Handbook of Mathematical Methods in Imaging

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Chapter 1

Sampling Methods

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1.1 Introduction and Historical Background

The topic of this chapter is devoted to *shape identification problems*, i.e. problems where the shape of an object has to be determined from indirect measurements. Such a situation typically occurs in problems of *tomography*, in particular electrical impedance tomography or optical tomography. For example, a current through a homogeneous object will in general induce a different potential than the same current through the same object containing an enclosed cavity. In impedance tomography the task is to determine the shape of the cavity from measurements of the potential on the boundary of the object. For survey articles on this subject we refer to [18], [55], and the contribution [Handbook:EIT] in this volume.

As a second of these fields we mention *inverse scattering problems* where one wants to detect - and identify - unknown objects through the use of acoustic, electromagnetic, or elastic waves. Similar to above, one of the important problems in inverse scattering theory is to determine the shape of the scattering obstacle from field measurements. Applications of inverse scattering problems occur in such diverse areas as medical imaging, material science, nondestructive testing, radar, remote sensing, or seismic exploration. A survey on the state of the art of the mathematical theory and numerical approaches for solving inverse time harmonic scattering problems until 1998 can be found in the standard monograph [36], see also [Handbook:Inverse Scattering] or [84] for an introduction and survey on inverse scattering problems.

Shape identification problems are intrinsically *nonlinear*, i.e. the measured quantities do not depend linearly on the shape. Even the notion of linearity does not make sense since, in general, the set of admissible shapes does not carry a linear structure. Traditional (and still very successful) approaches describe the objects by appropriate parameterizations and compute

the parameters by *iterative schemes* as, e.g., Newton-type methods. Newton-type methods are attractive because of their fast convergence, although they require a good initial guess to converge. Still, these methods are widely used – partly because techniques from shape optimization theory can be used to characterize the required first or second order derivatives. We refer to [85, 90] for general references, and to [58, 59, 66] for applications in inverse scattering theory.

While classical iterative algorithms use explicit parameterizations of the object, new shape optimization methods have been developed since around 1995 which completely avoid the use of parameterizations and replace the classical Fréchet derivative by a geometrically motivated *topological derivative*, see, e.g., [51] for the application of these methods in the inverse scattering context. Yet these methods have the shortcoming that they are not able to change the number of connectivity components during the algorithm. This has led to the development of *level set methods* which are based on implicit representations of the unknown object involving an "evolution parameter" t. We refer to [25] for a recent survey.

While very successful in many cases, iterative methods for shape identification problems – may they use classical tools as the Fréchet derivative or more recent techniques such as domain derivatives, level curves, or topological derivatives – are computationally very expensive since they require the solution of a direct problem in every step. Furthermore, for many important cases the convergence theory is still missing. This is due to the fact that these problems are not only nonlinear but also because their linearizations are *improperly posed*. Although there exist many results on the convergence of (regularized) iterative methods for solving nonlinear improperly posed problems (see, e.g., [40, 65]) the assumptions for convergence are not met in the applications to shape identification problems.[1]

These difficulties and disadvantages of iterative schemes gave rise to the development of different classes of *non-iterative* methods which avoid the solution of a sequence of direct problems. We briefly mention *decomposition methods* (according to the notion of [37]) which consist of an analytic continuation step (which is linear but highly improperly posed) and a nonlinear step of finding the boundary of the unknown domain by forcing the boundary condition to hold. We refer to Section 4.2 of [Handbook:Inverse Scattering].

This chapter will focus on a different class of non-iterative methods, the so-called *sampling methods*. The common idea of these methods is the construction of criteria on the known data to decide whether a given test object (a point or a curve or a set) is inside or outside the unknown domain D. Then, a grid of "sampling" points z is chosen to place these objects in a region that is known to contain the unknown domain D, in order to compute the (approximate) characteristic function of D. The different kinds of sampling methods differ in the way of defining the criteria and in the type of test objects.

One of the first methods which falls into this class has been developed by David Colton and one of the authors (A.K.) in 1996 ([35]), now known as the *Linear Sampling Method*. Its origin goes back to the *Dual Space Method* developed between 1985 and 1990 (see, e.g., [36]). The numerical implementation of the Linear Sampling Method is extremely simple and fast because sampling is done by points z only. For every sampling point z one has to compute the field of a point source in z with respect to the background medium[2] (if this is constant the response is even given analytically) and evaluate a series, i.e., a finite sum in practice.

A problem with the Linear Sampling Method from the mathematical point of view is that the computable criterion is only a sufficient condition which is, in general, not necessary. The *Fac-*

[1] Or, at least, it is unknown whether these assumptions are fulfilled or not.

[2] Essentially, one has to compute the fundamental solution of the underlying differential operator.

torization Method overcomes this drawback and yields a criterion for z which is both, necessary and sufficient. Therefore, this method succeeds to provide a simple formula for the characteristic function of D which can easily be used for numerical computations.

The Factorization Method consists of three components. First, a "measurement operator" M is factorized in three factors of the form

$$M \;=\; AGA^* \,, \tag{1.1.1}$$

where A^* is the dual operator of A with respect to the L^2 topology. Second, the range of A is characterized by the obstacle D, and vice versa. Third, if the operator G satisfies a certain coercivity condition then the range of A can be determined by the given operator M. This requires some functional analytic results on range identities which we have collected in an appendix.

Combining these three steps yields an explicit characterization of the unknown obstacle D by the measurement operator M.

The outline of this chapter is as follows. First, in Section 1.2, we present the Factorization Method for two different settings in the impedance tomography context. In the very first setting we deal with insulating inclusions, and this allows for a very elementary presentation of the method. Afterwards, in Section 1.3, we turn to applications from inverse acoustic and (full 3D) electromagnetic scattering. Finally, we give a brief overview of other sampling type methods in Section 1.4, including the original Linear Sampling Method and MUSIC type methods.

1.2 The Factorization Method in Impedance Tomography

We start with the impedance tomography problem. Consider an object, that fills a simply connected domain $\Omega \subset \mathbb{R}^n$ with Lipschitz continuous boundary, where $n = 2$ or $n = 3$, respectively. We assume that the object is a homogeneous and isotropic conductor, except for a finite number m of so-called inclusions, given by domains $D_i \subset \Omega$, $i = 1, \ldots, m$, with Lipschitz continuous boundary ∂D_i. We assume that these domains are well separated, i.e., $\overline{D}_i \cap \overline{D}_j = \emptyset$ when $i \neq j$, and that the complement of the closure $\overline{D}$ of $D = \bigcup_{i=1}^m D_i$ is connected. In impedance tomography, currents are imposed through the boundary of the object and the resulting boundary potentials are measured. Linear independent boundary currents yield independent pieces of information, which can be used as input data to determine the unknown shapes and positions of the inclusions.

In practice, at least in most medical applications, the boundary currents have a frequency in the kHz range (5-500 kHz), and the dc approximation with a positive real conductivity σ (or possibly a positive definite tensor) serves as a suitable physical model. Without loss of generality, we can always assume that the homogeneous conductivity of the object equals $\sigma = 1$, whereas $\sigma \neq 1$ within the inclusions.

Below we will consider two specific scenarios. In the first one, we assume that the inclusions are insulating, formally corresponding to the case where $\sigma = 0$. Our analysis of the Factorization Method for the corresponding inverse problem will be somewhat nonstandard; in particular, we employ a factorization in only two factors instead of three as in (1.1.1), but this allows for a most elementary treatment of the method.

Subsequently, we show how to deal with conducting obstacles with a conductivity tensor σ. Of particular interest is the setting where the object under consideration can be modelled as

a half space: examples of this sort arise in geophysics, cf. [79], and in medicine, e.g., when a planar device is used for mammography examinations, cf. [93]. Another interesting application for the half space problem has recently been considered in [17]. We therefore briefly describe the differences that arise in this context (mainly in the theoretical justification of the method).

We conclude our case studies with a setting where the inclusion degenerates to a crack, i.e., an $n-1$ dimensional smooth manifold within Ω. This application requires some care in the appropriate implementation of the Factorization Method.

