A NOTE ON THE NONLINEAR LANDWEBER ITERATION

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Dedicated to Heinz W. Engl on the occasion of his 60th birthday

Abstract. We reconsider the Landweber iteration for nonlinear ill-posed problems. It is known that this method beomes a regularization method in case the iteration is terminated as soon as the residual drops below a certain multiple of the noise level in the data. So far, all known estimates of this factor are greater than two. Here we derive a smaller factor that may be arbitrarily close to one depending on the type of nonlinearity of the underlying operator equation.

Key words. Nonlinear ill-posed problems, Landweber iteration, discrepancy principle, monotonicity rule

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1. Introduction. At about 1990 Heinz W. Engl and his group at the Johannes Kepler University in Linz/Austria started to push forward a theory concerning the regularization of *nonlinear* ill-posed problems. Together with Karl Kunisch and Andreas Neubauer he wrote a seminal paper [5] on the Tikhonov regularization method for nonlinear problems, establishing a first order optimal error bound as they were well-known for linear problems at the time; other papers soon followed (cf., e.g., the monograph [4] for additional references).

While having been a young postdoc with Heinz in Linz during the winter term 1992/93 we took up this momentum and developed a convergence analysis for the nonlinear Landweber iteration ([7], jointly with Andreas Neubauer and Otmar Scherzer). Later, further people from Linz, especially Barbara Blaschke-Kaltenbacher got interested in iterative regularization methods, most notably Bakushinskii's iteratively regularized Gauß-Newton scheme; see [8] for a compilation of the corresponding results. The analysis of some of these methods utilizes ideas that have first been formulated in [7], especially the monotonicity argument that was essential for our proof of convergence.

This argument, in turn, was borrowed from a paper by Defrise and de Mol [3] on the discrepancy principle for the linear Landweber method,

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but may have its roots in work done by McCormick and Rodrigue [9]; see also Alifanov and Rumjancev [2]. In its original formulation, cf. [10], Morozov's discrepancy principle suggests to terminate the iteration as soon as the data fit is on the order of the (presumably known) noise level in the data. The Defrise/de Mol argument, however, only applies as long as the residual is twice that large, and this shortcoming manifested itself in our stopping rule for the nonlinear Landweber iteration.

It was few years later, in 1999, that Tautenhahn and Hämarik [11] developed the so-called monotonicity rule for choosing regularization parameters in various regularization methods for linear problems, again. For the Landweber iteration their argument can be seen as a refinement of the original Defrise/de Mol estimate, powerful enough to extend their result to the discrepancy principle in Morozov's original spirit; see also Alifanov, Artyukhin, and Rumyantsev [1, pp. 65]. In this note we show that a similar refinement of our argument in [7] leads to a comparable improvement of the stopping criterion for the nonlinear Landweber iteration, and, at the same time, to better accuracy of the computed approximations. An improvement of this sort had already been envisaged by Hämarik [6].

2. The setting. Consider an ill-posed problem

$$F(x) \;=\; y\,, \tag{2.1}$$

where F is a nonlinear operator with open domain $\mathcal{D}(F)$ in a Hilbert space $\mathcal{X}$, and with images in another Hilbert space $\mathcal{Y}$. The task is to determine the exact solution $x^\dagger \in \mathcal{D}(F)$ of (2.1) from approximate data y^δ with

$$\|y^\delta - y\| \;\leq\; \delta\,, \tag{2.2}$$

where $\delta < \|y\|$ is reasonably small. To apply the nonlinear Landweber iteration

$$x_{k+1}^\delta \;=\; x_k^\delta \;+\; F'(x_k^\delta)^*\bigl(y^\delta - F(x_k^\delta)\bigr)\,, \qquad k = 0,1,2,\dots\,, \tag{2.3}$$

we need to assume that F is Fréchet differentiable, and that

$$\|F'(x)\| \;\leq\; \gamma \leq 1 \tag{2.4}$$

holds for every x in a neighborhood $\mathcal{B}_\rho(x^\dagger) \subset \mathcal{X}$, i.e., an open ball of radius ρ around $x^\dagger$. Note that (2.4) can always be achieved by a proper scaling of the problem.

