Hyperbolic conservation laws

A light introduction

Mária Lukáčová

Institute of Mathematics, Johannes Guttenberg-University Mainz, German

http://www.mathematik.uni-mainz.de/Members/lukacova

Outline

Fluids are everywhere

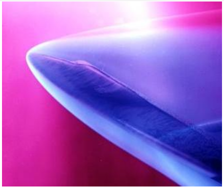
Compressible inviscid fluids: known results and challenges

Mathematical properties

Numerical approximation

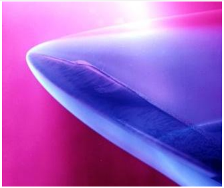
Panta rhei . . .

- flow around an airplane



Panta rhei . . .

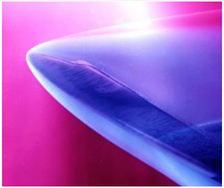
- flow around an airplane



- hydraulic jumps

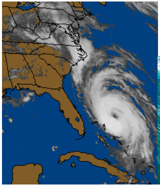

Panta rhei . . .

- flow around an airplane



- hydraulic jumps

- geophysical flows



Sources: efluid.com & Gallery of Fluid Mechanics (M.S.Cramer)

Can we find a common way to describe fluid motion ?

Yes:

Can we find a common way to describe fluid motion ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

Can we find a common way to describe fluid motion ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

- conservation of mass
- conservation of momentum . . . second Newton's law $F = m \cdot a$
- conservation of energy

Can we find a common way to describe fluid motion ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

- conservation of mass
- conservation of momentum ... second Newton's law $F = m \cdot a$
- conservation of energy

What are the quantities we need to consider ?

Can we find a common way to describe fluid motion ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

- conservation of mass
- conservation of momentum . . . second Newton's law $F = m \cdot a$
- conservation of energy

What are the quantities we need to consider ?

- density . . . $\rho(\boldsymbol{x}, t)$
- velocity vector . . . $\boldsymbol{u}(\boldsymbol{x}, t)$
- energy . . . $e(\boldsymbol{x}, t)$

Can we find a common way to describe fluid motion ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

- conservation of mass
- conservation of momentum ... second Newton's law $F = m \cdot a$
- conservation of energy

What are the quantities we need to consider ?

- density ... $\rho(\boldsymbol{x}, t)$
- velocity vector ... $\boldsymbol{u}(\boldsymbol{x}, t)$
- energy ... $e(\boldsymbol{x}, t)$

mass conservation

$$\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(\rho\, \boldsymbol{u})(\boldsymbol{x}, t) = 0$$

Can we find a common way to describe them ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

- conservation of mass
- conservation of momentum . . . second Newton's law $F = m \cdot a$
- conservation of energy

- density . . . $\rho(\boldsymbol{x}, t)$
- velocity vector . . . $\boldsymbol{u}(\boldsymbol{x}, t)$
- energy . . . $e(\boldsymbol{x}, t)$

conservation of momentum: Newton's second law

$$\frac{\partial \rho u_k}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(\rho u_k \boldsymbol{u})(\boldsymbol{x}, t) + \frac{\partial p}{\partial x_k}(\boldsymbol{x}, t) = \sum_{\ell=1}^{3} \frac{\partial \boldsymbol{\tau}_{k\ell}}{\partial x_\ell}, \ k = 1, 2, 3$$

Can we find a common way to describe them ?

Yes: *"Mathematics is the language of nature" (G. Galilei)*

- conservation of mass
- conservation of momentum . . . second Newton's law $F = m \cdot a$
- conservation of energy

- density . . . $\rho(\boldsymbol{x}, t)$
- velocity vector . . . $\boldsymbol{u}(\boldsymbol{x}, t)$
- energy . . . $e(\boldsymbol{x}, t)$

conservation of energy

$$\frac{\partial e}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}\left((e + p)\boldsymbol{u}\right)(\boldsymbol{x}, t) = \sum_{k,\ell=1}^{3} \frac{\partial \boldsymbol{\tau}_{\ell k} u_k}{\partial x_\ell} + K \Delta T$$

- mass conservation

$$\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(\rho\, \boldsymbol{u})(\boldsymbol{x}, t) = 0 \tag{1}$$

- conservation of momentum

$$\frac{\partial \rho u_k}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(\rho u_k \boldsymbol{u})(\boldsymbol{x}, t) + \frac{\partial p}{\partial x_k}(\boldsymbol{x}, t) = \sum_{\ell=1}^{3} \frac{\partial \boldsymbol{\tau}_{k\ell}}{\partial x_\ell}, k = 1, 2, 3 \tag{2}$$

