Numerical aspects of parabolic regularization for resonant hyperbolic balance laws

Marcus Kraft and Mária Lukáčová - Medviďová

Hamburg University of Technology
Institute of Numerical Simulation
Schwarzenbergstrasse 95
21073 Hamburg
Germany
kraft@tu-harburg.de, lukacova@tu-harburg.de

1 FVEG schemes and hyperbolic balance laws

Many problems arising in geophysics and engineering yield hyperbolic balance laws. Let us mention, for example, the compressible duct flow, multiphase flow or shallow water flow with variable bottom topography. All of them can be written in the following form

$$\boldsymbol{u}_t + \sum_{i=1}^{d} (\boldsymbol{f}(\boldsymbol{u}))_{x_i} = \boldsymbol{b}(\boldsymbol{u}), \tag{1}$$

where d is the space dimension. This system belongs to the class of *non-conservative systems*, i.e. systems which cannot be written in the divergence form. The non-conservative products cannot be defined in the distributional way and new tools have to used for theoretical investigations, see Dal Maso, LeFloch and Murat [DLM95]. Additionally, the system is *nonstrictly hyperbolic* with *resonant behaviour*, i.e. we have for the extended system with an additionally equation $\boldsymbol{b}(\boldsymbol{u})_t = 0$, cf. (9), some coinciding eigenvalues and linearly dependent eigenvectors.

As showed by several authors one can construct nonunique solutions of the one-dimensional Riemann problem, which locally satisfy entropy criterion, see, e.g., Chinnaya, LeRoux and Seguin [CLS04], Andrianov and Warnecke [AW04a], [AW04b]. For further theoretical results on resonant systems see also Goatin and LeFloch [GL04].

Despite these theoretical difficulties several numerical schemes have been proposed in order to solve problem (1). One possibility is to use the operator-splitting technique which yields two subsystems

$$\boldsymbol{u}_t + \sum_{i=1}^{d} (\boldsymbol{f}(\boldsymbol{u}))_{x_i} = 0 \tag{2}$$

and

$$\boldsymbol{u}_t = \boldsymbol{b}(\boldsymbol{u}). \tag{3}$$

They can be solved by some finite volume scheme and a suitable ODE-solver. Anyway, in the case where the desired solution is close to the equilibrium state, i.e.

$$\sum_{i=1}^{d} (\boldsymbol{f}(\boldsymbol{u}))_{x_i} = \boldsymbol{b}(\boldsymbol{u}), \tag{4}$$

the operator-splitting approach is not appropriate. In fact, using two different types of discretizations for (2) and (3) will produce unbalanced higher order errors. As a result the balance state (4) can be obtained only for very fine discretizations, which is not efficient. In the recent literature one can find the so-called well-balanced schemes, in which the balance between the gradient of fluxes and the source terms is dictated by the construction of the scheme, see, e.g., [GL96], [ABBKP04], [BKLL04], [NPPN06].

In [LV05] and [LNK06] the well-balanced finite volume evolution Galerkin (FVEG) scheme has been proposed for the shallow water equations with source terms. The FVEG scheme is a two step predictor corrector method. In the corrector step the classical finite volume update is done. In order to evaluate fluxes on cell interfaces the so called approximate evolution operator $E_{\Delta t/2}$ is used. Such an evolution operator is based on the theory of bicharacteristics and takes all of the infinitely many directions of wave propagation into account. Derivation of evolution operators can be found in [LMW00], [LS03], [LSW02], [LMW04] and [LNK06]. Thus, in the predictor step the values on cell interfaces are evolved at certain integrations points at the intermediate time $t_{n+1/2} := t_n + \Delta t/2$. Let us denote by $E_{\Delta t/2}\boldsymbol{U}^n$ the predicted value of the approximate solution at time $t_{n+1/2}$. Then the fluxes at the cell interfaces can be approximated by

$$\bar{\boldsymbol{f}}_k^{n+1/2} := \sum_j \omega_j \boldsymbol{f}_k(E_{\Delta t/2}\boldsymbol{U}^n(\boldsymbol{x}^j(\mathcal{E}))), \tag{5}$$

where the ω_j denote the weights of the integration rule and $\mathcal{E}$ is the cell interface.