More restrictive is the so-called *weak Scherzer condition*

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \le \eta \|F(\tilde{x}) - F(x)\| \tag{2.5}$$

for the Taylor remainder of F; here, x and $\tilde{x}$ are arbitrary elements from $\mathcal{B}_\rho(x^\dagger)$, and $\eta < 1/2$ is an appropriate constant. We refer to [4] for an interpretation of this condition; we also quote from loc. cit. that (2.5) implies the alternative bound

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \le \frac{\eta}{1-\eta} \|F'(x)(\tilde{x} - x)\|, \tag{2.6}$$

as well as

$$\|F(\tilde{x}) - F(x)\| \le \frac{1}{1-\eta} \|F'(x)(\tilde{x} - x)\| \tag{2.7}$$

for x and $\tilde{x}$ as above. Note that $\eta > 0$ unless F is affine linear in $\mathcal{B}_\rho(x^\dagger)$.

According to [7] the nonlinear Landweber iteration (2.3) converges for every x_0 sufficiently close to $x^\dagger$ to some solution $\widehat{x}$ of (2.1) if the data are exact, i.e., if $\delta = 0$ in (2.2), and if F satisfies (2.4) and (2.5) for some $0 < \eta < 1/2$; the limit $\widehat{x} = \widehat{x}(x_0)$, however, will depend on the initial guess x_0 and need not coincide with $x^\dagger$, if $F'(x^\dagger)$ happens to have a nontrivial null space. For inexact data y^δ, on the other hand, the Landweber iteration can be turned into a regularization method, if the iteration is terminated appropriately: More precisely, if $\{y^\delta\}_{\delta>0} \subset \mathcal{Y}$ is a family of approximations of the exact data y subject to (2.2), and if the Landweber iteration is terminated after k^δ iterations, where $k^\delta \in \mathbb{N}$ is determined by the inequality chain

$$\|y^\delta - F(x^\delta_{k^\delta})\| \le \tau\delta < \|y^\delta - F(x^\delta_k)\|, \qquad 0 \le k < k^\delta, \tag{2.8}$$

then it has been shown in [7] that there holds

$$x^\delta_{k^\delta} \to \widehat{x} \qquad \text{as} \quad \delta \to 0, \tag{2.9}$$

provided that the parameter τ is coupled to the constant η in (2.5) via

$$\tau > \tau^*(\eta) := 2\,\frac{1+\eta}{1-2\eta}. \tag{2.10}$$

In (2.9) the limit $\widehat{x}$ is the same as the limit of the Landweber iteration in the exact data case.

Rule (2.8) is known as discrepancy principle: it states that the iteration is to be terminated as soon as the norm of the residual starts to

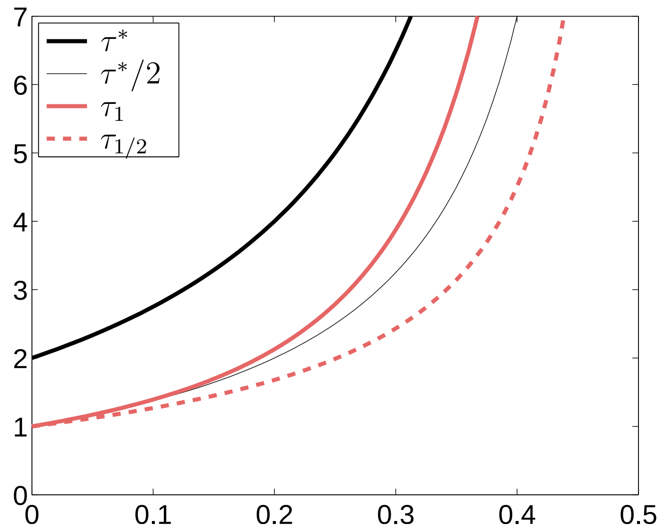

FIG. 2.1. *The graphs of τ^* from* (2.10) *and τ_γ from* (2.11) *as functions of $\eta \in (0, 1/2)$, using $\gamma = 1$, resp. $\gamma = 1/2$, for the latter. Also included is $\tau^*/2$.*

drop below a certain multiple τ of the noise level. Ideally, this parameter should be $\tau = 1$ as suggested by Morozov, or close to one; however, the right-hand side of (2.10) is always greater than two, and gets arbitrarily large when η comes close to $1/2$.