1.2.1 Impedance Tomography in the Presence of Insulating Inclusions

To begin with, we take up the case where Ω is a bounded domain, and the domains $D_i \subset \Omega$, $i = 1, \ldots, m$, correspond to insulating inclusions. Within the dc model the potential u_0 induced by a boundary current f is given by

$$\begin{aligned} \Delta u_0 = 0 \quad \text{in } \Omega \setminus \overline{D}\,, \qquad \frac{\partial}{\partial \nu} u_0 = 0 \quad \text{on } \partial D\,, \\ \frac{\partial}{\partial \nu} u_0 = f \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} u_0 \, \mathrm{d}s = 0\,, \end{aligned} \tag{1.2.1}$$

where the normal vectors ν on $\partial\Omega$ and ∂D are pointing into the exterior of Ω and D, respectively. In order to make the forward problem (1.2.1) well-posed we restrict f to be square integrable with vanishing mean on $\partial\Omega$. The corresponding set of admissible boundary currents is

$$L^2_\diamond(\partial\Omega) \;=\; \left\{ f \in L^2(\partial\Omega) \,:\, \int_{\partial\Omega} f \, \mathrm{d}s \;=\; 0 \right\}. \tag{1.2.2}$$

Under these assumptions problem (1.2.1) has a unique (weak) solution

$$u_0 \;\in\; H^1_\diamond(\Omega \setminus \overline{D}) \;=\; \left\{ u \in H^1(\Omega \setminus \overline{D}) \,:\, \int_{\partial\Omega} u \, \mathrm{d}s \;=\; 0 \right\}.$$

The last condition in (1.2.1) normalizes this boundary potential to have vanishing mean; without this condition, the solution would only be unique up to additive constants, reflecting the fact that only the voltage, i.e., the difference between the potential at two different points is a well-defined physical quantity.

Therefore, the **direct problem** is to determine the field u_0 when f and D are given.

The quantity that is measured in impedance tomography is the trace $g_0 = u_0|_{\partial\Omega}$, i.e., the boundary potential. The corresponding measurement operator

$$\Lambda_0 \,:\, \left\{ \begin{array}{ccc} L^2_\diamond(\partial\Omega) & \to & L^2_\diamond(\partial\Omega)\,, \\ f & \mapsto & g_0 = u_0|_{\partial\Omega}\,, \end{array} \right. \tag{1.2.3}$$

i.e., the so-called *Neumann-Dirichlet operator*, is usually referred to as *absolute data* in impedance tomography.

The **inverse problem** is to determine the shape of D from the measurement operator Λ_0.

For the Factorization Method we employ *relative data*, that is the difference between the above Neumann-Dirichlet operator and the corresponding one for a completely homogeneous object in Ω. To be precise, let $u_{\mathbb{1}}$ be the reference solution for the homogeneous object, given the same boundary current $f \in L^2_\diamond(\partial\Omega)$,

$$\Delta u_{\mathbb{1}} = 0 \quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial \nu} u_{\mathbb{1}} = f \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} u_{\mathbb{1}} \, \mathrm{d}s = 0\,, \tag{1.2.4}$$

and denote by $\Lambda_{\mathbb{1}} : f \mapsto g_{\mathbb{1}} = u_{\mathbb{1}}|_{\partial\Omega}$ the Neumann-Dirichlet map associated with (1.2.4). It is the relative data $M = \Lambda_0 - \Lambda_{\mathbb{1}}$ that later enters in (1.1.1) to lay the grounds for the setting of the Factorization Method.

We refer to [Handbook:EIT] for a more elaborate treatment of the impedance tomography problem, but we will see below that $\Lambda_0 - \Lambda_{\mathbb{1}}$ is a bounded and positive self adjoint operator. We also do not discuss practical issues such as electrode models that should be incorporated into a realistic problem setting. For the same reason we do not comment on how to obtain relative data in practice; the generation of accurate reference data is indeed a difficult subject, and some workarounds have therefore been suggested for this purpose[3]. Our specification of the impedance tomography problem is thus a purely mathematical one, although it can be shown to be a pretty reasonable approximation of the real case, cf., e.g., [61, 78].

Before we continue we pause to comment on the nature of the relative data introduced above. Any function h in the range $\mathcal{R}(\Lambda_0 - \Lambda_{\mathbb{1}})$ of $\Lambda_0 - \Lambda_{\mathbb{1}}$ corresponds to a suitable input current $f \in L^2_\diamond(\partial\Omega)$, such that h is the trace of $w = u_0 - u_{\mathbb{1}} : \Omega \setminus \overline{D} \to \mathbb{R}$, where u_0 and $u_{\mathbb{1}}$ are the solutions of (1.2.1) and (1.2.4), respectively. As u_0 and $u_{\mathbb{1}}$ are both harmonic in $\Omega \setminus \overline{D}$, the same holds true for w; on top of that, like u_0 and $u_{\mathbb{1}}$, w has finite H^1 norm on $\Omega \setminus \overline{D}$, as well as vanishing mean on $\partial\Omega$. Moreover, w has homogeneous Neumann boundary conditions on $\partial\Omega$, as u_0 and $u_{\mathbb{1}}$ both satisfy the same Neumann boundary condition. And finally, on ∂D_i, $i = 1, \ldots, m$, we have

$$\int_{\partial D_i} \frac{\partial}{\partial \nu} w \, \mathrm{d}s \;=\; -\int_{\partial D_i} \frac{\partial}{\partial \nu} u_{\mathbb{1}} \, \mathrm{d}s \;=\; 0$$

by virtue of Green's formula. Accordingly, the range of $\Lambda_0 - \Lambda_{\mathbb{1}}$ consists of traces of potentials w from

$$\mathcal{W} \;=\; \left\{ w \in H^1_\diamond(\Omega \setminus \overline{D}) \,:\, \Delta w = 0,\; \frac{\partial}{\partial \nu} w = 0 \text{ on } \partial\Omega, \int_{\partial D_i} \frac{\partial}{\partial \nu} w = 0,\; i = 1, \ldots, m \right\} . \quad (1.2.5)$$

It is well known that harmonic functions have infinite smoothness. Moreover, as the elements of $\mathcal{W}$ have a vanishing Neumann derivative on $\partial\Omega$, the "variation" of w on $\partial\Omega$ can only be caused by their behavior near the boundary of D – unless the boundary of Ω is non-smooth. In other words, the (local) variation of the trace of some function $w \in \mathcal{W}$ is an indicator for the (local) width of the domain $\Omega \setminus \overline{D}$. In fact, as we will show next, it is possible to characterize D completely, if the set of all traces of $\mathcal{W}$ on $\partial\Omega$ were known.[4]

To this end, we introduce the Neumann function $N(\,\cdot\,, z)$ associated with the Laplacian in the domain Ω, which is given as the (distributional) solution of the problem

$$\begin{gathered} -\Delta N(x,z) = \delta(x-z) \quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial \nu} N(x,z) = -\frac{1}{|\partial\Omega|} \quad \text{on } \partial\Omega\,, \\ \int_{\partial\Omega} N(x,z) \, \mathrm{d}s(x) = 0\,, \end{gathered} \quad (1.2.6)$$

where $z \in \Omega$ is kept fixed, and the differential operators act on the x-variable only. To achieve a unique solution we have normalized $N(\,\cdot\,, z)$ to have vanishing mean on $\partial\Omega$. The directional derivative

$$U_z(x) \;=\; p \cdot \operatorname{grad}_z N(x,z) \quad (1.2.7)$$

[3] We like to highlight one recent approach from [57], where different frequencies are used in the experimental setup to obtain relative data. This approach, however, leads to a different variant of the Factorization Method than the one that is described here.

[4] For one insulating inclusion it is even known that the trace of one single potential $w \in \mathcal{W}$ is enough to identify D, cf., e.g., [16]. For conducting obstacles (with known conductivity) the corresponding uniqueness problem is still open.

with respect to z of N in direction p (of unit length) yields the potential of a dipole source in z with moment p in the presence of an insulated boundary $\partial\Omega$: We refer to U_z as the dipole potential, tacitly assuming the dipole moment to be fixed. (All subsequent results hold true for an arbitrary choice of $p \in \mathbb{R}^n$ with $|p| = 1$, and it appears that there is still space to improve the numerical performance of the method, especially in three space dimensions, provided that this property is exploited in an optimal way.) We remark that U_z behaves like

$$U_z(x) \sim \begin{cases} \dfrac{1}{2\pi} \dfrac{(x-z)\cdot p}{|x-z|^2}, & n = 2, \\ \dfrac{1}{4\pi} \dfrac{(x-z)\cdot p}{|x-z|^3}, & n = 3, \end{cases} \qquad \text{as } x \to z, \tag{1.2.8}$$

and, in fact, U_z agrees with the right-hand side of (1.2.8) up to a harmonic function. This statement holds true for every fixed $z \in \Omega$.

Now we are ready to formulate the characterization of the inclusion D as it has been established by Brühl in his dissertation [21] (see also [22]), and which constitutes the basis for the Factorization Method.

