical explosion problem. The computational domain is the square $[-1, 1] \times [-1, 1]$. The initial data read,

$$\rho = 1, \; u = 0, \; v = 0, \; p = 1, |\boldsymbol{x}| < 0.4, \; \rho = 0.125, \; u = 0, \; v = 0, \; p = 0.1, \text{else}.$$

The solution is computed at time $T = 0.2$ with a CFL number 0.7. The solution exhibits a circular shock and circular contact discontinuity moving away from the centre of the circle and circular rarefaction wave moving in the opposite direction. The isolines of the density, $x-$component of the velocity, $y-$component of the velocity and pressure are given in Figure 3.

Next two test cases are two-dimensional Riemann problems. The computational domain $[-1, 1] \times [-1, 1]$ is divided into four quadrants. The initial data consist of single constant states in each of these four quadrants. These constant values are chosen in such a way that each pair

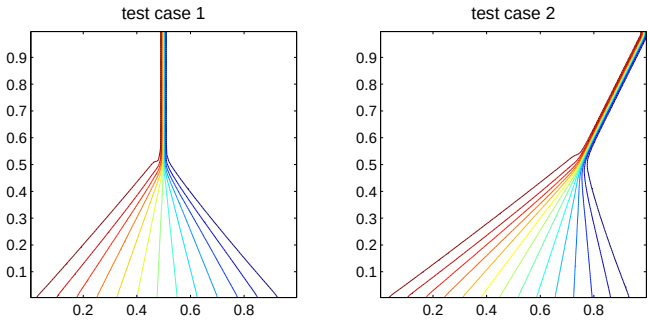

Figure 2: Burgers equation test cases on a 128×128 mesh

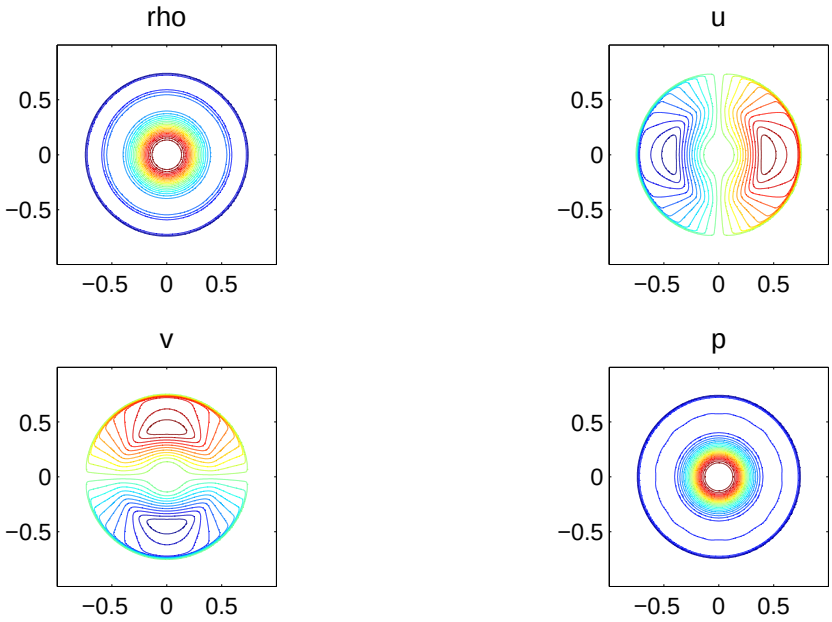

Figure 3: Cylindrical explosion problem: Isolines of the solution calculated on a 400×400 mesh at time $T = 0.2$

of quadrants defines a one-dimensional Riemann problem.

Test case 4: In this test problem we choose the initial data in such a way that two forward moving shocks and two standing slip lines are produced. The initial data read,

$$\begin{array}{lllll}
\rho = 0.5313, & u = 0.0, & v = 0.0, & p = 0.4, & \text{if } x > 0,\ y > 0, \\
\rho = 1.0, & u = 0.0, & v = 0.7276, & p = 1.0, & \text{if } x > 0,\ y < 0, \\
\rho = 1.0, & u = 0.7276, & v = 0.0, & p = 1.0, & \text{if } x < 0,\ y > 0, \\
\rho = 0.8, & u = 0.0, & v = 0.0, & p = 1.0, & \text{if } x < 0,\ y < 0.
\end{array}$$

The solution is computed at time $T = 0.52$ with a CFL number 0.7. The isolines of the density and pressure are given in Figure 4.

