

On the condition number of the antireflective transform

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Abstract

Deconvolution problems with a finite observation window require appropriate models of the unknown signal in order to guarantee uniqueness of the solution. For this purpose it has recently been suggested to impose some kind of antireflectivity of the signal. With this constraint, the deconvolution problem can be solved with an appropriate modification of the fast sine transform, provided that the convolution kernel is symmetric. The corresponding transformation is called the antireflective transform. In this work we determine the condition number of the antireflective transform to first order, and use this to show that the so-called reblurring variant of Tikhonov regularization for deconvolution problems is a regularization method. Moreover, we establish upper bounds for the regularization error of the reblurring strategy that hold uniformly with respect to the size n of the algebraic system, even though the condition number of the antireflective transform grows with n . We briefly sketch how our results extend to higher space dimensions.

Key words: Deconvolution, boundary conditions, antireflective transform, fast sine transform, Tikhonov regularization

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1. Introduction

We consider the one-dimensional space invariant deconvolution problem, where the signal $g : \mathcal{I} \rightarrow \mathbb{R}$, $\mathcal{I} \subset \mathbb{R}$, is formed by

$$g(x) = \int_{\mathbb{R}} k(x - x')f(x') dx'. \quad (1)$$

Without loss of generality we fix $\mathcal{I} = [0, \pi]$ for our convenience. We assume that the source f is continuous over \mathbb{R} , and that the kernel k is a continuous and

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symmetric (even) function. While the latter is certainly a restriction, there are prominent examples where the kernel is indeed symmetric, or close to symmetric at least.

Using the collocation method with the rectangular quadrature rule over equidistant grids, the integral equation (1) is reduced to an algebraic system of equations with finitely many equations for infinitely many unknowns, i.e., sampled values of f . To reduce the number of unknowns to the number of given equations, one can use appropriate *models*, or *boundary conditions*, for f , to obtain a square linear system

$$K\mathbf{f} = \mathbf{g}, \quad (2)$$

where $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ contain values of f and g at the individual grid points, and the entries of $K \in \mathbb{R}^{n \times n}$ depend on the convolution kernel k and the corresponding boundary conditions.

For several of these models the discrete problem can be solved very efficiently in only $O(n \log n)$ operations. For example, imposing periodic boundary conditions, the matrix K can be diagonalized with the discrete Fourier transform; with reflective boundary conditions and a symmetric kernel k , K can be diagonalized by the discrete cosine transform, cf. Ng, Chan, and Tang [7]. For symmetric kernels Serra-Capizzano suggested in [8] as an alternative the use of an *antireflective boundary condition*, meaning that

$$f(x) + f(-x) = c_0 \quad \text{and} \quad f(x) + f(\pi - x) = c_\pi$$

are both constants. For antireflective boundary conditions the structure of K is quite more involved, see Section 3, but still we can resort to a fast inversion algorithm with $O(n \log n)$ operations on the grounds of the discrete sine transform, and which yields the so-called *antireflective transform* [1].

While the discrete Fourier and cosine transforms are unitary, and therefore have condition number equal to one (with respect to the Euclidean norm), the antireflective transform is a rank two modification of a unitary transform, and its condition number grows to infinity as the number of grid points goes to infinity, which may affect the stability of the antireflective transform. In this paper we determine the leading order term of the asymptotic growth of this condition number, which is shown to be $\sqrt{2n}$, when n is the dimension of the vectors, i.e., the number of grid points.

The convolution problem (1) is known to be ill-posed, unless the convolution kernel has singularities. As a consequence the linear system (2) can be very ill-conditioned, and some kind of regularization is needed to stabilize the problem. The best known method of this sort may be *Tikhonov regularization*, which amounts to solving

$$(K^T K + \alpha I)\tilde{\mathbf{f}}_\alpha = K^T \mathbf{g} \quad (3)$$

for a useful approximation $\tilde{\mathbf{f}}_\alpha$ of \mathbf{f} . This approach is fairly well understood, and error bounds for $\|\tilde{\mathbf{f}}_\alpha - \mathbf{f}\|$ are readily available, cf., e.g., Groetsch [6]. However, it is known at least since Varah's paper [10] that Tikhonov regularization may result in erratic boundary effects, and this, in fact, appears to be the general

case for our deconvolution problem with antireflective boundary conditions, as illustrated in [2, 4].

Two different remedies have been suggested in [2, 4] to deal with these boundary effects. In [2] the original problem has been transformed to a problem where f and g have homogeneous boundary values, and Tikhonov regularization has been applied to solve this new problem, using the (unitary) fast sine transform. In [4], on the other hand, the authors have developed the so-called *reblurring strategy* by replacing K^T in (3) by K , i.e., they suggest to take

$$\mathbf{f}_\alpha = (K^2 + \alpha I)^{-1} K \mathbf{g} \quad (4)$$

as approximation of \mathbf{f} . We note that $K^T = K$ for a symmetric kernel and periodic or reflective boundary conditions, but $K^T \neq K$ in the antireflective case, as the corresponding transform is not a unitary one; thus (4) may be seen as one reasonable extension of Tikhonov regularization to the deconvolution problem for antireflective boundary conditions.

Unfortunately, as K^2 is not symmetric, the standard regularization theory from [5, 6] does not apply to the reblurring strategy, but we can use the results of this paper to derive estimates which are similar to the usual results from [5, 6]. In particular, we establish that the reblurring strategy is a regularization method.