Theorem 1.2.1 *A point $z \in \Omega$ belongs to D, if and only if the trace $\phi_z = U_z|_{\partial\Omega}$ coincides with the trace of some potential $w \in \mathcal{W}$.*

Proof: First, let $z \in D$. Then the dipole potential U_z is harmonic in $\Omega \setminus \{z\}$, i.e., in $\Omega \setminus \overline{D}$ and in a neighborhood of ∂D. Accordingly U_z belongs to $H^1_\diamond(\Omega \setminus \overline{D})$. As $N(x,z)$ has the same Neumann boundary data for any $z \in \mathbb{R}^n$, its directional derivative with respect to z has vanishing Neumann data on $\partial\Omega$. Moreover, according to Green's formula,

$$\int_{\partial D_i} \frac{\partial}{\partial \nu} U_z \, \mathrm{d}s \;=\; 0 \tag{1.2.9}$$

for every component D_i of D which does not contain z; however, as the total flux of U_z across $\partial(\Omega \setminus \overline{D})$ vanishes as well, (1.2.9) must also hold true for that component D_i of D which does contain z. Therefore, $U_z \in \mathcal{W}$, and its trace belongs to the corresponding trace space.

Now, let $z \notin \overline{D}$, and assume that the trace ϕ_z of the dipole potential U_z is the trace of a potential $w \in \mathcal{W}$. As we have seen in the first part of this proof, U_z and w thus have the same Cauchy data on $\partial\Omega$, and it follows from the uniqueness of solutions of the Cauchy problem for the Poisson equation that U_z and w coincide in $\Omega \setminus (\overline{D} \cup \{z\})$, where both are harmonic.[5] Now, w extends as a harmonic function into the point z, and hence, is bounded near z whereas U_z is not, cf. (1.2.8). This provides the desired contradiction.

In the last case, where z sits on the boundary of D we can use the same argument as before to show that w and U_z coincide in $\Omega \setminus \overline{D}$. According to (1.2.5), U_z must therefore have a finite H^1-norm on $\Omega \setminus \overline{D}$, which contradicts the asymptotic behaviour (1.2.8) near $z \in \partial D$.[6] □

It turns out that the potentials $w = u_0 - u_{\mathbb{1}}$ which provide the given relative data, have additional features that are not captured by the description of the set $\mathcal{W}$ of (1.2.5). For example, if the boundaries of the domains D_i are smooth, then the potential u_0 of (1.2.1) can be extended by reflection to a certain subset of D, showing that w has a harmonic extension to a larger

[5] It is here where the assumption on the connectedness of $\Omega \setminus \overline{D}$ is needed.

[6] This argument requires the Lipschitz continuity of ∂D, because this assumption makes sure that we can find an open cone $\mathcal{C} \subset \Omega \setminus \overline{D}$ with vertex in z, and hence, that the integral $\int_{\mathcal{C}} |\operatorname{grad} U_z|^2 \, \mathrm{d}x$ is unbounded.

domain than just $\Omega \setminus \overline{D}$, see [55, Appendix]. Therefore the space spanned by the relative data is *smaller* than the trace space of $\mathcal{W}$ in general. Still, there is a means to deduce this trace space from the given relative data – and the appropriate tool is the Factorization Method.

At this point we deviate from the usual presentation of the Factorization Method to opt for a more elementary derivation of the main results: Instead of the usual factorization of the data map in three factors as in (1.1.1) we follow the approach in [23], and factor the relative data in only two parts, namely

$$\Lambda_0 - \Lambda_{\mathbb{1}} \;=\; K^* K\,, \tag{1.2.10}$$

where K^* is an appropriate adjoint of the operator K given by

$$K \,:\, f \mapsto \begin{cases} u_0 - u_{\mathbb{1}} & \text{in } \Omega \setminus \overline{D}\,, \\ c_i - u_{\mathbb{1}} & \text{in } \overline{D}_i,\ i = 1, \ldots, m\,, \end{cases} \tag{1.2.11}$$

and the real numbers c_i in (1.2.11) are the means of the potential u_0 at the boundaries of the insulating inclusions, i.e.,

$$c_i \;=\; \frac{1}{|\partial D_i|} \int_{\partial D_i} u_0 \, \mathrm{d}s\,, \qquad i = 1, \ldots, m\,. \tag{1.2.12}$$

We claim (see Theorem 1.2.3 below for a proof) that K is a continuous operator from $L^2_\diamond(\partial\Omega)$ to $\mathcal{X}$, where

$$\mathcal{X} \;=\; \left\{ v : \Omega \to \mathbb{R} \,:\, v|_{\Omega\setminus\overline{D}} \in H^1_\diamond(\Omega \setminus \overline{D}),\, v|_D \in H^1(D),\, \int_{\partial D_i} [v] \, \mathrm{d}s = 0,\, i = 1, \ldots, m \right\}. \tag{1.2.13}$$

In this definition, again, the subscript $\diamond$ indicates that any $v \in \mathcal{X}$ is required to have vanishing mean on $\partial\Omega$, and

$$[v] \;=\; v^+|_{\partial D} \,-\, v^-|_{\partial D}$$

denotes the jump of v across the boundary of the inclusion(s), defined in the appropriate trace spaces. Here and below we denote by v^+ and v^- the restriction of a generic element $v \in \mathcal{X}$ to $\Omega \setminus \overline{D}$ and D, respectively. We equip $\mathcal{X}$ with the inner product

$$(\, v, w \,)_{\mathcal{X}} \;=\; \int_{\Omega\setminus\partial D} \operatorname{grad} v \cdot \operatorname{grad} w \, \mathrm{d}x \;=\; \int_D \operatorname{grad} v^- \cdot \operatorname{grad} w^- \, \mathrm{d}x \,+\, \int_{\Omega\setminus\overline{D}} \operatorname{grad} v^+ \cdot \operatorname{grad} w^+ \, \mathrm{d}x\,, \tag{1.2.14}$$

which turns $\mathcal{X}$ into a Hilbert space. Take note that $H^1_\diamond(\Omega)$, i.e., the set of all functions from $H^1(\Omega)$ with vanishing mean on $\partial\Omega$, is a subset of $\mathcal{X}$.

Lemma 1.2.2 *Let $\mathcal{K} \subset \mathcal{X}$ be the set of all elements $w \in \mathcal{X}$ that are harmonic in $\Omega \setminus \partial D$, and satisfy*

$$\frac{\partial}{\partial\nu} w = 0 \quad \text{on } \partial\Omega \qquad \text{and} \qquad \left[\frac{\partial}{\partial\nu} w\right] = 0 \quad \text{on } \partial D\,.$$

Then $\mathcal{K}$ is the orthogonal complement of $H^1_\diamond(\Omega)$ in $\mathcal{X}$.

Proof: Using Green's formula for any $v \in H^1_\diamond(\Omega)$ and any $w \in \mathcal{X}$ that is harmonic in $\Omega \setminus \partial D$ we obtain

$$\begin{aligned} \int_{\Omega\setminus\partial D} \operatorname{grad} v \cdot \operatorname{grad} w \, \mathrm{d}x \;&=\; \int_{\partial\Omega} v \, \frac{\partial w}{\partial\nu} \, \mathrm{d}s - \int_{\partial D} v \, \frac{\partial w^+}{\partial\nu} \, \mathrm{d}s + \int_{\partial D} v \, \frac{\partial w^-}{\partial\nu} \, \mathrm{d}s \\ &=\; \int_{\partial\Omega} v \, \frac{\partial w}{\partial\nu} \, \mathrm{d}s - \int_{\partial D} v \left[\frac{\partial w}{\partial\nu}\right] \mathrm{d}s\,, \end{aligned} \tag{1.2.15}$$

as v has a well-defined unique trace on ∂D. Now, if we choose $w \in \mathcal{K}$ then both integrals vanish, and hence $w \perp v$ with respect to the scalar product in $\mathcal{X}$.