- energy conservation

$$\frac{\partial e}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}((e + p)\boldsymbol{u})(\boldsymbol{x}, t) = \sum_{k,\ell=1}^{3} \frac{\partial \boldsymbol{\tau}_{\ell k} u_k}{\partial x_\ell} + K \Delta T \tag{3}$$

- $p(t, \boldsymbol{x}) \ldots$ pressure
- $T(t, \boldsymbol{x}) \ldots$ temperature
- $(\boldsymbol{\tau}_{\ell k})_{\ell,k=1}^{3} \ldots$ stress tensor

Outline

Fluids are everywhere

Compressible inviscid fluids: known results and challenges

Mathematical properties

Numerical approximation

Compressible inviscid fluids

• negligible viscous effect $\longrightarrow$
Example: Euler equations of gas dynamics

Compressible inviscid fluids

- negligible viscous effect $\longrightarrow$

Example: Euler equations of gas dynamics

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{3} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = 0$$

$$\boldsymbol{U} = (\rho, \rho \boldsymbol{u}, e)$$

"The world is hyperbolic." (J. Nečas)

Outline

Fluids are everywhere

Compressible inviscid fluids: known results and challenges

Mathematical properties

Numerical approximation

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = 0, \tag{4}$$

where $\boldsymbol{U} \in R^N, \ \boldsymbol{F}_k : R^N \longrightarrow R^N$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = 0, \qquad (4)$$

where $\boldsymbol{U} \in R^N$, $\boldsymbol{F}_k : R^N \longrightarrow R^N$

Defintion 1

Let $\boldsymbol{A}_k(\boldsymbol{U}) \equiv \frac{D\boldsymbol{F}_k(\boldsymbol{U})}{D\boldsymbol{U}}$ ... $N \times N$ Jacobian matrix. The system (4) is called hyperbolic iff

$$\boldsymbol{P}(\boldsymbol{U}, \boldsymbol{\nu}) = \sum_{j=1}^{d} \nu_k \, \boldsymbol{A}_k \, (\boldsymbol{U}) \, ,$$

$\forall \boldsymbol{U} \in R^N$, $\forall \boldsymbol{\nu} \in R^d$, $|\boldsymbol{\nu}| = 1$

N real eigenvalues $\lambda_1(\boldsymbol{U}, \boldsymbol{\nu}), \ldots, \lambda_N(\boldsymbol{U}, \boldsymbol{\nu})$.

- $d = N = 1 \ldots$ scalar linear advection eq.:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in R,\ t \geq 0 \tag{5}$$

$$u(x,0) = u_0(x). \tag{6}$$

• $d = N = 1$ ... scalar linear advection eq.:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in R,\ t \geq 0 \tag{5}$$

$$u(x,0) = u_0(x). \tag{6}$$

charact. curve : $(x(s), t(s))$, s - parameter

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = a$$

• $d = N = 1 \ldots$ scalar linear advection eq.:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in R,\ t \geq 0 \tag{5}$$
$$u(x,0) = u_0(x). \tag{6}$$

charact. curve : $(x(s), t(s))$, s - parameter

$$\frac{dt(s)}{ds} = 1, \quad \frac{dx(s)}{ds} = a$$

Theorem 1

The function u is a solution of (5), (6) $\Longleftrightarrow$ u is constant along the characteristics.

Definition 2 (weak solution)

Let $\boldsymbol{U}_0 \in L^1_{loc}(\mathbf{R}^d)$.
A function $\boldsymbol{U}(\boldsymbol{x}, t) \in L^1_{loc}(\mathbf{R}^d \times [0, \infty))$ is a weak solution of the Cauchy problem (7)

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = 0$$
$$\boldsymbol{U}(\boldsymbol{x}, 0) = \boldsymbol{U}_0, \tag{7}$$

iff $\forall \varphi \in C_0^\infty(R^d \times [0, \infty))$

$$\int_0^\infty \int_{R^d} \left(\boldsymbol{U} \frac{\partial \varphi}{\partial t} + \sum_{k=1}^{d} \boldsymbol{F}_k(\boldsymbol{U}) \frac{\partial \varphi}{\partial x_k} \right) + \int_{R^d} \boldsymbol{U}_0 \, \varphi(\cdot, 0) = 0.$$

Nonlinear Example 1: Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0, \quad x \in R, \; t \geq 0$$