The outline of this paper is as follows. In Section 2 we describe in more detail the setting of our problem, and recast the usual way of deriving error bounds for Tikhonov regularization for this context. Then, in Sections 3 and 4 we focus on the antireflective transform and determine the dominating term of its condition number. Finally, in Section 5, we apply these results to determine useful bounds for the regularizing properties of the reblurring strategy. The paper ends with a brief discussion of the extension of our results to higher space dimensions.

2. Problem setting

Throughout this paper, we use bold faced letters like \mathbf{z} for the n -dimensional vector with the samples of the function $z : \mathcal{I} = [0, \pi] \rightarrow \mathbb{R}$ on the equidistant grid

$$\Delta_h = \{(j-1)h : j = 1, \dots, n\} \subset \mathcal{I}$$

with mesh size $h = \pi/(n-1)$. To achieve error bounds that are independent of the grid size n we also introduce the rescaled Euclidean norm

$$\|\mathbf{z}\| = \frac{1}{\sqrt{n}} \|\mathbf{z}\|_2. \quad (5)$$

Note that this norm approximates the \mathcal{L}^2 -norm of z , i.e.,

$$\|\mathbf{z}\|^2 \approx \frac{1}{\pi} \int_{\mathcal{I}} |z(x)|^2 dx.$$

The analysis of regularization methods needs to take noise into account. To this end we employ the following noise model: We assume that the data are perturbed by some bounded function $e : \mathcal{I} \rightarrow \mathbb{R}$ with

$$\varepsilon = \sup_{x \in \mathcal{I}} |e(x)|. \quad (6)$$

With these assumptions the equidistant sample \mathbf{g}^ε of the function g perturbed by noise satisfies

$$\mathbf{g}^\varepsilon = \mathbf{g} + \mathbf{e} \quad \text{with} \quad \|\mathbf{e}\| \leq \|\mathbf{e}\|_\infty \leq \varepsilon.$$

A family of approximations $\{\mathbf{f}_\alpha : \alpha > 0\}$ is called a *regularization method* if, for every $\alpha > 0$, \mathbf{f}_α depends continuously on the data, and

$$\lim_{\varepsilon \rightarrow 0} \|\mathbf{f}_{\alpha(\varepsilon)}^\varepsilon - \mathbf{f}\| = 0$$

for a particular choice of $\alpha = \alpha(\varepsilon)$; see, for example, [5]. Throughout, we denote by $\mathbf{f}_\alpha^\varepsilon$ the corresponding approximations for the perturbed data \mathbf{g}^ε . Here, we investigate the family $\{\mathbf{f}_\alpha : \alpha > 0\}$ obtained from (4), which coincides with Tikhonov regularization for the periodic and reflective boundary conditions, and which defines the reblurring strategy for antireflective boundary conditions.

To investigate whether these schemes are regularization methods we introduce the spectral decomposition

$$K = X\Lambda X^{-1} \quad (7)$$

of the matrix K , where Λ is the diagonal matrix of the eigenvalues of K , and X can be identified with the discrete Fourier transform in the periodic case, the discrete cosine transform in the reflective case, and with the antireflective transform in the antireflective case, respectively. In particular, X is a unitary matrix in the first two cases, but fails to be unitary in the antireflective case. Throughout, we assume that K is invertible, i.e., that all eigenvalues of K are nonzero.

Now we proceed as follows. On the grounds of the triangle inequality

$$\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}\| \leq \|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}_\alpha\| + \|\mathbf{f}_\alpha - \mathbf{f}\|, \quad (8)$$

we can estimate the two terms on the right-hand side of (8) separately. We start with the first term, the *propagated data error*, which can be rewritten as

$$\mathbf{f}_\alpha^\varepsilon - \mathbf{f}_\alpha = (K^2 + \alpha I)^{-1} K (\mathbf{g}^\varepsilon - \mathbf{g}) = X(\Lambda^2 + \alpha I)^{-1} \Lambda X^{-1} \mathbf{e}. \quad (9)$$

The diagonal entries of the diagonal matrix $(\Lambda^2 + \alpha I)^{-1} \Lambda$ are given by $\lambda_i / (\lambda_i^2 + \alpha)$, where λ_i are the eigenvalues of K , and hence,

$$\|(\Lambda^2 + \alpha I)^{-1} \Lambda\|_2 \leq \sup_{\lambda \in \mathbb{R}} \left| \frac{\lambda}{\lambda^2 + \alpha} \right| \leq \frac{1}{2\sqrt{\alpha}}.$$

From this we obtain

$$\begin{aligned}\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}_\alpha\| &\leq \|X\|_2 \|(\Lambda^2 + \alpha I)^{-1} \Lambda\|_2 \|X^{-1}\|_2 \|\mathbf{e}\| \\ &\leq \frac{1}{2\sqrt{\alpha}} \mu(X) \|\mathbf{e}\|,\end{aligned}\tag{10}$$

where $\mu(X) = \|X\|_2 \|X^{-1}\|_2$ is the condition number of X . In particular, in the periodic and the reflective case we conclude that

$$\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}_\alpha\| \leq \frac{\varepsilon}{2\sqrt{\alpha}},\tag{11}$$

as $\mu(X) = 1$ for a unitary matrix.