Vice versa, pick $w \in \mathcal{X}$ from the orthogonal complement of $H^1_\diamond(\Omega)$, and let v be a C^∞ function with compact support in $\Omega \setminus \overline{D}$, then Green's formula yields

$$\begin{aligned}\int_{\Omega\setminus\overline{D}} w\Delta v \,\mathrm{d}x &= \int_{\partial\Omega} w\,\frac{\partial v}{\partial\nu}\,\mathrm{d}s - \int_{\partial D} w\,\frac{\partial v}{\partial\nu}\,\mathrm{d}s - \int_{\Omega\setminus\overline{D}} \operatorname{grad} w\cdot\operatorname{grad} v\,\mathrm{d}x \\ &= \int_{\partial\Omega} w\,\frac{\partial v}{\partial\nu}\,\mathrm{d}s - \int_{\partial D} w\,\frac{\partial v}{\partial\nu}\,\mathrm{d}s - \int_{\Omega\setminus\partial D} \operatorname{grad} w\cdot\operatorname{grad} v\,\mathrm{d}x\,,\end{aligned}$$

and all three integrals in the bottom line are zero by construction. Thus, w is harmonic in $\Omega\setminus\overline{D}$ according to Weyl's Lemma. The same kind of argument also shows that w is harmonic in D. Accordingly, as above, (1.2.15) holds true for any $v \in H^1_\diamond(\Omega)$, where now the left hand side of (1.2.15) is zero because of the orthogonality. A standard variational argument then shows that the normal derivative of w on $\partial\Omega$ and the flux of w across ∂D must vanish. □

We briefly mention that every potential w from $\mathcal{W}$ of (1.2.5) has a unique continuation to a potential $w \in \mathcal{K}$, and the restriction of a nontrivial element from $\mathcal{K}$ to $\Omega \setminus \overline{D}$ is a nonzero element from $\mathcal{W}$. Accordingly, the set of traces on $\partial\Omega$ of potentials from $\mathcal{W}$ and $\mathcal{K}$, respectively, are the same.

Theorem 1.2.3 *The operator $K : L^2_\diamond(\partial\Omega) \to \mathcal{X}$ defined in* (1.2.11) *is bounded, injective, and its range lies dense in the subset $\mathcal{K}$ introduced in Lemma 1.2.2. The adjoint operator $K^* : \mathcal{X} \to L^2_\diamond(\partial\Omega)$ satisfies*

$$K^*v = \begin{cases} v|_{\partial\Omega}\,, & v \in \mathcal{K}\,, \\ 0\,, & v \in H^1_\diamond(\Omega)\,. \end{cases}$$

In particular, there holds $K^*K = \Lambda_0 - \Lambda_{\mathbb{1}}$, i.e., (1.2.10).

Proof: We recall that the two Neumann problems (1.2.1) and (1.2.4) have well-defined unique solutions u_0 and $u_{\mathbb{1}}$ in the space $H^1_\diamond(\Omega\setminus\overline{D})$ and $H^1_\diamond(\Omega)$, respectively, that are given by the corresponding weak formulations

$$\begin{aligned}\int_{\Omega\setminus\overline{D}} \operatorname{grad} u_0\cdot\operatorname{grad} v_0\,\mathrm{d}x &= \int_{\partial\Omega} f v_0\,\mathrm{d}s \qquad \text{for every } v_0 \in H^1_\diamond(\Omega\setminus\overline{D})\,, \\ \int_{\Omega} \operatorname{grad} u_{\mathbb{1}}\cdot\operatorname{grad} v\,\mathrm{d}x &= \int_{\partial\Omega} f v\,\mathrm{d}s \qquad \text{for every } v \in H^1_\diamond(\Omega)\,.\end{aligned} \tag{1.2.16}$$

Moreover, the two solutions depend continuously (in H^1) on the given boundary data $f \in L^2_\diamond(\partial\Omega)$. Accordingly, $w = Kf$ is a well defined element of $\mathcal{X}$ and K a bounded linear operator from $L^2_\diamond(\partial\Omega)$ to $\mathcal{X}$: The jump condition $\int_{\partial D_i}[w]\,\mathrm{d}s = 0$ is a consequence of the definition (1.2.12) of c_i and the uniqueness of the trace of $u_{\mathbb{1}}$ on ∂D.

Now, choose any $f \in L^2_\diamond(\partial\Omega)$, and denote by u_0 and $u_{\mathbb{1}}$ the corresponding solutions of (1.2.1) and (1.2.4). As in the definition of Kf we can extend u_0 to a function

$$\hat{u}_0 = \begin{cases} u_0 & \text{in } \Omega\setminus\overline{D}\,, \\ c_i & \text{in } \overline{D}_i,\ i = 1,\ldots,m\,, \end{cases}$$

in $\mathcal{X}$, such that $Kf = \hat{u}_0 - u_{\mathbb{1}}$. First, for $v \in H^1_\diamond(\Omega)$ we have

$$(\,Kf, v\,)_{\mathcal{X}} \;=\; (\,\hat{u}_0, v\,)_{\mathcal{X}} - (\,u_{\mathbb{1}}, v\,)_{\mathcal{X}} \;=\; \int_{\Omega\setminus\overline{D}} \operatorname{grad} u_0 \cdot \operatorname{grad} v \,\mathrm{d}x \;-\; \int_{\Omega} \operatorname{grad} u_{\mathbb{1}} \cdot \operatorname{grad} v \,\mathrm{d}x \;=\; 0$$

by virtue of (1.2.16), and hence, $\mathcal{R}(K) \perp H^1_\diamond(\Omega)$. It thus follows from Lemma 1.2.2 that $\mathcal{R}(K) \subset \mathcal{K}$ and $\mathcal{N}(K^*) = \overline{\mathcal{R}(K)}^{\perp} \supset H^1_\diamond(\Omega)$ and, in particular, that $K^*v = 0$ for every $v \in H^1_\diamond(\Omega)$.
Second, for $v \in \mathcal{K}$ we compute

$$(\,Kf, v\,)_{\mathcal{X}} \;=\; (\,\hat{u}_0, v\,)_{\mathcal{X}} - (\,u_{\mathbb{1}}, v\,)_{\mathcal{X}} \;=\; (\,\hat{u}_0, v\,)_{\mathcal{X}}\,,$$

since $u_{\mathbb{1}}$ and v are orthogonal to each other according to Lemma 1.2.2. Together with (1.2.16) thus follows that

$$(\,Kf, v\,)_{\mathcal{X}} \;=\; \int_{\Omega\setminus\overline{D}} \operatorname{grad} u_0 \cdot \operatorname{grad} v \,\mathrm{d}x \;=\; \int_{\partial\Omega} f v \,\mathrm{d}s \;=\; (\,f, v\,)_{L^2(\partial\Omega)}\,,$$

i.e., that $K^*v = v|_{\partial\Omega}$. In particular, for $v = Kf = \hat{u}_0 - u_{\mathbb{1}} \in \mathcal{K}$ we obtain

$$K^*Kf \;=\; K^*(\hat{u}_0 - u_{\mathbb{1}}) \;=\; (u_0 - u_{\mathbb{1}})|_{\partial D}\,,$$

and hence, the assertion (1.2.10) follows, cf. (1.2.3),

Assume now, that $\mathcal{R}(K)$ were not dense in $\mathcal{K}$. Then there is some $0 \neq v \in \mathcal{K} \cap \overline{\mathcal{R}(K)}^{\perp} = \mathcal{K} \cap \mathcal{N}(K^*)$, and since $0 = K^*v = v|_{\partial\Omega}$ this function v has vanishing Dirichlet boundary values on $\partial\Omega$. Moreover, as v belongs to $\mathcal{K}$, it is harmonic in $\Omega \setminus \overline{D}$ with vanishing Neumann boundary values on $\partial\Omega$, see Lemma 1.2.2. Thus, $v^+ = v|_{\Omega\setminus\overline{D}} = 0$ because of the unique solvability of the Cauchy problem for harmonic functions. Using Lemma 1.2.2 once more, it follows that $v^- = v|_D$ is also harmonic with vanishing Neumann boundary values on ∂D, and hence, v^- is constant on each D_i, say $v^-|_{D_i} = v_i^-$, $i = 1, \ldots, m$. Since $\int_{\partial D_i} [v] \,\mathrm{d}s = -v_i^- |\partial D|$, and as v belongs to $\mathcal{X}$, these constants must all be zero. This is a contradiction to $v \neq 0$, and hence, $\mathcal{R}(K)$ is dense in $\mathcal{K}$.
Finally, to show injectivity of K we assume $Kf = 0$ for some $f \in L^2_\diamond(\partial\Omega)$. Then $u_0 = u_{\mathbb{1}}$ in $\Omega \setminus \overline{D}$ and $u_{\mathbb{1}} = c_i$ in D_i, $i = 1, \ldots, m$. Since $u_{\mathbb{1}}$ is harmonic in all of the domain Ω the field must be constant in Ω (principle of unique continuation) and the flux $f = \partial u_{\mathbb{1}} / \partial\nu$ vanishes on $\partial\Omega$. □

This theorem – together with Lemma 1.2.2 – reveals that the range of K^* consists of all traces of potentials $w \in \mathcal{K}$, whereas the range of $\Lambda_0 - \Lambda_{\mathbb{1}}$ only consists of a dense subset of this set. Accordingly, we need to find a way to deduce the range of K^* from the given data to decrypt the information hidden in these traces according to Theorem 1.2.1.
To this end we exploit the so-called *Picard criterion*, a formulation of which can be found in the appendix (Theorem 1.5.1) for the ease of completeness. The Picard criterion is based on the singular value decomposition of the operator K, which is largely equivalent to the spectral decomposition of the operator $K^*K = \Lambda_0 - \Lambda_{\mathbb{1}}$.