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 \leq x. \end{cases}$$

Nonlinear Example 1: Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0, \quad x \in R, \ t \geq 0$$

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 \leq x. \end{cases}$$

... method of characteristics:

$$\frac{d}{dt} x\,(t) = u\,(x\,(t)\,, t)$$

Nonlinear Example 1: Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0, \quad x \in R,\ t \geq 0$$

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 \leq x. \end{cases}$$

... method of characteristics:

$$\frac{d}{dt} x(t) = u(x(t), t)$$

$\Rightarrow$ $u(x,t)$ is constant along each characteristic

- solution for $t < 1$

$$u(x,t) = \begin{cases} 1, & x \le t, \\ (1-x)/(1-t), & t \le x \le 1, \\ 0, & x \ge 1 \end{cases}$$

- solution for $t < 1$

$$u(x,t) = \begin{cases} 1, & x \le t, \\ (1-x)/(1-t), & t \le x \le 1, \\ 0, & x \ge 1 \end{cases}$$

- solution for $t \geq 1$: discontinuity on $x = (t+1)/2$

- solution for $t < 1$

$$u(x,t) = \begin{cases} 1, & x \le t, \\ (1-x)/(1-t), & t \le x \le 1, \\ 0, & x \ge 1 \end{cases}$$

- solution for $t \ge 1$: discontinuity on $x = (t+1)/2$

$$u(x,t) = \begin{cases} 1, & x < t + \frac{1}{2}, \\ 0, & x > t + \frac{1}{2} \end{cases}$$

Shock relations

Theorem 2: (Rankine–Hugoniot condition)

Let $\boldsymbol{U} : \mathbf{R}^d \times \langle 0, \infty) \to D \subset \mathbf{R}^N$ be a piecewise-smooth function. Then $\boldsymbol{U}$ is a weak solution in $\mathbf{R}^d \times (0, \infty)$ if and only if the following conditions hold:

Theorem 2: (Rankine–Hugoniot condition)

Let $\boldsymbol{U} : \mathbf{R}^d \times \langle 0, \infty) \to D \subset \mathbf{R}^N$ be a piecewise-smooth function. Then $\boldsymbol{U}$ is a weak solution in $\mathbf{R}^d \times (0, \infty)$ if and only if the following conditions hold:

- $\boldsymbol{U}$ is a classical solution in any domain, where $\boldsymbol{U}$ is a C^1 function;

Theorem 2: (Rankine–Hugoniot condition)

Let $\boldsymbol{U} : \mathbb{R}^d \times \langle 0, \infty) \to D \subset \mathbb{R}^N$ be a piecewise-smooth function. Then $\boldsymbol{U}$ is a weak solution in $\mathbb{R}^d \times (0, \infty)$ if and only if the following conditions hold:

- $\boldsymbol{U}$ is a classical solution in any domain, where $\boldsymbol{U}$ is a C^1 function;
- jump condition:
  $\left(\boldsymbol{U}^+ - \boldsymbol{U}^-\right) n_t + \sum_{j=1}^{d} \left(\boldsymbol{F}_j\left(\boldsymbol{U}^+\right) - \boldsymbol{F}_j\left(\boldsymbol{U}^-\right)\right) n_j = 0$,
  where $(n_{x_1}, \dots, n_{x_d}, n_t)$ is an outer normal to any hypersurface of discontinuity Γ.

• let us verify the Rankine-Hugoniot relation for our example:

- let us verify the Rankine-Hugoniot relation for our example:
- discontinuity $\Gamma = \left\{ (x,t)\,;\; x = \dfrac{t+1}{2},\; t > 1 \right\}$

- let us verify the Rankine-Hugoniot relation for our example:

- discontinuity $\Gamma = \left\{(x,t)\,;\; x = \frac{t+1}{2},\; t > 1\right\}$

- outer normal to the discontinuity Γ :

$\boldsymbol{n} = (n_x, n_t) = (1, -s)$, where $s = \dfrac{d\xi(t)}{dt}$.

• let us verify the Rankine-Hugoniot relation for our example:

• discontinuity $\Gamma = \left\{ (x,t)\,;\; x = \frac{t+1}{2},\; t > 1 \right\}$

• outer normal to the discontinuity Γ :
$\boldsymbol{n} = (n_x, n_t) = (1, -s)$, where $s = \frac{d\xi(t)}{dt}$.