The second term in (8) concerns the *approximation error* that is independent of noise:

$$\begin{aligned}\mathbf{f} - \mathbf{f}_\alpha &= \mathbf{f} - (K^2 + \alpha I)^{-1} K \mathbf{g} = \mathbf{f} - (K^2 + \alpha I)^{-1} K^2 \mathbf{f} \\ &= \alpha (K^2 + \alpha I)^{-1} \mathbf{f} = X \alpha (\Lambda^2 + \alpha I)^{-1} X^{-1} \mathbf{f}.\end{aligned}$$

The same estimate as before thus yields

$$\|\mathbf{f}_\alpha - \mathbf{f}\| \leq \mu(X) \omega(\alpha) \sup_{x \in \mathcal{I}} |f(x)| \quad \text{with} \quad \omega(\alpha) = \sup_{\lambda \in \mathbb{R}} \frac{\alpha}{\lambda^2 + \alpha},$$

but unfortunately, the quantity $\omega(\alpha)$ is only bounded by one and cannot be shown to be $o(1)$, independently of λ (and thus n). Convergent bounds can be obtained with the usual remedy from the theory of ill-posed problems, which consists in so-called smoothness assumptions, the most simple one being as follows.

Assumption 1. *Let f be itself a blurred version of a continuous signal w , i.e.,*

$$f(x) = \int_{\mathbb{R}} k(x - x') w(x') dx', \quad x \in \mathbb{R},\tag{12}$$

where w satisfies the same boundary conditions as f (i.e., periodic, reflective, or antireflective ones).

Remark 2. We mention that it is easy to see that if w of (12) satisfies one of these three boundary conditions, then so does f as well. Vice versa, if f satisfies one of these boundary conditions and (12) holds true then w must satisfy the same boundary condition, unless the integral equation (1) has multiple solutions.

On the grounds of Assumption 1 we may therefore assume that

$$\mathbf{f} = K \mathbf{w} \quad \text{for some } \mathbf{w} \in \mathbb{R}^n\tag{13}$$

with a moderate bound

$$\|\mathbf{w}\| \leq \|\mathbf{w}\|_\infty \leq \varrho.\tag{14}$$

For instance, ϱ could be the maximum of w over \mathcal{I} . Inserting (13) into the representation of the approximation error then we get

$$\mathbf{f} - \mathbf{f}_\alpha = \alpha(K^2 + \alpha I)^{-1} K \mathbf{w} = X \alpha (\Lambda^2 + \alpha I)^{-1} \Lambda X^{-1} \mathbf{w}, \quad (15)$$

and we can improve the previous estimate to obtain

$$\|\mathbf{f}_\alpha - \mathbf{f}\| \leq \mu(X) \frac{\sqrt{\alpha}}{2} \|\mathbf{w}\|. \quad (16)$$

Combining (8), (11), (14), and (16) we conclude that the error of the approximation (4) is bounded by

$$\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}\| = O\left(\frac{\varepsilon}{\sqrt{\alpha}} + \sqrt{\alpha\varrho}\right) = O(\sqrt{\varepsilon\varrho}), \quad (17)$$

if the regularization parameter is chosen to be $\alpha = \alpha(\varepsilon) = \varepsilon/\varrho$. Hidden in the $O(\cdot)$ notation, however, is the condition number of X . Therefore, the bound (17) is independent of the dimension of the problem for the periodic and reflective boundary conditions only, whereas it grows to infinity with n in the antireflective case, as will be shown in Section 4.

Once we have estimated the growth rate of this condition number, however, we can use this result to improve on the above estimate for the reblurring strategy, cf. Section 5.

Finally, we like to mention that the analysis extends to the algorithm used in [2], again with a unitary matrix X . Therefore this is also a regularization method.

3. The antireflective transform

We assume that the convolution kernel k is symmetric, and denote by $k_j = hk(jh)$ the values of k at the grid points. Then, as shown in [8], the matrix K for the antireflective model has the form

$$K = \begin{bmatrix} s_0 & \mathbf{0} & 0 \\ \mathbf{s} & K_0 & J\mathbf{s} \\ 0 & \mathbf{0} & s_0 \end{bmatrix}, \quad (18)$$

where

$$K_0 = \begin{bmatrix} k_0 & k_1 & \cdots & \cdots & k_{n-4} & k_{n-3} \\ k_1 & k_0 & k_1 & & & k_{n-4} \\ \vdots & k_1 & k_0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & k_2 \\ k_{n-4} & & & \ddots & k_0 & k_1 \\ k_{n-3} & k_{n-4} & \cdots & k_2 & k_1 & k_0 \end{bmatrix} - \begin{bmatrix} k_2 & k_3 & \cdots & k_{n-3} & 0 & 0 \\ k_3 & & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & k_{n-3} \\ k_{n-3} & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & & k_3 \\ 0 & 0 & k_{n-3} & \cdots & k_3 & k_2 \end{bmatrix},$$

$$s_i = k_i + 2 \sum_{j=i+1}^{n-3} k_j, \quad i = 0, \dots, n-3, \quad \mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_{n-3} \\ 0 \end{bmatrix},$$

and J is the *antidiagonal unit matrix*

$$J = \begin{bmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{(n-2) \times (n-2)}.$$

Associated with the entries of K is the so-called *symbol*

$$\kappa(t) = k_0 + 2 \sum_{j=1}^{n-3} k_j \cos(jt).$$

In particular, we have $s_0 = \kappa(0) \approx \int_{-\pi}^{\pi} k(x) dx$.