Corollary 1.2.4 *The operator $\Lambda_0 - \Lambda_{\mathbb{1}}$ is a compact and self adjoint operator from $L^2_\diamond(\partial\Omega)$ into itself. As such, $L^2_\diamond(\partial\Omega)$ has an orthonormal eigenbasis $\{f_j\}$ and associated eigenvalues λ_j, such that*

$$(\Lambda_0 - \Lambda_{\mathbb{1}}) f_j \;=\; \lambda_j f_j\,, \qquad n \in \mathbb{N}\,. \tag{1.2.17}$$

These eigenvalues are positive, and converge to zero as $n \to \infty$. Throughout we shall assume that they are sorted in non-increasing order.

Proof: That Λ_0 and $\Lambda_{\mathbb{1}}$ are compact operators can be seen from the fact that the trace space of $H^1(\Omega\setminus\overline{D})$ on $\partial\Omega$, i.e., $H^{1/2}(\partial\Omega)$, is compactly embedded in $L^2(\partial\Omega)$. Accordingly, the difference operator $\Lambda_0-\Lambda_{\mathbb{1}}$ is compact as well as self adjoint, as follows readily from (1.2.10). Accordingly, one can find an orthonormal eigenbasis of $\Lambda_0-\Lambda_{\mathbb{1}}$ and the associated eigenvalues converge to zero for $j\to\infty$. It remains to prove that they are all positive; this follows from (1.2.10) and the injectivity of K by Theorem 1.2.3. □

As we have mentioned before, a point $z\in\Omega$ belongs to D, if and only if the trace ϕ_z of U_z is the trace of a potential in $\mathcal{K}$, i.e., if it belongs to the range of K^*. As we show in the appendix, cf. Corollary 1.5.2, this can be tested in the following way.

Theorem 1.2.5 *Let $\{f_j\}$ and $\{\lambda_j\}$ be the eigenbasis and eigenvalues of $\Lambda_0-\Lambda_{\mathbb{1}}$. Then, for any point $z\in\Omega$,*

$$z\in D \iff \sum_{n=1}^{\infty}\frac{|(\,\phi_z,f_j\,)_{L^2(\partial\Omega)}|^2}{\lambda_j} \;<\; \infty \tag{1.2.18}$$

with $\phi_z=U_z|_{\partial\Omega}$ from (1.2.7).

Remark: With the notations $1/\infty=0$ and $\operatorname{sign}\alpha=\begin{cases}\alpha/|\alpha|, & \alpha\neq 0,\\ 0, & \alpha=0,\end{cases}$ for any $\alpha\in\mathbb{C}$ we note that

$$\chi_D(z) \;=\; \operatorname{sign}\left[\sum_j \frac{\left|(\phi_z,f_j)_{L^2(\partial\Omega)}\right|^2}{\lambda_j}\right]^{-1}, \qquad z\in\Omega\,,$$

is the characteristic function of D. In particular, this result provides a constructive proof of the uniqueness of the inverse problem.

1.2.2 Conducting Obstacles

Next, we turn to the case of anisotropic conducting obstacles. To this end we assume that for each $x\in\Omega$ the conductivity $\sigma(x)$ is a real, symmetric positive definite $n\times n$-matrix, measurable and essentially bounded as a function of x, and that the associated quadratic form is bounded from below by some positive constant $c>0$, i.e.

$$p\cdot\big(\sigma(x)p\big)\;\geq\; c \text{ for every } x\in\overline{D} \text{ and every } p\in\mathbb{R}^n \text{ with } |p|=1 \text{ and } \quad \sigma(x)\;=\;I \text{ on } \Omega\setminus\overline{D}\,, \tag{1.2.19}$$

where D denotes the obstacles, that are assumed to have the same topological properties as before. Another assumption that seems to be necessary for the validity of the Factorization Method[7] is that

$$p\cdot\big(\sigma(x)p\big)\;\leq\;\kappa<1 \qquad \text{for every } p\in\mathbb{R}^n \text{ with } |p|=1, \text{ and every } x\in D\,, \tag{1.2.20}$$

which states that the background conductivity of the object is strictly larger than within the inclusions. Instead of (1.2.20) one can alternatively require that the conductivity within the inclusions is strictly larger than in the background, with straightforward modifications of the analysis; however, we will stick to the above assumption for the ease of simplicity. We mention

[7]It is an open problem whether the Factorization Method is applicable, if inequality (1.2.20) holds in some obstacles, while $p\cdot\big(\sigma(x)p\big)\geq\gamma>1$ in other inlusions; numerically, the method does not seem to deterior in this "mixed case".

that the assumption that the background conductivity be *strictly* larger (or smaller) than within the object can be relaxed to just being larger (or smaller), for the prize that the outcome of the method is unspecified for sampling points right on the boundary of the inclusions, cf. [43].

With conducting obstacles the potential corresponding to a boundary current $f \in L^2_\diamond(\partial\Omega)$ is given as the (weak) solution $u \in H^1_\diamond(\Omega)$ of the boundary value problem

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0 \quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial\nu} u = f \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} u\,\mathrm{d}s = 0\,, \tag{1.2.21}$$

which replaces the model (1.2.1) from Section 1.2.1 above. Accordingly we denote by Λ the Neumann-Dirichlet map associated with (1.2.21), i.e., $\Lambda : f \mapsto g = u|_{\partial\Omega}$.

As before, the corresponding **inverse problem** is to determine the shape of the obstacles D from the relative data $\Lambda - \Lambda_{\mathbb{1}}$. Here, again, $\Lambda_{\mathbb{1}}$ corresponds to the "unperturbed" case $\sigma = \sigma_{\mathbb{1}} = 1$ everywhere in Ω.

We mention that the problem whether not only D but the conductivity σ itself is uniquely determined by these data is completely settled when $n = 2$ – as long as σ is isotropic, cf. [14]. For $n = 3$ this question is still open for general scalar $L^\infty-$conductivities. Partial answers are known, we refer to [Handbook:EIT]. However, the set D is uniquely determined as we will see below in Theorem 1.2.9.

Now we proceed to derive a factorization of $\Lambda - \Lambda_{\mathbb{1}}$ in *three* factors as in (1.1.1), i.e.,

$$\Lambda - \Lambda_{\mathbb{1}} \;=\; AGA^*\,. \tag{1.2.22}$$

To this end we imagine the effect of a *virtual source* φ on the boundary of the obstacle D, given that the boundary of the object Ω is insulated: The corresponding potential v is the solution of the boundary value problem

$$\begin{aligned} \Delta v = 0 \quad \text{in } \Omega \setminus \overline{D}\,, \qquad -\frac{\partial}{\partial\nu} v = \varphi \quad \text{on } \partial D\,, \\ \frac{\partial}{\partial\nu} v = 0 \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} v\,\mathrm{d}s = 0\,. \end{aligned} \tag{1.2.23}$$

Recall that the normal vector ν on ∂D has been fixed to point into the interior of $\Omega \setminus \overline{D}$, and therefore the minus sign in front of the normal derivative on ∂D reflects the fact that φ is considered to be a source, and not a sink. We will require that this source has vanishing mean on *each* connected component D_i of D, i.e.,

$$\varphi \;\in\; H_*^{-1/2}(\partial D) \;=\; \left\{ \varphi \in H^{-1/2}(\partial D) \;:\; \int_{\partial D_i} \varphi\,\mathrm{d}s = 0\,,\; i = 1, \ldots, m \right\}, \tag{1.2.24}$$

where the integrals have to be interpreted as dual pairings between $H^{-1/2}$ functions and the unit constant from $H^{1/2}$. For later use we remark that the dual space of $H_*^{-1/2}(\partial D)$ can be identified with the subspace

$$H_*^{1/2}(\partial D) \;=\; \left\{ \psi \in H^{1/2}(\partial D) \;:\; \int_{\partial D_i} \psi\,\mathrm{d}s = 0\,,\; i = 1, \ldots, m \right\} \tag{1.2.25}$$

of $H^{1/2}(\partial D)$.