Rankine–Hugoniot condition gives:

$$s\left(u^+ - u^-\right) = f\left(u^+\right) - f\left(u^-\right)$$

in our case at $(x,t) = (1,1)$ we have:
$u^+ - u^- = 1,\; f\left(u^+\right) - f\left(u^-\right) = \frac{1}{2}$

- let us verify the Rankine-Hugoniot relation for our example:

- discontinuity $\Gamma = \left\{ (x,t)\,;\; x = \dfrac{t+1}{2},\; t > 1 \right\}$

- outer normal to the discontinuity Γ :

$\boldsymbol{n} = (n_x, n_t) = (1, -s)$, where $s = \dfrac{d\xi(t)}{dt}$.

Rankine–Hugoniot condition gives:

$$s\left(u^+ - u^-\right) = f\left(u^+\right) - f\left(u^-\right)$$

in our case at $(x,t) = (1,1)$ we have:
$u^+ - u^- = 1,\; f\left(u^+\right) - f\left(u^-\right) = \frac{1}{2}$

By the definition of Γ, $s = \dfrac{1}{2}$
$\Rightarrow$ R-H conditions is satisfied.

Nonlinear Example 2: Burgers equation

-Uniqueness ?

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0, \quad x \in R, \; t \geq 0 \tag{9}$$

$$u(x,0) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0, \end{cases} \tag{10}$$

Nonlinear Example 2: Burgers equation

-Uniqueness ?

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0, \quad x \in R, \; t \geq 0 \tag{9}$$

$$u(x,0) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0, \end{cases} \tag{10}$$

$$u_1(x,t) = \begin{cases} 0 & x < 0, \\ x/t, & 0 \leq x < t, \\ 1, & x \geq t \end{cases} \tag{11}$$

- Discontinuity at $x = t/2$: $s = 1/2$, $F(u) = u^2/2$

RH: $s := (F(u^+) - F(u^-))/(u^+ - u^-)$

$$u_2(x,t) = \begin{cases} 0, & x < t/2, \\ 1, & x > t/2. \end{cases}$$

- Discontinuity at $x = t/2$: $s = 1/2$, $F(u) = u^2/2$

RH: $s := (F(u^+) - F(u^-))/(u^+ - u^-)$

$$u_2(x,t) = \begin{cases} 0, & x < t/2, \\ 1, & x > t/2. \end{cases}$$

So what is THE right weak solution ?

Definition 3:

A convex function $\eta : \mathbf{R}^N \to \mathbf{R}$ is said to be entropy of our hyperbolic conservation law, if there are some entropy fluxes $\Phi_1, \ldots, \Phi_d : D \to \mathbf{R}$, s.t.

$$(\nabla\eta(\boldsymbol{U}))^T \cdot \boldsymbol{A}_j(\boldsymbol{U}) = \nabla\Phi_j(\boldsymbol{U}), \qquad j = 1, 2, \ldots, d.$$

Definition 3:

A convex function $\eta : \mathbf{R}^N \to \mathbf{R}$ is said to be entropy of our hyperbolic conservation law, if there are some entropy fluxes $\Phi_1, \ldots, \Phi_d : D \to \mathbf{R}$, s.t.

$$(\nabla \eta (\boldsymbol{U}))^T \cdot \boldsymbol{A}_j (\boldsymbol{U}) = \nabla \Phi_j (\boldsymbol{U}), \qquad j = 1, 2, \ldots, d.$$

*H.W. It is easy to show that if **U** is a weak solution that is smooth (i.e. C^1), then the following equality holds:*

$$\int_0^\infty \int_R \left(\eta(\boldsymbol{U}) \frac{\partial \varphi}{\partial t} + \sum_{k=1}^{d} \Phi_k (\boldsymbol{U}) \frac{\partial \varphi}{\partial x_k} \right) = 0$$

Definition 3:

A convex function $\eta : \mathbf{R}^N \to \mathbf{R}$ is said to be entropy of our hyperbolic conservation law, if there are some entropy fluxes $\Phi_1, \ldots, \Phi_d : D \to \mathbf{R}$, s.t.

$$(\nabla \eta (\boldsymbol{U}))^T \cdot \boldsymbol{A}_j (\boldsymbol{U}) = \nabla \Phi_j (\boldsymbol{U}), \qquad j = 1, 2, \ldots, d.$$

*H.W. It is easy to show that if **U** is a weak solution that is smooth (i.e. C^1), then the following equality holds:*

$$\int_0^\infty \int_R \left(\eta(\boldsymbol{U}) \frac{\partial \varphi}{\partial t} + \sum_{k=1}^{d} \Phi_k (\boldsymbol{U}) \frac{\partial \varphi}{\partial x_k} \right) = 0$$

Q: But what happens along discontinuities ?