As shown in [1] the eigenvectors of K are given by homogeneous sine vectors and first order polynomials. More precisely, let $Q = [q_{ij}]$ be the sine transform matrix of order $n-2$ with entries

$$q_{ij} = \sqrt{\frac{2}{n-1}} \sin\left(\frac{ij\pi}{n-1}\right), \quad i, j = 1, \dots, n-2.$$

Then the antireflective transform can be defined by the matrix

$$A = \left[\mathbf{1} \mid \begin{array}{c} \mathbf{0} \\ Q \\ \mathbf{0} \end{array} \mid \boldsymbol{\ell} \right], \quad (19)$$

where

$$\mathbf{1} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\ell} = \frac{\sqrt{3}}{\sqrt{n(n^2-1)}} \begin{bmatrix} 1-n \\ 3-n \\ \vdots \\ n-1 \end{bmatrix}.$$

Note that $\mathbf{1}$ and $\boldsymbol{\ell}$ differ from the corresponding vectors used both in [1] and [2]. They have been chosen here to form an *orthonormal* basis of the grid samples of all linear polynomials.

According to [1] the spectral decomposition of K of (18) can now be written as

$$K = A\Lambda A^{-1}, \quad (20)$$

where the diagonal entries λ_{jj} of Λ are given by

$$\lambda_{jj} = \begin{cases} \kappa\left(\frac{j-1}{n-1}\pi\right), & 1 \leq j < n, \\ \kappa(0), & j = n. \end{cases}$$

The spectral decomposition (20) is useful to adopt other spectral filtering methods than Tikhonov regularization to the reblurring strategy, cf. [1]. Moreover, using (20) it is easy to prove that all antireflective matrices form an algebra. This algebra is not closed by transposition, though, which may be seen as the underlying reason for the aforementioned boundary artifacts when using classical Tikhonov regularization.

4. The condition number of the antireflective transform

To compute the condition number of the antireflective transform and to investigate the regularizing properties of the reblurring strategy we need to determine the *singular value decomposition* of the matrix A . To this end we determine the spectral decomposition of the matrix $A^T A$. From (19) we obtain

$$A^T A = \begin{bmatrix} 1 & \mathbf{a}^T & 0 \\ \mathbf{a} & I & \mathbf{b} \\ 0 & \mathbf{b}^T & 1 \end{bmatrix} \quad (21)$$

with

$$\mathbf{a} = Q\mathbf{1}' \quad \text{and} \quad \mathbf{b} = Q\boldsymbol{\ell}', \quad (22)$$

where $\mathbf{1}'$ and $\boldsymbol{\ell}'$ refer to the inner $n - 2$ components of $\mathbf{1}$ and $\boldsymbol{\ell}$, respectively. Using trigonometric identities we find that the entries of \mathbf{a} and \mathbf{b} are given by

$$a_k = \begin{cases} \left(\frac{2}{n(n-1)}\right)^{1/2} \cot\left(\frac{k\pi}{2(n-1)}\right), & k \text{ odd,} \\ 0, & k \text{ even,} \end{cases} \quad (23)$$

and

$$b_k = \begin{cases} 0, & k \text{ odd,} \\ -\left(\frac{6}{n(n+1)}\right)^{1/2} \cot\left(\frac{k\pi}{2(n-1)}\right), & k \text{ even,} \end{cases} \quad (24)$$

$k = 1, \dots, n - 2$. In particular, we have $\mathbf{a}^T \mathbf{b} = 0$.

Lemma 3. *There holds $\|\mathbf{a}\|_2 < 1$ and $\|\mathbf{b}\|_2 < 1$, and, more precisely,*

$$\|\mathbf{a}\|_2 = 1 - \frac{1}{n} + O(n^{-2}) \quad \text{and} \quad \|\mathbf{b}\|_2 = 1 - \frac{3}{n} + O(n^{-2})$$

as $n \rightarrow \infty$.

Proof. The first assertion readily follows from (22), as Q is an orthogonal matrix and the boundary values of the two normalized vectors $\mathbf{1}$ and $\boldsymbol{\ell}$ are both nonzero. For the second assertion we assume without loss of generality that n is even.

From (23) follows that

$$\begin{aligned}\|\mathbf{a}\|_2^2 &= \frac{2}{n(n-1)} \sum_{j=1}^{n/2-1} \cot^2\left(\frac{(2j-1)\pi}{2(n-1)}\right) \\ &= \frac{2}{n(n-1)} \left(\sum_{j=1}^{n/2-1} t_j^{-2} + \sum_{j=1}^{n/2-1} [\cot^2(t_j) - t_j^{-2}] \right), \quad t_j = \frac{(2j-1)\pi}{2n-2}.\end{aligned}$$

Note that, up to the factor h , the last sum is the (second order accurate) compound mid point rule with mesh width h for the integral

$$\int_0^{\pi/2-h/2} (\cot^2 t - t^{-2}) dt = [t^{-1} - \cot t - t]_0^{\pi/2-h/2} = \frac{2}{\pi} - \frac{\pi}{2} + O(h).$$