Here we will prove that this number τ can be chosen significantly smaller than in (2.10), and hence, further iterations are possible without loosing the aforementioned theoretical properties of the method. In fact, these additional iterations improve the accuracy of the approximations, as we will see below. More precisely, we show that we can choose

$$\tau \;>\; \tau_\gamma(\eta) \;:=\; \alpha_\gamma^+(\eta)\,\frac{1+\eta}{1-2\eta} \tag{2.11}$$

in (2.8), with

$$\alpha_\gamma^+(\eta) \;=\; \frac{\sqrt{1-4\eta+(5+\gamma^4)\eta^2-2\eta^3}+\eta\gamma^2}{1-\eta}\,, \tag{2.12}$$

where γ has been introduced in (2.4). The two bounds τ^* and τ_γ, as functions of η, are illustrated in Figure 2.1.

We show next how to adapt the arguments from [4] to this modified stopping rule. To this end we start with an auxiliary result.

LEMMA 2.1. *Let $r_k^\delta = y^\delta - F(x_k^\delta)$ and $A_k = F'(x_k^\delta)$, $k = 0, 1, 2, \ldots$. Assume further that F satisfies* (2.4) *and* (2.5) *with some constant* $0 <$

$\eta < 1/2$. Then there holds

$$\|(I - A_k A_k^*) r_k^\delta\| \;\geq\; \|r_{k+1}^\delta\| \;-\; \frac{\eta\gamma^2}{1-\eta}\, \|r_k^\delta\|$$

with γ of (2.4)*.*

Proof. We have

$$\begin{aligned} r_{k+1}^\delta \;&=\; y^\delta - F(x_{k+1}^\delta) \;=\; y^\delta - F(x_k^\delta + A_k^* r_k^\delta) \\ &=\; r_k^\delta + F(x_k^\delta) - F(x_k^\delta + A_k^* r_k^\delta) \\ &=\; (I - A_k A_k^*) r_k^\delta - \Bigl(F(x_k^\delta + A_k^* r_k^\delta) \;-\; F(x_k^\delta) \;-\; F'(x_k^\delta) A_k^* r_k^\delta \Bigr)\,. \end{aligned}$$

The Taylor remainder term within the big parentheses can be estimated by means of (2.5) and its corollary (2.6), to obtain

$$\|(I - A_k A_k^*) r_k^\delta\| \;\geq\; \|r_{k+1}^\delta\| \;-\; \frac{\eta}{1-\eta}\, \|A_k A_k^* r_k^\delta\| \;\geq\; \|r_{k+1}^\delta\| \;-\; \frac{\eta\gamma^2}{1-\eta}\, \|r_k^\delta\|\,,$$

where we have used assumption (2.4) for the final estimate. $\square$

3. Monotonicity of the Landweber iterates. The following result provides the key improvement of the analysis in [7], resp. Proposition 11.2 from [4].

PROPOSITION 3.1. *Let the assumptions* (2.2), (2.4)*, and* (2.5) *hold with $0 < \eta < 1/2$, and denote by x any solution of* (2.1)*. Then, a sufficient condition for x_{k+1}^δ to be a better approximation of x than x_k^δ is that*

$$\|y^\delta - F(x_{k+1}^\delta)\| \;>\; \tau_\gamma(\eta)\, \delta \tag{3.1}$$

with $\tau_\gamma(\eta)$ of (2.11)*.*

Proof. We adopt the notation from Lemma 2.1. As in [4] we start by reformulating the difference

$$\Delta \;=\; \|x_{k+1}^\delta - x\|^2 \;-\; \|x_k^\delta - x\|^2$$

between the squared norms of two consecutive iteration errors; we want to show that Δ will be negative, if (3.1) holds. Following [4] we have