Definition 4 (Entropy solution)

Let $\boldsymbol{U}$ be a weak solution of the Cauchy problem (7), then $\boldsymbol{U}$ is a weak entropy solution, iff

$$\int_0^\infty \int_R \left(\eta(\boldsymbol{U}) \frac{\partial \varphi}{\partial t} + \sum_{k=1}^{d} \Phi_k(\boldsymbol{U}) \frac{\partial \varphi}{\partial x_k} \right) \geq 0$$

$$\forall \varphi \in C_0^\infty(R \times \langle 0, \infty)), \varphi \geq 0,$$
$$\forall \eta, (\Phi_1, \ldots, \Phi_d) - \text{ Entropy / Entropy fluxes}$$

Entropy in fluid dynamics:

η - mathematical entropy,
S - physical entropy, $S = c_v \ln \frac{p}{\rho^\kappa} \longrightarrow$

$$\eta = -\rho S, \qquad \Phi_k = -\rho u_k S, \qquad k = 1, \ldots d.$$

Entropy in fluid dynamics:

η - mathematical entropy,
S - physical entropy, $S = c_v \ln \frac{p}{\rho^\kappa} \longrightarrow$

$$\eta = -\rho S, \qquad \Phi_k = -\rho u_k S, \qquad k = 1, \ldots d.$$

• We have now a selection criterion for detecting a suitable solution, i.e. **weak entropy solution.**

• For our example: u_1 is the right weak entropy solution; it is a rarefaction wave

Viscosity solution

- yet another concept of choosing a suitable physically reasonable solution

Viscosity solution

- yet another concept of choosing a suitable physically reasonable solution

- the original physical problem (compressible Navier-Stokes eqs.) has small viscosity terms

- **Is there any connection between entropy solution and the solution of viscous problem ?**

Viscosity solution

- yet another concept of choosing a suitable physically reasonable solution

- the original physical problem (compressible Navier-Stokes eqs.) has small viscosity terms

- **Is there any connection between entropy solution and the solution of viscous problem ?**

$\Rightarrow$ Yes !

... vanishing viscosity method (P. D. Lax (1954))

Theorem 3 (vanishing viscosity solution)

- $\eta \in C^2\left(\mathbf{R}^N; \mathbf{R}\right)$ is a convex entropy
- $\Phi_j \in C^1\left(\mathbf{R}^N, \mathbf{R}\right),\ j = 1, 2, \ldots, d \ldots$ entropy fluxes

Let $\boldsymbol{U}^\varepsilon$ be a sufficiently smooth solution of

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = \varepsilon \Delta \boldsymbol{U}^\varepsilon. \qquad (12)$$

Theorem 3 (vanishing viscosity solution)

- $\eta \in C^2(\mathbf{R}^N;\mathbf{R})$ is a convex entropy
- $\Phi_j \in C^1(\mathbf{R}^N,\mathbf{R})$, $j = 1, 2, \ldots, d \ldots$ entropy fluxes

Let $\boldsymbol{U}^\varepsilon$ be a sufficiently smooth solution of

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = \varepsilon \Delta \boldsymbol{U}^\varepsilon. \qquad (12)$$

i) $\exists c > 0 \forall \varepsilon$

$$\|\boldsymbol{U}^\varepsilon\|_{L^\infty\left(\mathrm{R}^d\times(0,\infty);\mathrm{R}^s\right)} \leq c,$$

ii)

$$\boldsymbol{U}^\varepsilon \to \boldsymbol{U} \qquad \text{as} \qquad \varepsilon \to 0 \qquad \text{a.e. in } \mathbf{R}^d \times [0,\infty).$$

Theorem 3 (vanishing viscosity solution)

- $\eta \in C^2\left(\mathbf{R}^N; \mathbf{R}\right)$ is a convex entropy
- $\Phi_j \in C^1\left(\mathbf{R}^N, \mathbf{R}\right),\ j = 1, 2, \ldots, d \ldots$ entropy fluxes

Let $\boldsymbol{U}^\varepsilon$ be a sufficiently smooth solution of

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = \varepsilon \Delta \boldsymbol{U}^\varepsilon. \qquad (12)$$

i) $\exists c > 0 \forall \varepsilon$

$$\|\boldsymbol{U}^\varepsilon\|_{L^\infty\left(\mathrm{R}^d \times (0,\infty); \mathrm{R}^s\right)} \leq c,$$

ii)

$$\boldsymbol{U}^\varepsilon \to \boldsymbol{U} \qquad \text{as} \qquad \varepsilon \to 0 \qquad \text{a.e. in } \mathbf{R}^d \times [0, \infty).$$

Then $\boldsymbol{U}$ is a weak solution of our problem AND

$$\frac{\partial \eta(\boldsymbol{U})}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \Phi_j(\boldsymbol{U}) \leq 0$$

in the sense of distributions on $\mathbf{R}^d \times (0, \infty)$.