It therefore follows that

$$\|\mathbf{a}\|_2^2 = \frac{2}{n} \left(\frac{4(n-1)}{\pi^2} \sum_{j=1}^{n/2-1} \frac{1}{(2j-1)^2} + \frac{1}{\pi} \left(\frac{2}{\pi} - \frac{\pi}{2} + O(n^{-1}) \right) \right). \quad (25)$$

Since

$$\int_{n/2}^{\infty} (2t-1)^{-2} dt \leq \sum_{j=n/2}^{\infty} (2j-1)^{-2} \leq \int_{n/2-1}^{\infty} (2t-1)^{-2} dt,$$

we conclude that

$$\sum_{j=n/2}^{\infty} (2j-1)^{-2} = \frac{1}{2(n-1)} + O(n^{-2})$$

and

$$\sum_{j=1}^{n/2-1} (2j-1)^{-2} = \frac{\pi^2}{8} - \sum_{j=n/2}^{\infty} (2j-1)^{-2} = \frac{\pi^2}{8} - \frac{1}{2(n-1)} + O(n^{-2}).$$

Inserting this into (25) we obtain

$$\begin{aligned}\|\mathbf{a}\|_2^2 &= \frac{2}{n} \left(\frac{n-1}{2} - \frac{2}{\pi^2} + \frac{2}{\pi^2} - \frac{1}{2} + O(n^{-1}) \right) \\ &= 1 - \frac{2}{n} + O(n^{-2}),\end{aligned}$$

and hence,

$$\|\mathbf{a}\|_2 = 1 - \frac{1}{n} + O(n^{-2}).$$

The estimate for $\|\mathbf{b}\|_2$ can be established along similar lines. From (24) we get

$$\begin{aligned}
\|\mathbf{b}\|_2^2 &= \frac{6}{n(n+1)} \sum_{j=1}^{n/2-1} \cot^2\left(\frac{j\pi}{n-1}\right) \\
&= \frac{6}{n(n+1)} \left(\sum_{j=1}^{n/2-1} \left(\frac{n-1}{j\pi}\right)^2 + \sum_{j=1}^{n/2-1} \left[\cot^2\left(\frac{j\pi}{n-1}\right) - \left(\frac{n-1}{j\pi}\right)^2 \right] \right) \\
&= \frac{6}{n(n+1)} \left(\frac{(n-1)^2}{\pi^2} \sum_{j=1}^{n/2-1} j^{-2} + \frac{n-1}{\pi} \left(\frac{2}{\pi} - \frac{\pi}{2} + O(n^{-1}) \right) \right), \quad (26)
\end{aligned}$$

where we have used that the last sum in the second line is, up to the factor h , the compound mid point rule with mesh width h for the integral

$$\int_{h/2}^{\pi/2} (\cot^2 t - t^{-2}) dt = [t^{-1} - \cot t - t]_{h/2}^{\pi/2} = \frac{2}{\pi} - \frac{\pi}{2} + O(h).$$

From

$$\int_{n/2}^{\infty} t^{-2} dt \leq \sum_{j=n/2}^{\infty} j^{-2} \leq \int_{n/2-1}^{\infty} t^{-2} dt,$$

we conclude that

$$\sum_{j=1}^{n/2-1} j^{-2} = \frac{\pi^2}{6} - \sum_{j=n/2}^{\infty} j^{-2} = \frac{\pi^2}{6} - \frac{2}{n} + O(n^{-2}),$$

and inserting this into (26) we obtain

$$\begin{aligned}
\|\mathbf{b}\|_2^2 &= \frac{6}{n(n+1)} \left(\frac{(n-1)^2}{6} - \frac{2}{\pi^2} \frac{(n-1)^2}{n} + (n-1) \left(\frac{2}{\pi^2} - \frac{1}{2} \right) + O(1) \right) \\
&= 1 - \frac{6}{n} + O(n^{-2}).
\end{aligned}$$

Taking the square root we finally achieve the desired estimate. \square

According to (21) the eigenvalues and eigenvectors of $A^T A$ satisfy

$$\lambda \begin{bmatrix} \gamma \\ \mathbf{x} \\ \delta \end{bmatrix} = A^T A \begin{bmatrix} \gamma \\ \mathbf{x} \\ \delta \end{bmatrix} = \begin{bmatrix} \gamma + \mathbf{a}^T \mathbf{x} \\ \gamma \mathbf{a} + \mathbf{x} + \delta \mathbf{b} \\ \mathbf{b}^T \mathbf{x} + \delta \end{bmatrix}. \quad (27)$$

Since $A^T A$ is a rank four correction of the identity, the eigenvalue $\lambda = 1$ occurs with multiplicity $n - 4$ and is associated to eigenvectors with $\delta = \gamma = 0$ and

$\mathbf{a}^T \mathbf{x} = \mathbf{b}^T \mathbf{x} = 0$. For the complementary eigenvectors we can thus use the ansatz $\mathbf{x} = \xi \mathbf{a} + \eta \mathbf{b}$, and then we obtain from (27) that

$$\lambda \begin{bmatrix} \gamma \\ \xi \mathbf{a} + \eta \mathbf{b} \\ \delta \end{bmatrix} = \begin{bmatrix} \gamma + \xi \|\mathbf{a}\|_2^2 \\ (\gamma + \xi) \mathbf{a} + (\delta + \eta) \mathbf{b} \\ \delta + \eta \|\mathbf{b}\|_2^2 \end{bmatrix}.$$