$$\Delta \;=\; 2\,\langle\, r_k^\delta, y^\delta - F(x_k^\delta) - F'(x_k^\delta)(x - x_k^\delta)\,\rangle \;-\; \langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle \;-\; \|r_k^\delta\|^2,$$

and, by inserting the exact data y and using (2.2) and (2.5), we obtain

$$\begin{aligned}\Delta &= 2\,\langle\, r_k^\delta, y - F(x_k^\delta) - F'(x_k^\delta)(x - x_k^\delta)\,\rangle \\ &\qquad\quad + 2\langle\, r_k^\delta, y^\delta - y\,\rangle - \langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle - \|r_k^\delta\|^2 \\ &\le 2\eta\,\|r_k^\delta\|\,\|y - F(x_k^\delta)\| + 2\delta\,\|r_k^\delta\| \\ &\qquad\quad - \langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle - \|r_k^\delta\|^2 \\ &\le (2\eta - 1)\,\|r_k^\delta\|^2 + 2(1+\eta)\delta\,\|r_k^\delta\| - \langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle\,. \end{aligned} \tag{3.2}$$

So far, this estimate is exactly the same as in [4]. The difference lies in the upcoming more subtle treatment of the very last term $\langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle$, which has simply been neglected in [4], by virtue of its nonnegativity due to (2.4). Here we perform a case-by-case analysis instead, depending on whether

$$\|r_{k+1}^\delta\| \ge \frac{\eta\gamma^2}{1-\eta}\,\|r_k^\delta\|\,, \tag{3.3}$$

or not, where $\gamma \le 1$ is the constant occurring in (2.4).

Assume first that (3.3) holds true. Then we use the refined estimate

$$\langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle = \|(I - A_k A_k^*)^{1/2} r_k^\delta\|^2 \ge \|(I - A_k A_k^*) r_k^\delta\|^2\,,$$

which follows from the fact that $\|I - A_k A_k^*\| \le 1$. By virtue of Lemma 2.1 we thus conclude that

$$\langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle \ge \Big(\|r_{k+1}^\delta\| - \frac{\eta\gamma^2}{1-\eta}\,\|r_k^\delta\|\Big)^2,$$

because the assumption (3.3) garantees that the term in parantheses is nonnegative. Inserting this last inequality into (3.2) we obtain

$$\begin{aligned}\Delta &\le \Big(2\eta - 1 - \frac{\eta^2\gamma^4}{(1-\eta)^2}\Big)\|r_k^\delta\|^2 + 2(1+\eta)\delta\,\|r_k^\delta\| \\ &\qquad\quad - \|r_{k+1}^\delta\|^2 + \frac{2\eta\gamma^2}{1-\eta}\,\|r_k^\delta\|\|r_{k+1}^\delta\| \\ &= 2\|r_k^\delta\|\Big((1+\eta)\delta - \alpha_\gamma^-(\eta)\|r_{k+1}^\delta\|\Big) - \Big(\|r_{k+1}^\delta\| - \beta(\eta)\|r_k^\delta\|\Big)^2, \end{aligned}$$

where $\beta(\eta) = \big(1 - 2\eta + \eta^2\gamma^4/(1-\eta)^2\big)^{1/2}$, and

$$\begin{aligned}\alpha_\gamma^-(\eta) &= \beta(\eta) - \frac{\eta\gamma^2}{1-\eta} = \frac{\sqrt{1 - 4\eta + (5+\gamma^4)\eta^2 - 2\eta^3} - \eta\gamma^2}{1-\eta} \\ &= \frac{1-2\eta}{\alpha_\gamma^+(\eta)} = \frac{1+\eta}{\tau_\gamma(\eta)} \end{aligned} \tag{3.4}$$

with α_γ^+ of (2.12) and $\tau_\gamma(\eta)$ of (2.11). We thus have established that

$$\Delta \;\leq\; 2\alpha_\gamma^-(\eta)\|r_k^\delta\|\Bigl(\tau_\gamma(\eta)\,\delta \;-\; \|r_{k+1}^\delta\|\Bigr)\,. \tag{3.5}$$