Well-posedness of the Cauchy problem for hyperbolic conservation laws

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{F}_k}{\partial x_k}(\boldsymbol{U}) = 0 \text{ on } \mathbf{R}^d \times \mathbf{R}^+$$

$$\boldsymbol{U}(\boldsymbol{x}, 0) = \boldsymbol{U}_0(\boldsymbol{x}) \text{ on } \mathbf{R}^d$$

$$\boldsymbol{U} = (U_1, \dots, U_N)$$

Well-posedness of the Cauchy problem for hyperbolic conservation laws

- $N = 1,\ d \geq 1 \ldots$ scalar equation: existence and uniqueness of the weak entropy solution (Kružkov '70)

Well-posedness of the Cauchy problem for hyperbolic conservation laws

- $N = 1,\ d \geq 1 \ldots$ scalar equation: existence and uniqueness of the weak entropy solution (Kružkov '70)

- $N \geq 1,\ d = 1 \ldots$ one-dim. system: existence and uniqueness of the weak entropy solution in the BV space (Lax, Glimm '60, Bressan '90)

Well-posedness of the Cauchy problem for hyperbolic conservation laws

- $N = 1,\ d \geq 1 \ldots$ scalar equation: existence and uniqueness of the weak entropy solution (Kružkov '70)

- $N \geq 1,\ d = 1 \ldots$ one-dim. system: existence and uniqueness of the weak entropy solution in the BV space (Lax, Glimm '60, Bressan '90)

- $N > 1,\ d > 1$
concept of monotone solutions and BV-spaces is not simply transformable to multi-d

Well-posedness of the Cauchy problem for hyperbolic conservation laws

- $N = 1,\ d \geq 1 \ldots$ scalar equation: existence and uniqueness of the weak entropy solution (Kružkov '70)

- $N \geq 1,\ d = 1 \ldots$ one-dim. system: existence and uniqueness of the weak entropy solution in the BV space (Lax, Glimm '60, Bressan '90)

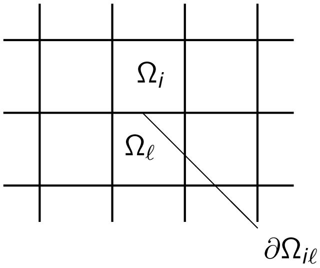
- $N > 1,\ d > 1$
concept of monotone solutions and BV-spaces is not simply transformable to multi-d
$\Longrightarrow$ well-posedness **OPEN PROBLEM !**
$\Longrightarrow$ we even do not know the appropriate functional setting

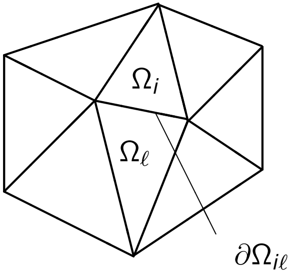
Outline

- the method of choice: **Finite Volume Methods (FVM)**

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}_1}{\partial x_1}(\boldsymbol{U}) + \frac{\partial \boldsymbol{F}_2}{\partial x_2}(\boldsymbol{U}) = 0 \qquad (13)$$

- $\Omega = \bigcup_i \Omega_i$, Ω_i - finite volumes:
- triangles, quadrilateral, polygons





$$\int_{\Omega_i}\int_{t_n}^{t_{n+1}} \frac{\partial \boldsymbol{U}}{\partial t} + \sum_{k=1}^{2}\int_{t_n}^{t_{n+1}}\int_{\Omega_i} \frac{\partial \boldsymbol{F}_k(\boldsymbol{U})}{\partial x_k} = 0$$

$\longrightarrow$

$$\int_{\Omega_i} \left(\boldsymbol{U}(\boldsymbol{x}, t_{n+1}) - \boldsymbol{U}(\boldsymbol{x}, t_n)\right) \mathrm{d}\boldsymbol{x} \tag{14}$$

$$+ \int_{t_n}^{t_{n+1}} \sum_{k=1}^{2} \sum_{\ell \in S(i)} \int_{\partial\Omega_{i\ell}} \boldsymbol{F}_k(\boldsymbol{U}(\boldsymbol{x}, t))\boldsymbol{n}_{i\ell,k} \mathrm{d}S\mathrm{d}t = 0,$$