From this we find two eigenvalues $\lambda_{1,2}$ and associated eigenvectors $\mathbf{y}_{1,2}$ by setting $\delta = \eta = 0$, namely

$$\lambda_{1,2} = 1 \pm \|\mathbf{a}\|_2, \quad \mathbf{y}_{1,2} = \frac{1}{\sqrt{2} \|\mathbf{a}\|_2} [\pm \|\mathbf{a}\|_2, \mathbf{a}^T, 0]^T. \quad (28)$$

Similarly, for $\gamma = \xi = 0$ we find the remaining two eigenvalues and associated eigenvectors

$$\lambda_{3,4} = 1 \pm \|\mathbf{b}\|_2, \quad \mathbf{y}_{3,4} = \frac{1}{\sqrt{2} \|\mathbf{b}\|_2} [0, \mathbf{b}^T, \pm \|\mathbf{b}\|_2]^T. \quad (29)$$

Finally, we determine the left singular vectors by multiplying A with its right singular vectors: Using (28), (29), and (22) we thus obtain

$$A \mathbf{y}_{1,2} = \lambda_{1,2}^{1/2} \frac{1}{\|\mathbf{a}\|_2} \left(\frac{1 \pm \|\mathbf{a}\|_2}{2} \right)^{1/2} \begin{bmatrix} \pm \frac{\|\mathbf{a}\|_2}{1 \pm \|\mathbf{a}\|_2} \frac{1}{\sqrt{n}} \\ \mathbf{1}' \\ \pm \frac{\|\mathbf{a}\|_2}{1 \pm \|\mathbf{a}\|_2} \frac{1}{\sqrt{n}} \end{bmatrix}, \quad (30)$$

and

$$A \mathbf{y}_{3,4} = \lambda_{3,4}^{1/2} \frac{1}{\|\mathbf{b}\|_2} \left(\frac{1 \pm \|\mathbf{b}\|_2}{2} \right)^{1/2} \begin{bmatrix} \pm \frac{\|\mathbf{b}\|_2}{1 \pm \|\mathbf{b}\|_2} l_1 \\ \boldsymbol{\ell}' \\ \pm \frac{\|\mathbf{b}\|_2}{1 \pm \|\mathbf{b}\|_2} l_n \end{bmatrix}, \quad (31)$$

where l_1 and l_n denote the boundary elements of $\boldsymbol{\ell}$.

We summarize these results in the following theorem, where we use the notation $\mathbf{y} \doteq \mathbf{z}$ if the two vectors \mathbf{y} and \mathbf{z} depend on n and for each entry y_i of \mathbf{y} and the corresponding entry z_i of \mathbf{z} there holds $y_i/z_i \rightarrow 1$ as $n \rightarrow \infty$.

Theorem 4. *The two dominant singular values of A are given by*

$$\sigma_1 \doteq \sigma_2 \doteq \sqrt{2},$$

where $\sqrt{2}$ is, in fact, a strict upper bound, and the two minimal singular values are given by

$$\sigma_{n-1} \doteq \frac{\sqrt{3}}{\sqrt{n}} \quad \text{and} \quad \sigma_n \doteq \frac{1}{\sqrt{n}},$$

respectively. The corresponding right singular vectors are

$$\mathbf{v}_1 \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mathbf{a} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \mathbf{b} \\ 1 \end{bmatrix}, \quad \mathbf{v}_{n-1} \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \mathbf{b} \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_n \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ \mathbf{a} \\ 0 \end{bmatrix},$$

and the left singular vectors are

$$\mathbf{u}_1 \doteq \begin{bmatrix} 1/(2\sqrt{n}) \\ \mathbf{1}' \\ 1/(2\sqrt{n}) \end{bmatrix}, \quad \mathbf{u}_2 \doteq \begin{bmatrix} l_1/2 \\ \boldsymbol{\ell}' \\ l_n/2 \end{bmatrix},$$

$$\mathbf{u}_{n-1} \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \sqrt{3/n} \boldsymbol{\ell}' \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_n \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ \mathbf{1}'/\sqrt{n} \\ -1 \end{bmatrix},$$

respectively. The remaining singular values are equal to one, and the corresponding left and right singular vectors have homogeneous boundary values.

Proof. The proof follows from (28), (29), (30), and (31), together with Lemma 3. \square

Now we are in the position to determine the condition number of the antireflective transform to first order.

Corollary 5. *The condition number of the antireflective transform satisfies*

$$\mu(A) \doteq \sqrt{2n}, \quad n \rightarrow \infty.$$

Remark 6. It is important to note that the ill-conditioned subspace of A^{-1} has dimension two, independent of n , since A has two singular values that decay like $1/\sqrt{n}$ while all others are between one and two. Also, A^{-1} only amplifies vectors that fail to be orthogonal to $\mathcal{U} = \text{span}\{\mathbf{u}_{n-1}, \mathbf{u}_n\}$. According to Theorem 4 the vectors from \mathcal{U} are essentially zero, except for their two boundary values, see Figure 1 for an illustration.

We finally mention that it is fairly easy to derive upper and lower bounds for the condition number of A that both grow like a constant multiple of \sqrt{n} , cf. Tablino-Possio [9]. Our intension here has been to determine the precise leading order term of this condition number.

5. Convergence of the reblurring strategy with antireflective boundary conditions

As we have seen in Corollary 5 the conditioning of the antireflective transform deteriorates as $n \rightarrow \infty$. As a consequence, the corresponding upper bounds (10) and (16) deteriorate with increasing n .