Let us consider, for simplicity, a regular rectangular mesh consisting of mesh cells $\Omega_{ij} = [x_i - \hbar/2, x_i + \hbar/2] \times [y_j - \hbar/2, y_j + \hbar/2]$, $i, j \in Z$, $\hbar$ is a mesh size. Further, let us denote the cell averages at time t_n by $\boldsymbol{U}_{ij}^n$ and the central difference operator in x_k-direction by $\delta_{x_k}^{ij}$, i.e. $\delta_{x_1}^{ij} u = u_{i+1/2,j} - u_{i-1/2,j}$. Then the finite volume update reads

$$\boldsymbol{U}_{ij}^{n+1} = \boldsymbol{U}_{ij}^n - \lambda \sum_{k=1}^{2} \delta_{x_k}^{ij} \bar{\boldsymbol{f}}_k^{n+1/2} + \lambda \boldsymbol{B}_{ij}^{n+1/2}, \tag{6}$$

where $\lambda = \Delta t/\hbar$ and $\boldsymbol{B}_{ij}$ stands for the approximation of the source term multiplied by the mesh size, $\hbar\boldsymbol{b}$. Note that in order to obtain a well-balanced scheme a care has to be taken in order to approximate the source term. In our scheme we are using the cell interface approach, see [J01], [LV05], [LNK06].

To solve the problems in numerical solutions arising by non-uniqueness of the Riemann problem for resonant hyperbolic balance laws we propose to use a parabolic regularization by adding a suitable viscosity term. This is motivated by an original physical problem from which the hyperbolic problem arises. Note that approach proposed in this paper is general and can be applied to any numerical scheme (typically finite volume scheme) for the problem (1). For the non-local regularization of phase-transition problems see, e.g., Rohde [R04], Schofer [S06] and the references therein.

2 Shallow water equations with bottom topography

Let us consider the shallow water equations with variable bottom topography. Such a system arises in many geophysical problems, for example in oceanography, river flow or atmospheric flows

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + (hu^2 + gh^2/2)_x + (huv)_y &= -ghb_x \\ (hv)_t + (huv)_x + (hv^2 + gh^2/2)_y &= -ghb_y. \end{aligned} \tag{7}$$

Here h denotes the depth of the shallow water, $(u,v)^T$ is the velocity vector, b represents the bottom topography and g is the gravitational constant.

In [LNK06] the following approximate evolution operator for (7) has been derived

$$\begin{aligned} h\,(P) &= -b(P) \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} (h\,(Q) + b(Q)) - \frac{\tilde{c}}{g}\left(u\,(Q)\,\mathrm{sgn}(\cos\theta) + v\,(Q)\,\mathrm{sgn}(\sin\theta)\right) \mathrm{d}\theta \\ u\,(P) &= \frac{1}{2\pi}\int_0^{2\pi} -\frac{g}{\tilde{c}}(h(Q)+b(Q))\mathrm{sgn}(\cos\theta)\mathrm{d}\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} u\,(Q)\left(\cos^2\theta + \frac{1}{2}\right) + v\,(Q)\sin\theta\cos\theta\mathrm{d}\theta \end{aligned} \tag{8}$$

with an analogous equation for the velocity v. Here $P = (x, y, t_n + \Delta t/2)$ is the apex of the so called bicharacteristic cone, i.e. an integration point for cell interface fluxes. The values $\tilde{c}$, $\tilde{u}$, $\tilde{v}$ are suitable linearizations around P and $Q = (x - \tilde{u}\Delta t/2 + \tilde{c}\Delta t/2\cos\theta, y - \tilde{v}\Delta t/2 + \tilde{c}\Delta t/2\sin\theta, t_n)$. This operator, denoted by $E^{const}_{\Delta t/2}$, is used in order to evolve piecewise constant approximate solutions. An analogous operator for bilinear data has been derived in [LNK06].

2.1 Transcritical states and parabolic regularization

We now turn our attention to transcritical flows, i.e. flows that change from $Fr > 1$ to $Fr < 1$ or vice versa, where $Fr = \|(u,v)^T\|_2/c$ denotes the Froude number and $c = \sqrt{gh}$ is the wave celerity. For better understanding let us consider one-dimensional shallow water equations and rewrite them in an extended form such that the bottom topography is formally a part of the variables

$$\begin{aligned} b_t &= 0, \\ h_t + (hu)_x &= 0 \\ (hu)_t + (hu^2 + gh^2/2)_x + ghb_x &= 0. \end{aligned} \tag{9}$$

We can rewrite these equations in a quasilinear form

$$\boldsymbol{w}_t + \boldsymbol{A}(\boldsymbol{w})\boldsymbol{w}_x = 0, \tag{10}$$

where

$$\boldsymbol{w} = \begin{pmatrix} b \\ h \\ hu \end{pmatrix}, \qquad \boldsymbol{A}(\boldsymbol{w}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ c^2 & c^2 - u^2 & 2u \end{pmatrix}. \tag{11}$$