On the other hand, if (3.3) fails to hold, then we estimate as in [4], i.e., drop the nonnegative term $\langle\, r_k^\delta, (I - A_k A_k^*) r_k^\delta\,\rangle$ in (3.2), and then use (the opposite of) (3.3), and (3.4), to obtain

$$\begin{aligned}\Delta \;&\leq\; 2(1+\eta)\delta\,\|r_k^\delta\| \;-\; (1-2\eta)\|r_k^\delta\|^2 \\ &\leq\; 2(1+\eta)\delta\,\|r_k^\delta\| \;-\; \frac{(1-\eta)(1-2\eta)}{\eta\gamma^2}\,\|r_k^\delta\|\,\|r_{k+1}^\delta\| \\ &=\; 2\alpha_\gamma^-(\eta)\,\|r_k^\delta\|\,\Bigl(\tau_\gamma(\eta)\,\delta \;-\; \frac{1-\eta}{2\eta\gamma^2}\,\alpha_\gamma^+(\eta)\,\|r_{k+1}^\delta\|\Bigr)\,.\end{aligned}$$

It is not difficult to check that

$$\alpha_\gamma^+(\eta) \;\geq\; \frac{2\eta\gamma^2}{1-\eta}\,, \qquad 0 < \eta < 1/2\,,$$

showing that (3.5) is valid in either case.

We thus conclude that if $\|r_{k+1}^\delta\| > \tau_\gamma(\eta)\delta$, the difference Δ of the two consecutive error terms will be negative, i.e., $\|x_{k+1}^\delta - x\| < \|x_k^\delta - x\|$, as has been claimed. ☐

Note that for exact data ($\delta = 0$) Proposition 3.1 implies that the iteration error decreases monotonically as function of $k \in \mathbb{N}_0$. But for the proof of convergence of the corresponding iterates $(x_k)_k$ to a solution $\widehat{x}$ of (2.1) as presented, e.g., in [4], the convergence of the series

$$\sum_{k=0}^{\infty} \|y^\delta - F(x_k)\|^2 \;<\; \infty \tag{3.6}$$

is another important ingredient. Under our relaxed assumptions this by-product of the proof of Proposition 3.1 is a little more difficult to deduce than in [4]; this is the content of the following corollary.

Corollary 3.2. *Under the assumptions of Proposition 3.1, if* $\|y^\delta - F(x_k^\delta)\| > \tau\delta$ *for all* $0 \leq k \leq k_*$ *with some* $\tau > \tau_\gamma(\eta)$*, then*

$$k_*\tau^2\delta^2 \;\leq\; \sum_{k=1}^{k_*} \|y^\delta - F(x_k^\delta)\|^2 \;\leq\; \frac{1}{1+\eta}\frac{\tau_\gamma(\eta)\tau}{\tau - \tau_\gamma(\eta)}\,\|x_0 - x^\dagger\|^2\,. \tag{3.7}$$

Proof. From (3.5) we conclude that for $0 \le k < k_*$ and $x = x^\dagger$

$$\begin{aligned}\|x_k^\delta - x^\dagger\|^2 \,-\, \|x_{k+1}^\delta - x^\dagger\|^2 & \\ &\ge\, 2\alpha_\gamma^-(\eta)\|r_k^\delta\|\big(\|r_{k+1}^\delta\| \,-\, \tau_\gamma(\eta)\,\delta\big) \\ &\ge\, 2\alpha_\gamma^-(\eta)\,\frac{\tau - \tau_\gamma(\eta)}{\tau}\,\|r_k^\delta\|\|r_{k+1}^\delta\|\,.\end{aligned}$$

