ERRATUM: AN INVERSE BACKSCATTER PROBLEM FOR ELECTRIC IMPEDANCE TOMOGRAPHY

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In the paper [1] we have claimed (in the concluding remarks) that the same arguments that we have used in the rest of the paper for insulating cavities can also be applied to establish that two (simply connected) perfectly conducting inclusions with the same backscatter data of impedance tomography are necessarily the same.

Unfortunately, while our uniqueness result for insulating cavities is absolutely correct, the corresponding statement on perfect conductors *fails* to be true, and we will provide a counterexample below. A correct statement is as follows. (Throughout, we say that a perfect conductor is supported in the closure of a domain Ω if the homogeneous Neumann condition on $\Gamma = \partial\Omega$ in the forward problem associated with the backscatter data, i.e., [1, (2.16)], is replaced by a homogeneous Dirichlet condition, and the normalizing condition on $T = \partial D$ is deleted.)

Theorem 1. *Assume that Ω is a simply connected domain with C^2-boundary, and that a perfect conductor is supported in $\overline{\Omega} \subset D$, where D is the unit disk. Let Φ be a conformal map that takes $D \setminus \overline{\Omega}$ onto a concentric annulus $\{x \in D : R < |x| < 1\}$, and define*

$$\beta'_R = -\frac{2}{\pi}\sum_{j=1}^{\infty} j\,\frac{R^{2j}}{1+R^{2j}}\,. \tag{1}$$

Then, if $\beta'_R \neq (1-k^2)/(12\pi)$ for all $k = 2, 3, \ldots$, then there is no other perfect conductor supported in the closure of a simply connected C^2-domain within D, such that the backscatter of the two conductors coincide.

Proof. Note that β'_R is a continuous and monotonic function of $R \in [0, 1)$, with $\beta'_0 = 0$ and $\lim_{R\to 1} \beta'_R = -\infty$; in fact, β'_R is the (constant) backscatter data corresponding to a discoidal perfect conductor of radius R centered at the origin (cf. [1, Example 2.1]). Now, for a fixed Ω as in the statement of the theorem, R, and thus β'_R, is uniquely defined, see [1]. Assuming that there is a perfect conductor with the same backscatter supported in some other simply connected set, we can proceed as in the proof of Theorem 4.1 of [1] to conclude that

$$k = 2\omega = \sqrt{1 - 12\pi\beta'_R}$$

is a positive integer. For the case of perfect conductors the right-hand side is, in fact, greater than one (R is necessarily positive), and hence, there are countably many candidates for choosing β'_R appropriately, namely $\beta'_R = (1-k^2)/(12\pi)$ for some $k = 2, 3, \ldots$. On the other hand, if β'_R is none of these numbers then there cannot exist a different conductor with the same backscatter. □

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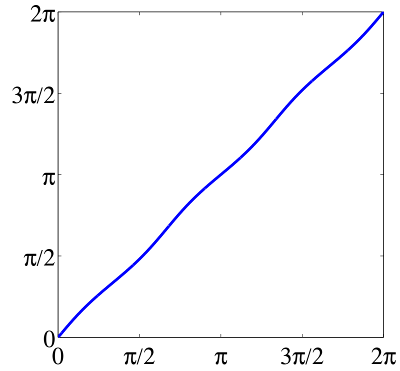

FIG. 1. *Graph of f for $k = 3$ and $\rho = 0.09$.*

We proceed by showing that Theorem 1 is sharp. Let $k \in \mathbb{N} \setminus \{1\}$ and $0 < \rho < 1$, and consider the function

$$F(z) \;=\; \Bigl(\frac{z^k - \rho}{1 - \rho z^k}\Bigr)^{1/k} \tag{2}$$

of the complex variable $z \in D$, where the kth root is defined in such a way that F is smooth near the boundary of D and the associated boundary map $f : [0, 2\pi] \to [0, 2\pi]$ is given by

$$f(\theta) \;=\; \arg F(e^{\mathrm{i}\theta}) \;=\; \frac{2}{k}\,\arctan\Bigl(\frac{1+\rho}{1-\rho}\,\tan\frac{k\theta}{2}\Bigr) \;+\; \frac{2\pi j}{k} \tag{3}$$

for

$$(2j-1)\frac{\pi}{k} \;<\; \theta \;<\; (2j+1)\frac{\pi}{k}\,, \qquad j = 0, \ldots, k\,.$$

The latter satisfies the differential equation (4.5) of [1], i.e.,

$$f'''(\theta) \;=\; \frac{3}{2}\,\frac{f''(\theta)^2}{f'(\theta)} \;+\; \gamma_2 f'(\theta) \;-\; \gamma_1 f'(\theta)^3 \tag{4}$$

with $\gamma_1 = \gamma_2 = k^2/2$, and attains the boundary values $f(0) = 0$ and $f(2\pi) = 2\pi$, see Figure 1.