The eigenvalues of the matrix $\boldsymbol{A}$ are $\lambda_1 = 0$, $\lambda_2 = u - c$ and $\lambda_3 = u + c$. The eigenvectors written in a matrix form give

$$\boldsymbol{R} = \begin{pmatrix} c^2 - u^2 & 0 & 0 \\ -c^2 & 1 & 1 \\ 0 & u - c & u + c \end{pmatrix} \quad \overset{c=u}{\Longrightarrow} \quad \boldsymbol{R} = \begin{pmatrix} 0 & 0 & 0 \\ -u^2 & 1 & 1 \\ 0 & 0 & 2u \end{pmatrix}. \tag{12}$$

So in the case $c = \pm u$ the matrix $\boldsymbol{R}$ is singular and the system is parabolic degenerate locally. This is called in the literature the resonant case, see, e.g., [GL04]. It has been pointed out by LeVeque [LeV98] that some schemes relying closely on the hyperbolic structure of the problem may show some deficiencies for this type of transcritical states. LeVeque showed, for example, that the wave propagation algorithm was not able to approximate correctly the steady transcritical shock. In the next section we will present results obtained by the FVEG scheme, which also yields some oscillations on transcritical shocks. This is a typical behavior for resonant systems. However, it has been pointed out by LeFloch [L99] that these shocks are sensitive on regularization. Thus we use a parabolic regularization and add a viscous term to the momentum equation. In [GP00] asymptotic derivation of the viscous shallow water equations was done. Note that in [GP00] no bottom topography has been considered. In an analogous way we propose the following form of the viscous term in the momentum equation for the shallow water system with a bottom topography

$$(hu)_t + (hu^2 + gh^2/2)_x = -ghb_x + 4\mu(hu_x)_x, \tag{13}$$

where μ is a viscosity parameter. Our aim is to choose μ in a numerical scheme in such a way, that it vanishes as the mesh is refined.

2.2 Numerical Experiments

Transcritical shock

Let us consider the one-dimensional transcritical flow problem firstly proposed by LeVeque in [LeV98]. The computational domain is $[0, 1]$ and the bottom topography is given by

$$b(x) := \begin{cases} 0.25(\cos(\pi(x-0.5)/0.1)+1) & \text{if } |x-0.5| < 0.1 \\ 0 & \text{otherwise} \end{cases}. \tag{14}$$

The initial data for the water depth h are $h(x, 0) = 1-b(x)$ and for the velocity $u(x, 0) = 0.3$. The gravitational constant is set to be $g = 1$. The numerical solution has been computed up to the final time $T = 5$, where a steady state is already formed. The interest of this example is a steady transcritical shock (hydraulic jump) at which the FVEG scheme produces oscillatory behavior which does not disappear as the grid size is refined.

In all our examples a second order accurate scheme is used and the minmod limiter is applied. We have used extrapolation boundary conditions at $x = 0$ and $x = 1$. Figure 1 shows the water depth h with bottom topography (left) and the Froude number Fr (right) calculated by the FVEG scheme without a parabolic regularization. From left to right the flow turns continuously from subcritical (fluvial) to supercritical (torrential) and then through a transcritical shock it goes back to subcritical. As can be noticed the oscillations arise under the great change in the bottom topography and a sudden change of the flow regime from the supercritical to subcritical flow.

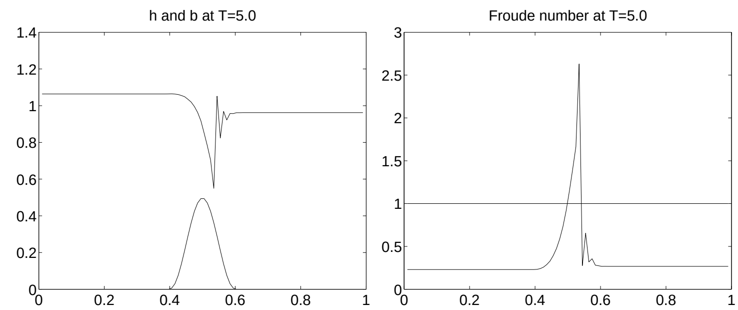

Fig. 1. Transcritical shock without parabolic regularization, 100 mesh cells.