Associated with (1.2.23) we define the operator

$$A \;:\; \left\{ \begin{array}{ccc} H_*^{-1/2}(\partial D) & \to & L^2_\diamond(\partial\Omega)\,, \\ \varphi & \mapsto & v|_{\partial\Omega}\,, \end{array} \right. \tag{1.2.26}$$

and remark that the adjoint operator $A^* : L^2_\diamond(\partial\Omega) \to H^{1/2}_*(\partial D)$ of A is easily seen to map $f \in L^2_\diamond(\partial\Omega)$ onto the trace of the solution u_0 of (1.2.1) on the boundary of the obstacle – after an appropriate renormalization of this trace on each component ∂D_i of ∂D. More precisely the following holds

$$(A^* f)(x) \;=\; u_0(x) - c_i \qquad \text{for } x \in \partial D_i\,,\;\; i = 1, \ldots, m\,, \tag{1.2.27}$$

with c_i as in (1.2.12).

In order to establish (1.2.22) it remains to determine the operator G in the middle. We define G via the weak solution w of the diffraction problem

$$\begin{gathered} \operatorname{div}(\sigma \operatorname{grad} w) = 0 \quad \text{in } \Omega \setminus \partial D\,, \qquad \frac{\partial}{\partial \nu} w = 0 \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} w \,\mathrm{d}s = 0\,, \\ \big[w\big]_{\partial D} = \psi\,, \qquad \big[\nu \cdot (\sigma \operatorname{grad} w)\big]_{\partial D} = 0\,, \end{gathered} \tag{1.2.28}$$

and the solution $w_{\mathbb{1}}$ of the corresponding problem with σ replaced by one everywhere. Again, the normal ν on ∂D is pointing into the exterior of D. Note that when $\sigma = 1$ throughout all of Ω, then the corresponding solution $w_{\mathbb{1}}$ of (1.2.28) can be represented as a modified double layer potential with density ψ and the Neumann function for the Laplacian as kernel, i.e.,

$$w_{\mathbb{1}}(x) \;=\; \int_{\partial D} \frac{\partial}{\partial_y \nu}\, N(x,y) \psi(y) \,\mathrm{d}s(y)\,, \qquad x \in \Omega \setminus \partial D\,.$$

For a general conductivity tensor the weak form of (1.2.28) is obtained by integrating the differential equation against any test function $v \in H^1(\Omega)$ and using partial integration, which yields

$$\int_{\Omega \setminus \partial D} \operatorname{grad} w \cdot \big(\sigma \operatorname{grad} v\big) \,\mathrm{d}x \;=\; 0 \qquad \text{for every } v \in H^1(\Omega)\,. \tag{1.2.29}$$

Now we can make the Ansatz $w = w_{\mathbb{1}} + \hat{w}$ with $\hat{w} \in H^1(\Omega)$ to rewrite this as a standard variational problem in $H^1(\Omega)$: Find $\hat{w} \in H^1(\Omega)$ such that

$$\int_\Omega \operatorname{grad} \hat{w} \cdot \big(\sigma \operatorname{grad} v\big) \,\mathrm{d}x \;=\; -\int_{\Omega \setminus \partial D} \operatorname{grad} w_{\mathbb{1}} \cdot \big(\sigma \operatorname{grad} v\big) \,\mathrm{d}x$$

for every $v \in H^1(\Omega)$. From this follows readily that problem (1.2.28) has a unique weak solution in $H^1(\Omega \setminus \partial D)$, provided that $\psi \in H^{1/2}(\partial D)$, i.e., that ψ belongs to the trace space of $H^1(D)$. In accordance with the definition of A^*, however, we will restrict ψ to $H^{1/2}_*(\partial D)$.

The flux of w and $w_{\mathbb{1}}$ across ∂D is well defined in $H^{-1/2}(\partial D)$, cf., e.g., [46, Thm. 2.5], and there holds

$$\begin{aligned} \int_{\partial D_i} \frac{\partial}{\partial \nu}(w^+ - w_{\mathbb{1}}^+) \,\mathrm{d}s \;&=\; \int_{\partial D_i} \nu \cdot (\sigma \operatorname{grad} w^-) \,\mathrm{d}s \;-\; \int_{\partial D_i} \frac{\partial}{\partial \nu} w_{\mathbb{1}}^- \,\mathrm{d}s \\ &=\; \int_{D_i} \operatorname{div}(\sigma \operatorname{grad} w) \,\mathrm{d}x \;-\; \int_{D_i} \Delta w_{\mathbb{1}} \,\mathrm{d}x \;=\; 0\,. \end{aligned}$$

We can therefore define the bounded operator G in the following way:

$$G \,:\, \left\{ \begin{array}{ccc} H^{1/2}_*(\partial D) & \to & H^{-1/2}_*(\partial D)\,, \\ \psi & \mapsto & \dfrac{\partial}{\partial \nu}(w^+ - w_{\mathbb{1}}^+)\Big|_{\partial D}\,. \end{array} \right. \tag{1.2.30}$$

Theorem 1.2.6 *With A and G defined as above, the difference $\Lambda - \Lambda_{\mathbb{1}}$ of the two Neumann-Dirichlet operators associated with* (1.2.21) *and* (1.2.4)*, respectively, satisfies*

$$\Lambda - \Lambda_{\mathbb{1}} \;=\; AGA^* \,.$$

Proof: Consider an arbitrary element $f \in L^2_\diamond(\partial\Omega)$ and the corresponding function $\psi = A^* f$, which satisfies

$$\psi|_{\partial D_i} \;=\; u_0|_{\partial D_i} - c_i \,,$$

where u_0 is given by (1.2.1), and c_i is as in (1.2.12). The function ψ belongs to $H^{1/2}_*(\partial D)$, and it is easy to verify that the associated solution $w_{\mathbb{1}}$ of (1.2.28) – where σ is replaced by one – is given by

$$w_{\mathbb{1}} \;=\; \begin{cases} u_0 - u_{\mathbb{1}} & \text{in } \Omega \setminus \overline{D} \,, \\ c_i - u_{\mathbb{1}} & \text{in } D_i, \;\; i = 1, \ldots, m \,, \end{cases}$$

where $u_{\mathbb{1}}$ is the solution of (1.2.4). Similarly, the solution w of (1.2.28) is given by

$$w \;=\; \begin{cases} u_0 - u & \text{in } \Omega \setminus \overline{D} \,, \\ c_i - u & \text{in } D_i, \;\; i = 1, \ldots, m \,, \end{cases}$$

with u from (1.2.21). Accordingly, $w^+ - w^+_{\mathbb{1}} = u^+_{\mathbb{1}} - u^+$, and hence,

$$\varphi \;=\; GA^* f \;=\; \frac{\partial}{\partial \nu}(u^+_{\mathbb{1}} - u^+)\Big|_{\partial D} \,.$$

If we insert this particular source term φ into (1.2.23) then we conclude readily that the associated solution v of (1.2.23) is given by $v = u^+ - u^+_{\mathbb{1}}$. It thus follows from (1.2.26) that

$$AGA^* f \;=\; A\varphi \;=\; g - g_{\mathbb{1}} \;=\; (\Lambda - \Lambda_{\mathbb{1}}) f$$

as required. □

At this occasion we recall that every function $w \in \mathcal{W}$ of (1.2.5) has a well-defined normal derivative $\varphi \in H^{-1/2}_*(\partial D)$ at the inner boundary ∂D, and hence, solves the corresponding boundary value problem (1.2.23). And vice versa, the solution of (1.2.23) for any $\varphi \in H^{-1/2}_*(\partial D)$ belongs to $\mathcal{W}$. Thus, we can reformulate Theorem 1.2.1 as follows.

Theorem 1.2.7 *A point $z \in \Omega$ belongs to D, if and only if the trace ϕ_z of the dipole potential U_z in z, defined by (1.2.7), belongs to $\mathcal{R}(A)$.*

As in the insulating case it remains to derive a constructive algorithm to test whether the trace of some dipole potential belongs to $\mathcal{R}(A)$, or not. The next step on our way towards this goal is an investigation of the functional analytic properties of the operator G. In the following we will often consider operators acting between a reflexive Banach space X and its dual space X^*. We will denote the action of an element $\ell \in X^*$ on an element $\psi \in X$ by $\langle \ell, \psi \rangle$ and the pair of spaces by $\langle X^*, X \rangle$ in order to indicate that the first argument belongs to X^* and the second to X. A particular example is the Sobolev space $H^{1/2}_*(\partial D)$ with dual space $H^{-1/2}_*(\partial D)$.