$S(i) := \{\ell \mid \partial\Omega_{i\ell} := \partial\Omega_i \cap \partial\Omega_\ell$ an edge of $\Omega_i\}$
$\boldsymbol{n}_{i\ell} = (n_{i\ell,1}, n_{i\ell,2})$ ... outer normalvector, $|\boldsymbol{n}_{i\ell}| = 1$
- $\boldsymbol{U}(\boldsymbol{x}, t_n)|_{\Omega_i} \approx \boldsymbol{U}_i^n$ p.w. constant

$\longrightarrow$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{1}{|\Omega_i|}\int_{t_n}^{t_{n+1}} \sum_{k=1}^{2} \sum_{\ell \in S(i)} \int_{\partial\Omega_{i\ell}} \boldsymbol{F}_k(\boldsymbol{U}(\boldsymbol{x}, t))\boldsymbol{n}_{i\ell,k}$$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{1}{|\Omega_i|} \int_{t_n}^{t_{n+1}} \sum_{k=1}^{2} \sum_{\ell \in S(i)} \int_{\partial\Omega_{i\ell}} \boldsymbol{F}_k(\boldsymbol{U}(\boldsymbol{x}, t)) \boldsymbol{n}_{i\ell,k}$$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{1}{|\Omega_i|} \int_{t_n}^{t_{n+1}} \sum_{k=1}^{2} \sum_{\ell \in S(i)} \int_{\partial \Omega_{i\ell}} \boldsymbol{F}_k(\boldsymbol{U}(\boldsymbol{x}, t)) \boldsymbol{n}_{i\ell,k}$$

$$\boldsymbol{U}_i^{n+1} := \boldsymbol{U}_i^n - \frac{\Delta t}{|\Omega_i|} \sum_{\ell \in S(i)} \boldsymbol{H}(\boldsymbol{U}_i^n, \boldsymbol{U}_\ell^n, \boldsymbol{n}_{i\ell})$$

numerical flux function $\boldsymbol{H}(\boldsymbol{U}_i^n, \boldsymbol{U}_\ell^n, \boldsymbol{n}_{i\ell})$:

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{1}{|\Omega_i|} \int_{t_n}^{t_{n+1}} \sum_{k=1}^{2} \sum_{\ell \in S(i)} \int_{\partial\Omega_{i\ell}} \boldsymbol{F}_k(\boldsymbol{U}(\boldsymbol{x}, t)) \boldsymbol{n}_{i\ell,k}$$

$$\boxed{\boldsymbol{U}_i^{n+1} := \boldsymbol{U}_i^n - \frac{\Delta t}{|\Omega_i|} \sum_{\ell \in S(i)} \boldsymbol{H}(\boldsymbol{U}_i^n, \boldsymbol{U}_\ell^n, \boldsymbol{n}_{i\ell})}$$

numerical flux function $\boldsymbol{H}(\boldsymbol{U}_i^n, \boldsymbol{U}_\ell^n, \boldsymbol{n}_{i\ell})$:

consistency

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{U}, \boldsymbol{n}) = \sum_{k=1}^{2} \boldsymbol{F}_k(\boldsymbol{U}) n_k$$

conservation

$$\boldsymbol{H}(\boldsymbol{U}_i, \boldsymbol{U}_\ell, \boldsymbol{n}_{i\ell}) = \boldsymbol{H}(\boldsymbol{U}_\ell, \boldsymbol{U}_i, -\boldsymbol{n}_{\ell i})$$

- classical methods: dimensional splitting methods

- **Godunov methods** $\Rightarrow$ 1D Riemann problem:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x}(\boldsymbol{U}) = 0,$$

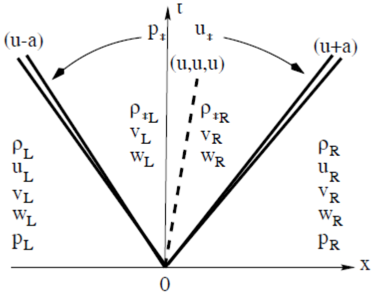
$$\boldsymbol{U}(x,0) := \begin{cases} \boldsymbol{U}_L(= \boldsymbol{U}_i^n), & x < 0 \\ \boldsymbol{U}_R(= \boldsymbol{U}_\ell^n), & x \geq 0. \end{cases}$$