Recall that we have denoted by u_i^n the cell average of u on the i-th cell at time t_n. Define the following finite difference operators

$$\delta_x u_i := u_{i+1/2} - u_{i-1/2}, \qquad \mu_x h_i := (h_{i+1/2} + h_{i-1/2})/2.$$

We approximate the viscous term by the central finite differences at the time t_n as follows

$$4\mu(hu_x)_x \approx 4\mu\delta_x(\mu_x h_i^n\, \delta_x u_i^n)/\hbar^2. \tag{15}$$

The viscosity parameter μ needs to be chosen as small as possible but big enough to damp the oscillations completely. Numerical experiments indicated that μ should be of the form

$$\mu = \alpha\hbar, \tag{16}$$

where α is a constant, $\alpha = \mathcal{O}(1)$, $\hbar$ is a mesh size. For this choice of α the regularization acts in a uniform manner for different mesh sizes. For our experiments we set $\alpha = 0.1$. Furthermore, the regularization term is a first order term and thus one wants to apply the regularization only if necessary. A suitable switch has to be found in order to localize the resonant phenomenon. In the examples presented in this paper the regularization has been applied over the whole computational domain.

In Figure 2 the water depth h is plotted for a mesh with 100 mesh cells. We can notice a stable steady transcritical shock without any oscillations. The shock is smeared slightly due the added numerical diffusion. For finer meshes

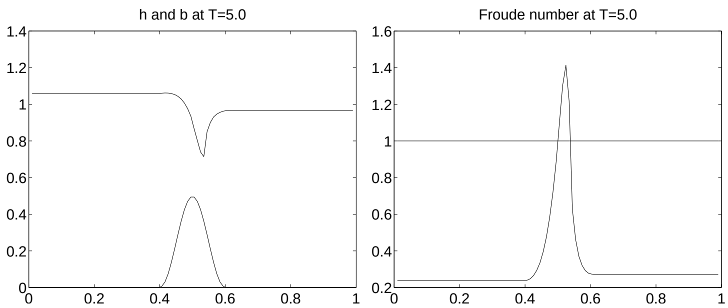

Fig. 2. Transcritical shock with parabolic regularization, 100 mesh cells.

the schock resolution is sharper, see Figure 3, where the results on a mesh with 1000 cells are shown.

Transcritical steady state without shock

Now, we consider the same problem as before but assume the following initial data for the velocity $u(x, 0) = 0.6$. We set the final time to $T = 10$. In this case the steady state is again transcritical but smooth. In Figure 4 the bottom topography and water depth are depicted (left) as well as the momentum hu (right). Note, that no regularization has been necessary now.

Acknowledgement

We want to thank Christian Rohde, University of Stuttgart, for a fruitful discussion on regularization aspects of conservation laws. The second au-

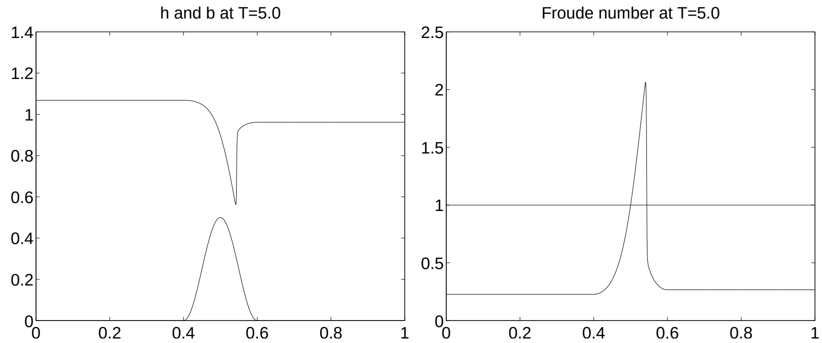

Fig. 3. Transcritical shock with parabolic regularization, 1000 mesh cells.

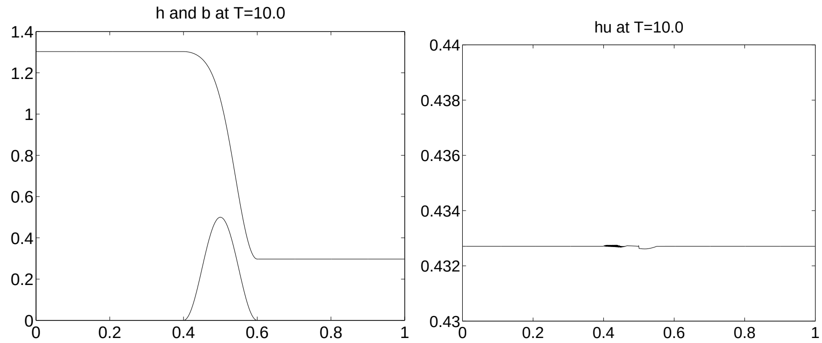

Fig. 4. Transcritical flow without parabolic regularization, 1000 mesh cells.

thor has been partially supported by the grant LU 1194/4-1 of the Deutsche Forschungsgemeinschaft as well as by the DAAD. She gratefully acknowledges these supports.