Theorem 1.2.8 *The operator $G : H_*^{1/2}(\partial D) \to H_*^{-1/2}(\partial D)$ is self adjoint*[8] *and coercive, i.e. there exists $\gamma > 0$ with*

$$\langle\, G\psi, \psi \,\rangle \;\geq\; \gamma \|\psi\|^2_{H^{1/2}(\partial D)} \quad \textit{for all } \psi \in H_*^{1/2}(\partial D)\,. \tag{1.2.31}$$

Here, $\langle\, \cdot\,, \cdot \,\rangle$ denotes the dual pairing in the dual system $\langle\, H_*^{-1/2}(\partial D), H_*^{1/2}(\partial D) \,\rangle$.

Proof: The proof proceeds in a couple of steps.

1. At first we establish the symmetry of G. Take any ψ and $\tilde\psi$ from $H_*^{1/2}(\partial D)$, define w and $w_{\mathbb{1}}$ as in the proof of Theorem 1.2.6, and – using $\tilde\psi$ instead of ψ in (1.2.28) – define $\tilde w$ and $\tilde w_{\mathbb{1}}$ accordingly. Then we conclude that

$$\begin{aligned}
\langle\, G\psi, \tilde\psi \,\rangle \;&=\; \int_{\partial D} \tilde\psi\, \frac{\partial}{\partial \nu} w^+ \,\mathrm{d}s \;-\; \int_{\partial D} \tilde\psi\, \frac{\partial}{\partial \nu} w_{\mathbb{1}}^+ \,\mathrm{d}s \\
&= \int_{\partial D} (\tilde w^+ - \tilde w^-)\, \frac{\partial}{\partial \nu} w^+ \,\mathrm{d}s \;-\; \int_{\partial D} (\tilde w_{\mathbb{1}}^+ - \tilde w_{\mathbb{1}}^-)\, \frac{\partial}{\partial \nu} w_{\mathbb{1}}^+ \,\mathrm{d}s \\
&= \int_{\partial D} \tilde w^+\, \frac{\partial}{\partial \nu} w^+ \,\mathrm{d}s \;-\; \int_{\partial D} \tilde w^- \big(\nu \cdot (\sigma \operatorname{grad} w^-)\big)\,\mathrm{d}s \\
&\qquad\qquad -\; \int_{\partial D} \tilde w_{\mathbb{1}}^+\, \frac{\partial}{\partial \nu} w_{\mathbb{1}}^+ \,\mathrm{d}s \;+\; \int_{\partial D} \tilde w_{\mathbb{1}}^-\, \frac{\partial}{\partial \nu} w_{\mathbb{1}}^- \,\mathrm{d}s\,.
\end{aligned}$$

Now we can use (1.2.28), and apply Green's formula in D or $\Omega \setminus \overline{D}$, respectively, in each of these integrals (care has to be taken concerning the orientation of the normal on ∂D), to obtain

$$\begin{aligned}
\langle\, G\psi, \tilde\psi \,\rangle \;&=\; -\int_{\Omega\setminus\overline{D}} \operatorname{grad} \tilde w \cdot \operatorname{grad} w \,\mathrm{d}x - \int_D \operatorname{grad} \tilde w \cdot \big(\sigma \operatorname{grad} w\big)\,\mathrm{d}x \\
&\qquad + \int_{\Omega\setminus\overline{D}} \operatorname{grad} \tilde w_{\mathbb{1}} \cdot \operatorname{grad} w_{\mathbb{1}} \,\mathrm{d}x + \int_D \operatorname{grad} \tilde w_{\mathbb{1}} \cdot \operatorname{grad} w_{\mathbb{1}} \,\mathrm{d}x \\
&=\; \int_{\Omega\setminus\partial D} \operatorname{grad} \tilde w_{\mathbb{1}} \cdot \operatorname{grad} w_{\mathbb{1}} \,\mathrm{d}x - \int_{\Omega\setminus\partial D} \operatorname{grad} \tilde w \cdot \big(\sigma \operatorname{grad} w\big)\,\mathrm{d}x\,,
\end{aligned} \tag{1.2.32}$$

from which the symmetry of G is obvious.

2. Turning to the coercivity assertion (1.2.31) we fix some $\psi \in H_*^{1/2}(\partial D)$ and employ the weak form (1.2.29) of (1.2.28) with $v = w - w_{\mathbb{1}} \in H^1(\Omega)$. Starting from (1.2.32) with $\psi = \tilde\psi$ we thus

[8] i.e. G coincides with $G^* : H_*^{1/2}(\partial D) \to H_*^{-1/2}(\partial D)$ if the bi-dual of $H_*^{1/2}(\partial D)$ is identified with itself.

obtain

$$\begin{aligned}
\langle\, G\psi, \psi\,\rangle \;&=\; \int_{\Omega\setminus\partial D} |\operatorname{grad} w_{\mathbb{1}}|^2 \,\mathrm{d}x \;-\; \int_{\Omega\setminus\partial D} \operatorname{grad} w \cdot \big(\sigma \operatorname{grad} w\big)\,\mathrm{d}x \\
&=\; \int_{\Omega\setminus\partial D} |\operatorname{grad} w_{\mathbb{1}}|^2 \,\mathrm{d}x \;-\; \int_{\Omega\setminus\partial D} \operatorname{grad} w \cdot \big(\sigma \operatorname{grad} w\big)\,\mathrm{d}x \\
&\qquad +\; 2\int_{\Omega\setminus\partial D} \operatorname{grad} w \cdot \big(\sigma \operatorname{grad}(w - w_{\mathbb{1}})\big)\,\mathrm{d}x \\
&=\; \int_{\Omega\setminus\partial D} |\operatorname{grad} w_{\mathbb{1}}|^2 \,\mathrm{d}x \;+\; \int_{\Omega\setminus\partial D} \operatorname{grad} w \cdot \big(\sigma \operatorname{grad} w\big)\,\mathrm{d}x \\
&\qquad -\; 2\int_{\Omega\setminus\partial D} \operatorname{grad} w \cdot \big(\sigma \operatorname{grad} w_{\mathbb{1}}\big)\,\mathrm{d}x \\
&=\; \int_{\Omega\setminus\partial D} \operatorname{grad} w_{\mathbb{1}} \cdot \big((1-\sigma)\operatorname{grad} w_{\mathbb{1}}\big)\,\mathrm{d}x \;+\; \int_{\Omega\setminus\partial D} \operatorname{grad}(w - w_{\mathbb{1}}) \cdot \big(\sigma \operatorname{grad}(w - w_{\mathbb{1}})\big)\,\mathrm{d}x \\
&\geq\; \int_{\Omega\setminus\partial D} \operatorname{grad} w_{\mathbb{1}} \cdot \big((1-\sigma)\operatorname{grad} w_{\mathbb{1}}\big)\,\mathrm{d}x\,.
\end{aligned}$$

The integrand of the last integral vanishes in $\Omega \setminus \overline{D}$, and can be bounded from below using the restriction (1.2.20) on the conductivity. Accordingly we have

$$\langle\, G\psi, \psi\,\rangle \;\geq\; (1-\kappa) \int_D |\operatorname{grad} w_{\mathbb{1}}|^2 \,\mathrm{d}x\,. \tag{1.2.33}$$

3. To accomplish the proof of (1.2.31) we need to show that

$$\|\operatorname{grad} w_{\mathbb{1}}\|_{L^2(D)} \;\geq\; c\|\psi\|_{H^{1/2}(\partial D)} \tag{1.2.34}$$

for some constant $c > 0$. Assume the contrary: Let $\psi^{(j)} \in H_*^{1/2}(\partial D)$ and the corresponding $w_{\mathbb{1}}^{(j)}$ be such that $\|\psi^{(j)}\|_{H^{1/2}(\partial D)} = 1$ for every j, and that $\|\operatorname{grad} w_{\mathbb{1}}^{(j)}\|_{L^2(D)}$ converges to zero as j tends to infinity. Define $\tilde{w}_{\mathbb{1}}^{(j)} \in H^1(\Omega \setminus \partial D)$ as

$$\tilde{w}_{\mathbb{1}}^{(j)} \;=\; \begin{cases} w_{\mathbb{1}}^{(j)} & \text{in } \Omega \setminus D\,, \\ w_{\mathbb{1}}^{(j)} - c_i^{(j)} & \text{in } D_i,\ i = 1, \dots, m\,, \end{cases}$$

with

$$c_i^{(j)} \;=\; \frac{1}{|\partial D_i|} \int_{\partial D_i} (w_{\mathbb{1}}^{(j)})^-\,\mathrm{d}s\,, \qquad i = 1, \dots, m\,.$$