Note that F is the kth root of a Möbius transformation, which maps D onto itself, with argument z^k. As such, one can convince oneself that F can be extended to a conformal map of $z \in D \setminus \Sigma$, where

$$\Sigma \;=\; \{\, z = re^{2\pi j\mathrm{i}/k} \;:\; 0 \le r \le \rho^{1/k},\, j = 0, \ldots, k-1 \,\} \tag{5}$$

is the k-fold "star" given by all kth complex roots of the real interval $[0, \rho]$, and

$$F(D \setminus \Sigma) \;=\; D \setminus \Sigma'\,,$$

with $\Sigma' = e^{\pi\mathrm{i}/k}\Sigma$; for k odd this simplifies to $\Sigma' = -\Sigma$, see Figure 2.

Now, define Ω as the complement in D of the preimage of $D \setminus B_R$ under F for some suitable $\rho^{1/k} < R < 1$, see Figure 2 again, and consider its closure to be the support

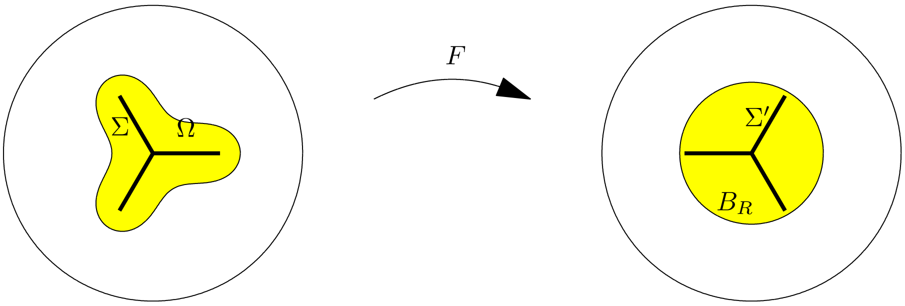

FIG. 2. *The conformal map F with the stars Σ and Σ', and the domain Ω.*

of a perfect conductor within a homogeneous body. Then it follows as in Theorem 3.3 of [1] that the corresponding backscatter of electric impedance tomography satisfies

$$b(\theta) \;=\; \beta'_R f'(\theta)^2 \;+\; \frac{1}{12\pi} \;-\; \frac{1}{12\pi}\, f'(\theta)^2 \;+\; \frac{1}{4\pi}\,\frac{f''(\theta)^2}{f'(\theta)^2} \;-\; \frac{1}{6\pi}\,\frac{f'''(\theta)}{f'(\theta)}\,. \tag{6}$$

Inserting (4) into (6) we conclude that

$$b(\theta) \;=\; \frac{1-k^2}{12\pi} \;+\; \Bigl(\beta'_R - \frac{1-k^2}{12\pi}\Bigr) f'(\theta)^2 \;=\; \beta'_R\,,$$

if $R = R(k)$ is chosen to be such that $\beta'_R = (1-k^2)/12\pi$, which is possible since $k \geq 2$. Observe that the choice of k fixes the radius R, but $\rho > 0$ remains a free parameter that must (and can) be chosen such that $D \setminus \overline{B}_R \subset D \setminus \Sigma'$, as required by the above construction.

It follows that for a perfect conductor with the shape of any of these sets Ω (the one shown in Figure 2 corresponds to $k = 3$ and $\rho = 0.09$, with associated radius $R(3) \approx 0.48$) the backscatter is a constant function of the angle θ, namely β'_R, which is the same as for the concentric perfectly conducting disk with the respective radius $R(k)$. In particular, the two perfectly conducting inclusions $\overline{B}_{R(3)}$ and $\overline{\Omega}$ shown in Figure 2 share the same constant backscatter $b = -2/(3\pi)$.

REFERENCES

[1] M. Hanke, N. Hyvönen, and S. Reusswig, An inverse backscatter problem for electric impedance tomography, *SIAM J. Math. Anal.* 41:1948–1966, 2009.