From Lemma 2.1 it follows that

$$\begin{aligned}\|r_{k+1}^\delta\| \,&\le\, \|(I - A_k A_k^*) r_k^\delta\| \,+\, \frac{\eta\gamma^2}{1-\eta}\,\|r_k^\delta\| \\ &\le\, \Big(1 \,+\, \frac{\eta\gamma^2}{1-\eta}\Big)\|r_k^\delta\| \,\le\, 2\,\|r_k^\delta\|\end{aligned}$$

for any $k \in \mathbb{N}_0$, any $0 < \eta < 1/2$, and any value of $\gamma \le 1$ in (2.4). Eliminating $\|r_k^\delta\|$ in this manner from the previous estimate, and inserting (3.4) we obtain

$$\|x_k^\delta - x^\dagger\|^2 \,-\, \|x_{k+1}^\delta - x^\dagger\|^2 \,\ge\, (1+\eta)\frac{\tau - \tau_\gamma(\eta)}{\tau_\gamma(\eta)\tau}\,\|r_{k+1}^\delta\|^2\,.$$

Adding these inequalities from $k = 0$ to $k_* - 1$ the assertion of this corollary follows. ◻

For exact data we now can let $k_* \to \infty$ in (3.7) to obtain the convergence of the series (3.6). For inexact data, on the other hand, we conclude from (3.7) that there must be a first index k_* where the additional assumption of this corollary fails to hold, which shows that the discrepancy principle (2.8) yields a well-defined finite stopping index $k^\delta \in \mathbb{N}$, as long as the parameter τ in (2.8) satisfies $\tau > \tau_\gamma(\eta)$ with $\tau_\gamma(\eta)$ of (2.11).

4. The regularizing properties of the improved stopping rule.

In the previous section we have seen that the discrepancy principle (2.8) with parameter $\tau > \tau_\gamma(\eta)$ is a well-defined stopping rule for the nonlinear Landweber iteration, and that the iteration error is monotonically decreasing up to the next-to-last iterate. Here we prove that it is also a regularization method.

THEOREM 4.1. *Let the assumptions* (2.2), (2.4), *and* (2.5) *hold with* $0 < \eta < 1/2$, *and fix* $\tau > \tau_\gamma(\eta)$ *of* (2.11). *If the Landweber iteration is stopped with* k^δ *according to the discrepancy principle* (2.8) *then* $x_{k^\delta}^\delta$ *converges to a solution* $\widehat{x}$ *of* (2.1) *as* $\delta \to 0$.

Proof. The proof is much like the proof of Theorem 11.5 in [4]: One has to consider finite accumulation points of k^δ as $\delta \to 0$, and subsequences of k^δ that go to infinity; while the finite accumulation points can be treated exactly as in [4], a more careful analysis is required in the other case.

Accordingly, let δ_n be a sequence going to zero as $n \to \infty$, and $k_n = k^{\delta_n}$ be corresponding stopping indices of (2.8) with

$$k_1 \,<\, k_2 \,<, \ldots \,<\, k_n \,\to\, \infty \qquad \text{as } n \to \infty\,.$$

Let $\widehat{x}$ be the solution of (2.1) to which the Landweber iteration converges in the exact data case. Then, for $n > m$, Proposition 3.1 yields

$$\|x_{k_n-1}^{\delta_n} - \widehat{x}\| \,\le\, \ldots \,\le\, \|x_{k_m}^{\delta_n} - \widehat{x}\| \,\le\, \|x_{k_m}^{\delta_n} - x_{k_m}\| \,+\, \|x_{k_m} - \widehat{x}\|\,,$$

and for any $\varepsilon > 0$ we can fix $m = m(\varepsilon)$ so large that the last term on the right-hand side is less than $\varepsilon/2$. Because of the stability of the Landweber iteration we also have $\|x_{k_m}^{\delta_n} - x_{k_m}\| < \varepsilon/2$ as soon as $n > n(\varepsilon) > m(\varepsilon)$, showing that

$$\|x_{k_n-1}^{\delta_n} - \widehat{x}\| \,\le\, \varepsilon$$

for $n > n(\varepsilon)$. But then we conclude from (2.3) and (2.4), with $r_n = y^{\delta_n} - F(x_{k_n-1}^{\delta_n})$, that

$$\begin{aligned}\|x_{k_n}^{\delta_n} - \widehat{x}\| \,&=\, \|x_{k_n-1}^{\delta_n} - \widehat{x} + F'(x_{k_n-1}^{\delta_n})^* r_n\| \\ &\le\, \varepsilon \,+\, \|F'(x_{k_n-1}^{\delta_n})^* r_n\| \,\le\, \varepsilon \,+\, \|r_n\|\,,\end{aligned}$$