- classical methods: dimensional splitting methods

- **Godunov methods** $\Rightarrow$ 1D Riemann problem:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x}(\boldsymbol{U}) = 0,$$

$$\boldsymbol{U}(x,0) := \begin{cases} \boldsymbol{U}_L (= \boldsymbol{U}_i^n), & x < 0 \\ \boldsymbol{U}_R (= \boldsymbol{U}_\ell^n), & x \geq 0. \end{cases}$$



Riemann solvers

- Exact Riemann solver (Godunov 1959)
- One-wave Rusanov solver (1961)
- Roe linearised Riemann solver (1981)
- Osher-Solomon (nonlinear) Riemann solver (1982)
- Flux-vector splitting solver (upwinding): Viyajasundaram (1986), van Leer (1983), Steger-Warming (1981)

Riemann solvers

- Exact Riemann solver (Godunov 1959)
- One-wave Rusanov solver (1961)
- Roe linearised Riemann solver (1981)
- Osher-Solomon (nonlinear) Riemann solver (1982)
- Flux-vector splitting solver (upwinding):
  Viyajasundaram (1986), van Leer (1983),
  Steger-Warming (1981)

- basic idea of approximate Riemann solvers: split flux into "positive" and "negative" part

$$\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{F}^+(\boldsymbol{U}) + \boldsymbol{F}^-(\boldsymbol{U}),$$

where corresponding Jacobians have either (only) positive or negative eigenvalues

$$\boldsymbol{A}^+ = \frac{D\boldsymbol{F}^+}{D\boldsymbol{U}}, \quad \boldsymbol{A}^- = \frac{D\boldsymbol{F}^-}{D\boldsymbol{U}}$$

Riemann solvers

- Exact Riemann solver (Godunov 1959)
- One-wave Rusanov solver (1961)
- Roe linearised Riemann solver (1981)
- Osher-Solomon (nonlinear) Riemann solver (1982)
- Flux-vector splitting solver (upwinding): Viyajasundaram (1986), van Leer (1983), Steger-Warming (1981)

- basic idea of approximate Riemann solvers: split flux into "positive" and "negative" part

$$\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{F}^{+}(\boldsymbol{U}) + \boldsymbol{F}^{-}(\boldsymbol{U}),$$

where corresponding Jacobians have either (only) positive or negative eigenvalues

$$\boldsymbol{A}^{+} = \frac{D\boldsymbol{F}^{+}}{D\boldsymbol{U}}, \quad \boldsymbol{A}^{-} = \frac{D\boldsymbol{F}^{-}}{D\boldsymbol{U}}$$

$$\boldsymbol{F}_{i+1/2} \equiv \boldsymbol{H}(\boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) := \boldsymbol{A}^{+}(\cdot)\boldsymbol{U}_i + \boldsymbol{A}^{-}(\cdot)\boldsymbol{U}_{i+1}$$

Central Schemes

- central finite difference approximation of fluxes

Central Schemes

- central finite difference approximation of fluxes
- **Lax-Friedrichs method (1960)** (in 1D)

a typical representative, first order accurate

$$
\begin{aligned}
\boldsymbol{U}_i^{n+1} &= \boldsymbol{U}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2} - \boldsymbol{F}_{i-1/2}\right) \\
\boldsymbol{F}_{i+1/2} &:= \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{U}_i^n) + \boldsymbol{F}(\boldsymbol{U}_{i+1}^n)\right) - \frac{1}{2}\frac{\Delta x}{\Delta t}(\boldsymbol{U}_{i+1}^n - \boldsymbol{U}_i^n)
\end{aligned}
$$

Central Schemes

- central finite difference approximation of fluxes
- **Lax-Friedrichs method (1960)** (in 1D)

a typical representative, first order accurate

$$\begin{aligned} \boldsymbol{U}_i^{n+1} &= \boldsymbol{U}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2} - \boldsymbol{F}_{i-1/2}\right) \\ \boldsymbol{F}_{i+1/2} &:= \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{U}_i^n) + \boldsymbol{F}(\boldsymbol{U}_{i+1}^n)\right) - \frac{1}{2}\frac{\Delta x}{\Delta t}(\boldsymbol{U}_{i+1}^n - \boldsymbol{U}_i^n) \end{aligned}$$

- Lax-Wendroff method (1964) ... second order accurate
- Nessyahu-Tadmor scheme (1990)
- FORCE flux schmes of Toro (1996)
- Kurganov-Tadmor scheme (2000), ...