We show now that the estimate that has led us to (10) and (16) can be improved so as to achieve upper bounds for the regularization error of the reblurring strategy that hold uniformly with respect to n . To this end we utilize the following result.

Lemma 7. *Let A be the matrix (19) of the antireflective transform. Then there is a constant c , independent of n , such that*

$$\|A^{-1}\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2 + c\|\mathbf{x}\|_\infty^2 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \quad (32)$$

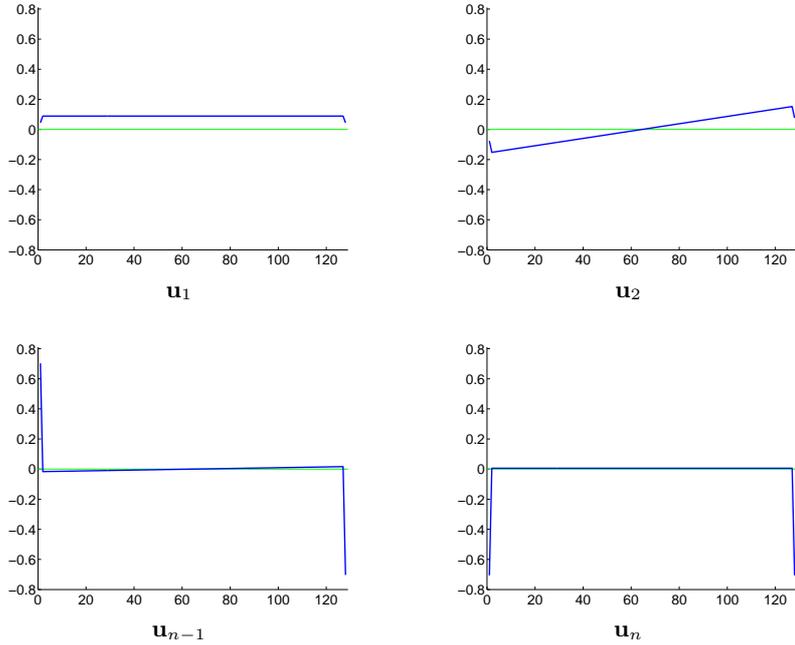


Figure 1: First two and last two right singular vector of A for $n = 128$.

Proof. Using the singular value decomposition of A from Theorem 4, we have

$$\|A^{-1}\mathbf{x}\|_2^2 \leq \|\mathbf{x}\|_2^2 + \frac{1}{\sigma_{n-1}^2} (\mathbf{u}_{n-1}^T \mathbf{x})^2 + \frac{1}{\sigma_n^2} (\mathbf{u}_n^T \mathbf{x})^2. \quad (33)$$

According to the particular form of \mathbf{u}_{n-1} we can estimate

$$|\mathbf{u}_{n-1}^T \mathbf{x}| \leq \frac{1}{\sqrt{2}} (|x_1| + |x_n|) + \left(\frac{3}{2n}\right)^{1/2} \left(\sum_{j=2}^{n-1} |\ell_j|\right) \|\mathbf{x}\|_\infty \leq c_1 \|\mathbf{x}\|_\infty,$$

where ℓ_j are the entries of ℓ and c_1 is a constant, independent of n . Similarly we obtain

$$|\mathbf{u}_n^T \mathbf{x}| \leq \frac{1}{\sqrt{2}} \left(|x_1| + |x_n| + \frac{n-2}{n} \|\mathbf{x}\|_\infty \right) \leq \frac{3}{\sqrt{2}} \|\mathbf{x}\|_\infty.$$

Inserting these two estimates into (33) and using Theorem 4 to estimate σ_{n-1} and σ_n the desired inequality follows immediately. \square

Now we show how to improve the bounds from Section 2. Again, we consider the propagated data error first. Starting from (9) – with the spectral decomposition (20) instead of (7) – we estimate

$$\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}_\alpha\| \leq \|A\|_2 \|(\Lambda^2 + \alpha I)^{-1} \Lambda\|_2 \|A^{-1} \mathbf{e}\| \leq \frac{1}{2\sqrt{\alpha}} \|A\|_2 \|A^{-1} \mathbf{e}\|,$$

and then, using Theorem 4 and Lemma 7 we conclude that

$$\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}_\alpha\| \leq \frac{1}{\sqrt{2\alpha}} \|A^{-1}\mathbf{e}\| \leq C_1 \frac{\varepsilon}{\sqrt{\alpha}},$$

where we have used the error model (6) for the last inequality and C_1 is a constant, independent of n .

Concerning the approximation error we proceed in a similar manner from (15), and estimate

$$\|\mathbf{f} - \mathbf{f}_\alpha\| \leq \alpha \|A\|_2 \|(\Lambda^2 + \alpha I)^{-1} \Lambda\|_2 \|A^{-1}\mathbf{w}\| \leq \frac{\sqrt{\alpha}}{\sqrt{2}} \|A^{-1}\mathbf{w}\|.$$

On the grounds of Assumption 1 we may again assume that (14) holds true, and then we conclude from Lemma 7 that

$$\|\mathbf{f} - \mathbf{f}_\alpha\| \leq \frac{\sqrt{1+c}}{\sqrt{2}} \sqrt{\alpha} \varrho.$$

Thus we have established the following result:

Theorem 8. *Let the exact solution \mathbf{f} of (2) satisfy (13) with (14). Then the total error of the reblurring strategy (4) with antireflective boundary conditions satisfies*

$$\|\mathbf{f}_\alpha^\varepsilon - \mathbf{f}\| = O(\sqrt{\varepsilon\varrho}),$$

for $\alpha = \alpha(\varepsilon) = \varepsilon/\varrho$, where the constant in the $O(\cdot)$ -notation is independent of the dimension n .