and it follows from (2.7) that

$$\begin{aligned}\|x_{k_n}^{\delta_n} - \widehat{x}\| \,&\le\, \varepsilon \,+\, \|y^{\delta_n} - y\| \,+\, \|y - F(x_{k_n-1}^{\delta_n})\| \\ &\le\, \varepsilon \,+\, \delta_n \,+\, \frac{1}{1-\eta}\,\varepsilon \,\le\, \frac{2}{1-\eta}\,\varepsilon\end{aligned}$$

by choosing n sufficiently large. This establishes our assertion. $\square$

In [7] we also investigated the order-optimality of the nonlinear Landweber iteration under additional assumptions on F. The corresponding result for our improved stopping rule is as follows.

THEOREM 4.2. *Assume that F satisfies* (2.4), *and that*

$$F'(x) \,=\, R_x F'(x^\dagger) \qquad \textit{for all } x \in \mathcal{B}_\rho(x^\dagger)$$

and some $\rho > 0$, where the map $x \mapsto R_x \in \mathscr{B}(\mathcal{Y})$ is Lipschitz continuous in $\mathcal{B}_\rho(x^\dagger)$. Then, if $\tau > 1$ and $x^\dagger - x_0 = \bigl(F'(x^\dagger)^* F'(x^\dagger)\bigr)^\nu f$ for some $0 < \nu \le 1/2$ and $f \in \mathcal{X}$ with $\|f\|$ sufficiently small, and if $\|y^\delta - y\| \le \delta$, then there is a constant $c > 0$ such that the stopping index k^δ of (2.8) is well defined, and there holds

$$k^\delta \;\le\; c\,(\|f\|/\delta)^{1/(2\nu+1)},$$

and

$$\|x^\delta_{k^\delta} - x^\dagger\| \;\le\; c\,\|f\|^{1/(2\nu+1)}\delta^{2\nu/(2\nu+1)}\,.$$

Concerning the proof we remark that the new assumption on F implies that (2.5) holds for any value of $\eta > 0$ provided that $\tilde{x}$ and x are sufficiently close to $x^\dagger$, i.e., in our region of interest, if $f \in \mathcal{X}$ as introduced in the statement of the theorem is sufficiently small. Since $\tau_\gamma(\eta)$ approaches one as $\eta \to 0$ it follows that $\tau > \tau_\gamma(\eta)$ for η sufficiently small, that is, for f sufficiently small, and therefore the stopping index (2.8) is well defined in that case.

With this in mind the proof of Theorem 4.2 is exactly the same as in [7], with obvious modifications of the corresponding constants, and is therefore omitted here.

5. Summary. We have shown that in the discrepancy principle (2.8) for the nonlinear Landweber iteration smaller parameters τ are admissible than those that have previously been used; new bounds for τ are given by $\tau_\gamma(\eta)$ of (2.11), where $0 < \eta < 1/2$ is the constant in the nonlinearity constraint (2.5) on the operator F, and $0 < \gamma \le 1$ is a bound for its Fréchet derivative, cf. (2.4). The value of $\tau_\gamma(\eta)$ depends monotonically on η and γ, and comes arbitrarily close to the optimal parameter $\tau = 1$ when η approaches zero, i.e., when the problem becomes 'more and more linear'.

Another simple computation reveals that the new bounds for τ are only half as large as the previously known ones, i.e., $\tau^*(\eta)$ of (2.10), provided that

$$\gamma^2 \;\le\; 1 - \eta\,. \tag{5.1}$$

In view of (2.4) this inequality can always be achieved by rescaling the problem, as such a rescaling does not alter the constant η in (2.5). However, as smaller values of γ lead to smaller step sizes in the Landweber iteration, this will typically slow down the convergence. On the other hand, the requirement (5.1) is not too dramatic, either, as its right-hand side is greater than 1/2, and hence, $\gamma = 1/\sqrt{2}$ will always be sufficient.

Once again, we refer to Figure 2.1 for a comparison of the old and new bounds for τ.

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