References

[ABBKP04] Audusse E., Bouchut F., Bristeau M.-O., Klein R., Perthame B.: A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows, to appear in SIAM J. Sci. Comp., (2004)

[BKLL04] Botta N., Klein R., Langenberg S., Lützenkirchen S.: Well balanced finite volume methods for nearly hydrostatic flows, J. Comp. Phys. **196(2)**, 539-565 (2004)

[CLS04] Chinnayya A., LeRoux A.-Y., Seguin N.: A well-balanced numerical scheme for the approximation of the shallow-water equations with topography: the resonance phenomenon, International Journal on Finite Volume (electronic), **1(1)**, 1-33 (2004)

[AW04a] Andrianov N., Warnecke G.: On the solution to the Riemann problem for the compressible duct flow, SIAM J. Appl. Math. **64(3)**, 878-901 (2004)

[AW04b] Andrianov N., Warnecke G.: The Riemann problem for the Baur-Nunziato model of two-phase flows, J. Comp. Phys. **195**, 434-464 (2004)

[DLM95] Dal Maso G., LeFloch P.G., Murat F.: Definition and weak stability of non conservative products, J. Math. Pures Appl. **74**, 483-548 (1995)

[GL04] Goatin P., LeFloch P.G.: The Riemann problem for a class of resonant hyperbolic systems of balance laws, Ann. Inst. Henri Poincare, Anal. Non Lineaire, **21(6)**, 881-902 (2004)

[GP00] Gerbeau J.-F., Perthame B.: Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation, Rapport de recherche **4084**, INRIA (2000)

[GL96] Greenberg J.M., LeRoux A.-Y.: A well-balanced scheme for numerical processing of source terms in hyperbolic equations, SIAM J. Numer. Anal., **33(1-16)** (1996)

[J01] Jin S.: A steady-state capturing method for hyperbolic systems with geometrical source terms, M^2AN, Math. Model. Numer. Anal., **35(4)**, 631-646 (2001)

[L99] LeFloch: An Introduction to Nonclassical Shocks of Systems of Conservative Laws. In: Kröner et al. (eds.), An Introduction to Recent Development in Theory and Numerics for Conservation Laws, 28-72, Springer (1999)

[LeV98] LeVeque R.J.: Balancing source terms and flux gradients in high-resolution Godunov methods: The quasi-steady wave propagation algorithm, J. Comp. Phys. **146**, 346-365 (1998)

[LMW00] Lukáčová-Medviďová M., Morton K.W., Warnecke G.: Evolution Galerkin methods for hyperbolic systems in two space dimensions, MathComp., **69**, 1355–1384 (2000)

[LMW04] Lukáčová-Medviďová M., Morton K.W., Warnecke G.: Finite volume evolution Galerkin (FVEG) methods for hyperbolic problems, SIAM J. Sci. Comput., **26(1)**, 1-30 (2004)

[LNK06] Lukáčová-Medviďová M., Noelle S., Kraft M.: Well-balanced Finite Volume Evolution Galerkin Methods for the Shallow Water Equations, in print J. Comp. Phys. (2006)

[LSW02] Lukáčová-Medviďová M., Saibertová J., Warnecke G.: Finite volume evolution Galerkin methods for nonlinear hyperbolic systems. J. Comp. Phys., **183**, 533-562 (2002)

[LS03] Lukáčová-Medviďová M., Saibertová J.: Genuinly multidimensional Evolution Galerkin schemes for the shallow water equations. In: F. Brezzi et al. (eds.) Numerical Mathematics and Advanced Applications ENUMATH 2001, Springer, 145-154 (2003)

[LV05] Lukáčová-Medviďová M., Vlk Z.: Well-balanced Finite Volume Evolution Galerkin Methods for the Shallow Water Equations with Source Terms, Int. J. Num. Fluids **47(10-11)**, 1165-1171 (2005)

[NPPN06] Noelle S., Pankratz N., Puppo G., Natvig J.: Well-balanced finite volume schemes of arbitrary order of accuracy for shallow water flows, J. Comput. Phys. **213**, 474-499 (2006)

[R04] Rohde Ch.: Approximation of Solutions of Conservation Laws by Non-Local Regularization and Discretization, Habilitation Thesis, University of Freiburg (2004).

[S06] Schofer D.: Approximation von komplexen Riemannlösungen für gemischt hyperbolisch-elliptische Systeme, Diploma Thesis, University of Freiburg (2006)