Note that the upper bound from Theorem 8 is the same as in (17) for Tikhonov regularization with reflective or periodic boundary conditions; only the constant hidden in the $O(\cdot)$ -notation may be somewhat larger for the reblurring strategy.

Remark 9. We mention that the comparison of the different boundary conditions is not completely fair, as the assumptions required for the antireflective boundary condition, namely that $\|\mathbf{e}\|_\infty$ and $\|\mathbf{w}\|_\infty$ be uniformly bounded independent of n , is somewhat stronger than the corresponding requirement for $\|\mathbf{e}\|$ and $\|\mathbf{w}\|$ required by the other two boundary conditions. This fact depends on the term $c\|\mathbf{x}\|_\infty^2$ in (32) which can only be removed if $\mathbf{x} \in \mathcal{U}^\perp$, where \mathcal{U} is the subspace introduced in Remark 6.

6. Two-dimensional analysis

Our analysis has a natural extension to higher dimensions, as we sketch next. We restrict our analysis to two space dimensions, though.

As in [1], we define the two-dimensional antireflective transform $A_{(2)} \in \mathbb{R}^{n^2 \times n^2}$ as

$$A_{(2)} = A \otimes A, \tag{34}$$

where \otimes denote the usual tensor product.

Lemma 10. *The condition number of the two-dimensional antireflective transform satisfies*

$$\mu(A_{(2)}) \doteq 2n, \quad n \rightarrow \infty.$$

Proof. The result follows directly from Theorem 4 and tensor product properties. Let $A = U\Sigma V^T$ be the singular value decomposition of A , then the singular value decomposition of $A_{(2)}$ is $A_{(2)} = U_{(2)}\Sigma_{(2)}V_{(2)}^T$, where $U_{(2)} = (U \otimes U)P^T$, $\Sigma_{(2)} = P(\Sigma \otimes \Sigma)P$, $V_{(2)}^T = P^T(V^T \otimes V^T)$, and P is the permutation matrix that arranges the diagonal entries of $\Sigma \otimes \Sigma$ in a non increasing order. It follows that the dominant singular value of $A_{(2)}$ is $\sigma_1 \doteq 2$ and the minimal singular value is $\sigma_{n^2} \doteq 1/n$. \square

The analysis in Section 5 can be generalized in much the same way, as the following analog of Lemma 7 holds true.

Lemma 11. *Let $A_{(2)}$ be the coefficient matrix (34) of the two-dimensional antireflective transform. Then there is a constant c , independent of n , such that*

$$\|A_{(2)}^{-1}\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2 + c\|\mathbf{x}\|_\infty^2 \quad \text{for all } \mathbf{x} \in \mathbb{R}^{n^2}.$$

Proof. Every vector $\mathbf{x} \in \mathbb{R}^{n^2}$ can be rewritten as $\mathbf{x} = \mathbf{y} \otimes \mathbf{z}$ with $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. We point out that $\|\mathbf{x}\| = \|\mathbf{y}\|\|\mathbf{z}\|$ and $\|\mathbf{x}\|_\infty = \|\mathbf{y}\|_\infty\|\mathbf{z}\|_\infty$; note that the rescaling factor in (5) is $1/\sqrt{n^2} = 1/n$ for the larger vector \mathbf{x} . From Lemma 7 follows that

$$\begin{aligned} \|A_{(2)}^{-1}\mathbf{x}\|^2 &= \|(A^{-1} \otimes A^{-1})(\mathbf{y} \otimes \mathbf{z})\|^2 \\ &= \|A^{-1}\mathbf{y}\|^2 \|A^{-1}\mathbf{z}\|^2 \\ &\leq (\|\mathbf{y}\|^2 + c_1\|\mathbf{y}\|_\infty^2)(\|\mathbf{z}\|^2 + c_2\|\mathbf{z}\|_\infty^2) \\ &\leq \|\mathbf{x}\|^2 + (c_1 + c_2 + c_1c_2)\|\mathbf{x}\|_\infty^2 \end{aligned}$$

and the desired inequality follows with $c = c_1 + c_2 + c_1c_2$. \square

7. Conclusions

The reblurring strategy has been suggested in [4] to regularize ill-conditioned deconvolution problems using antireflective boundary conditions. This is a particularly fast method that requires only $O(n \log n)$ operations to solve and regularize problems of this sort over an equidistant grid with n unknowns. In the present paper we have come up with an analysis to provide some theoretical justification of this approach. In particular, in Section 5 we have obtained upper bounds for the regularization error that hold uniformly with respect to the size of the algebraic linear system. It has thus been shown that the reblurring strategy is a useful regularization method for ill-posed deconvolution problems with antireflective boundary conditions. Alternatively, in [3] the reblurring strategy has been interpreted as a discretization of a regularized continuous problem. Both arguments can be used to explain the good performance of the method.

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