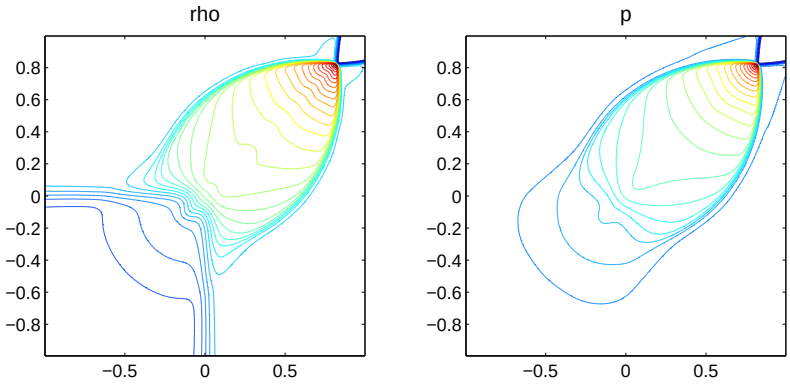

Figure 4: Two dimensional Riemann problem containing two shocks/two slip lines. Isolines of density and pressure calculated on a 400×400 mesh at time $T = 0.52$

Test case 5: This test case is a two dimensional Riemann problem that produces two forward moving shocks and two backward moving shocks. Initial data are taken as

$$\begin{array}{lllll}
\rho = 1.1, & u = 0.0, & v = 0.0, & p = 1.1, & \text{if } x > 0,\ y > 0, \\
\rho = 0.5065, & u = 0.0, & v = 0.8939, & p = 0.35, & \text{if } x > 0,\ y < 0, \\
\rho = 0.5065, & u = 0.8939, & v = 0.0, & p = 0.35, & \text{if } x < 0,\ y > 0, \\
\rho = 1.1, & u = 0.8939, & v = 0.8939, & p = 1.1, & \text{if } x < 0,\ y < 0.
\end{array}$$

Computation are done until time $T = 0.25$ with a CFL number 0.7. The density and pressure isolines are given in Figure 5.

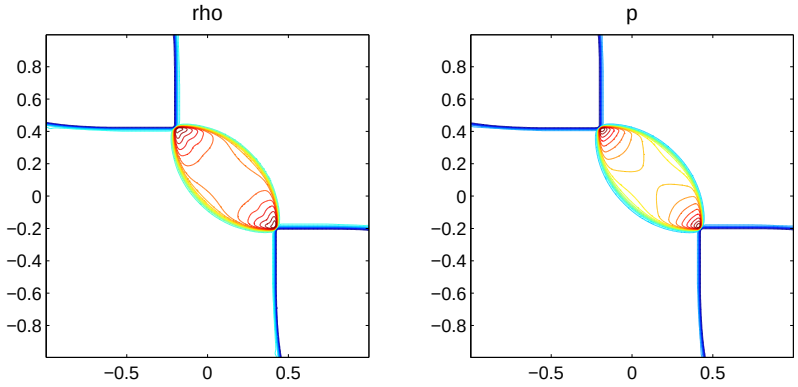

Figure 5: Two dimensional Riemann problem with four shocks. Isolines of density and pressure calculated on a 400×400 mesh at time $T = 0.25$

5 Conclusions

A novel genuinely multi-dimensional relaxation scheme is presented, based on a multi-dimensional relaxation system in which the foot of the characteristics traverses all quadrants in an isotropic way. This scheme is tested on some bench-mark problems for scalar and vector conservation laws in two dimensions and the results demonstrate its efficiency in capturing the flow features accurately.

Acknowledgment. The authors sincerely thank the Department of Science and Technology (DST), Government of India and the German Academic Exchange Service (DAAD) for providing the financial support for collaborative research under the DST-DAAD Project Based Personnel Exchange Programme. K.R. Arun would like to express his gratitude to the Council of Scientific & Industrial Research (CSIR) for supporting his research at the Indian Institute of Science under the grant-09/079(2084)/2006-EMR-1.

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