Then $\tilde{w}_{\mathbb{1}}^{(j)}|_{D_i}$ has vanishing mean on ∂D_i, and $\|\operatorname{grad} \tilde{w}_{\mathbb{1}}^{(j)}\|_{L^2(D_i)} \to 0$ for every $i = 1, \dots, m$ as $j \to \infty$. By virtue of the Poincaré inquality this implies that $\tilde{w}_{\mathbb{1}}^{(j)}$ tends to zero in $H^1(D)$. From (1.2.28) thus follows that the normal derivative $\frac{\partial}{\partial \nu}\tilde{w}_{\mathbb{1}}^{(j)}$ at ∂D (from either side) tends to zero in $H^{-1/2}(\partial D)$, and hence, that $\tilde{w}_{\mathbb{1}}^{(j)}|_{\Omega\setminus\overline{D}}$ converges in $H^1(\Omega \setminus \overline{D})$ to the solution of the homogeneous Neumann problem, normalized at the outer boundary. In other words, $\tilde{w}_{\mathbb{1}}^{(j)}$ converges to zero in $H^1(D)$ and in $H^1(\Omega \setminus \overline{D})$ as $j \to \infty$. Recurring to (1.2.28) once again, we observe that

$$\psi^{(j)}|_{\partial D_i} + c_i^{(j)} \;=\; [w_{\mathbb{1}}^{(j)}]_{\partial D_i} + c_i^{(j)} \;=\; [\tilde{w}_{\mathbb{1}}^{(j)}]_{\partial D_i}\,, \tag{1.2.35}$$

and since $\psi^{(j)} \in H_*^{1/2}(\partial D)$ it follows by integration over ∂D_i that

$$c_i^{(j)} \;=\; \frac{1}{|\partial D_i|}\int_{\partial D_i}[\tilde{w}_{\mathbb{1}}^{(j)}]_{\partial D_i}\,\mathrm{d}s - \frac{1}{|\partial D_i|}\int_{\partial D_i}\psi^{(j)}\,\mathrm{d}s \;=\; \frac{1}{|\partial D_i|}\int_{\partial D_i}[\tilde{w}_{\mathbb{1}}^{(j)}]_{\partial D_i}\,\mathrm{d}s \;\to\; 0$$

as j runs to infinity. Inserting this into (1.2.35) we conclude that

$$\psi^{(j)}|_{\partial D_i} \;=\; [\tilde{w}_{\mathbb{1}}^{(j)}]_{\partial D_i} \;-\; c_i^{(j)} \to 0\,, \qquad j\to\infty\,,$$

in $H^{1/2}(\partial D_i)$, $i=1,\ldots,m$, but this contradicts $\|\psi^{(j)}\|_{H^{1/2}(\partial D)}=1$. Therefore (1.2.34) is true for some $c>0$ and every $\psi\in H_*^{1/2}(\partial D)$, and hence, (1.2.31) follows from (1.2.33) and (1.2.34). □

By virtue of Theorem 1.2.8 all assumptions of Corollary 1.5.6 are satisfied for the factorization of the relative data $\Lambda-\Lambda_{\mathbb{1}}$ established in Theorem 1.2.7. Therefore we can now conclude the main result of this section.

Theorem 1.2.9 *Let $z\in\Omega$ and ϕ_z be defined as before.*

Then:
$$z\in D \iff \sum_{n=1}^{\infty}\frac{|(\,\phi_z, f_j\,)_{L^2(\partial\Omega)}|^2}{\lambda_j} \;<\; \infty\,,$$

where f_j and λ_j are the orthonormal eigenfunctions and eigenvalues of $\Lambda-\Lambda_{\mathbb{1}}$.

1.2.3 Local Data

It is an important feature of the Factorization Method that it can be easily adapted to applications where the given data correspond to what is called the local Neumann-Dirichlet map Λ^ℓ. This is the map that takes Neumann boundary values supported on some relatively open subset $\Gamma\subset\partial\Omega$ only, and returns the corresponding boundary potentials on the very same subset (normalized to have vanishing mean, say). The local Neumann-Dirichlet map occurs whenever part of the boundary is inaccessible to measurements, in which case Γ corresponds to that part of the boundary of Ω where electrodes can be attached. Mathematically, the local Neumann-Dirichlet map can thus be interpreted as a Galerkin projection

$$\Lambda^\ell \;=\; P\Lambda P^* \tag{1.2.36}$$

of the full Neumann-Dirichlet map, where

$$P:\left\{\begin{array}{ccc} L^2_\diamond(\partial\Omega) & \to & L^2_\diamond(\Gamma)\,,\\ g & \mapsto & g|_\Gamma - \frac{1}{|\Gamma|}\int_\Gamma g\,\mathrm{d}s\,,\end{array}\right. \tag{1.2.37}$$

and P^* is its L^2 adjoint, i.e.,

$$P^*f \;=\; \begin{cases} f & \text{on } \Gamma\,,\\ 0 & \text{on } \partial D\setminus\Gamma\,.\end{cases}$$

From Theorem 1.2.6 we immediately conclude that if the conductivity distribution satisfies (1.2.19) and (1.2.20), then the difference of the two local Neumann-Dirichlet maps Λ^ℓ and $\Lambda^\ell_{\mathbb{1}}$ can be factorized in the form

$$\Lambda^\ell-\Lambda^\ell_{\mathbb{1}} \;=\; (PA)G(PA)^*$$

with A and G as before. Moreover, the coercivity of G allows a constructive way to check whether a given function belongs to $\mathcal{R}(PA)$, considered as an operator from $H_*^{-1/2}(\partial D)$ to $L^2_\diamond(\Gamma)$. Note that it is obvious from Theorem 1.2.7, that the function $P\phi_z$ belongs to $\mathcal{R}(PA)$ when $z\in D$; the converse statement requires a little more efforts.

Theorem 1.2.10 *Let Γ be a relatively open subset of $\partial\Omega$, and let P be the projector defined in* (1.2.37)*. Then $z \in D$, if and only if $P\phi_z \in \mathcal{R}(PA)$.*

Proof: According to the definition (1.2.26) of A the test function $P\phi_z$ belongs to $\mathcal{R}(PA)$, if and only if ϕ_z coincides on Γ (up to a constant) with the trace of a solution v of (1.2.23). In this case, however, the dipole potential U_z and the function v are both harmonic functions in $\Omega \setminus (\overline{D} \cup \{z\})$, and have the same Cauchy data on Γ (again, up to a constant). Now we choose a connected subset Ω' of $\Omega \setminus (\overline{D} \cup \{z\})$, whose boundary contains a portion of Γ that is also a relatively open subset of $\partial\Omega$. Then U_z and v coincide up to a constant in Ω' according to Holmgren's theorem, and hence, near all of $\partial\Omega$. This shows that $\phi_z \in \mathcal{R}(A)$, and hence, the assertion follows from Theorem 1.2.7. □

Accordingly, if Γ is a relatively open subset of $\partial\Omega$ then Theorem 1.2.9 also extends readily to the local situation, if the eigenfunctions and eigenvalues of $\Lambda - \Lambda_{\mathbb{1}}$ are replaced by those of $\Lambda^\ell - \Lambda^\ell_{\mathbb{1}}$.

Note that Theorem 1.2.10 requires that Γ is a relatively open subset of $\partial\Omega$, and in fact, the Factorization Method no longer applies for discrete measurements or finitely many boundary currents. Still, this is precisely the situation that is encountered in practice, as data are always finite dimensional. Due to the rapid decay of the eigenvalues of $\Lambda - \Lambda_{\mathbb{1}}$, however, the full relative data can be very well approximated by operators of finite rank, such as those corresponding to real data; see [55] for detailed numerical examples.

1.2.4 Other Generalizations

The half space problem

The Factorization Method can also be applied to a related inverse electrostatic problem in full space with near field data, if the same manifold of codimension one is used to generate a source *and* to measure the resulting change of the potential. In fact, this problem which has been studied in [53] and [76], is very similar to the setting for the Helmholtz equation that we will consider in the following section. We also like to refer to [15] where this approach has been applied to some real two dimensional data.

For quite a few applications, however, the impedance tomography problem is more appropriately modelled in a half space, rather than in the full space or within a bounded domain. For this setting new difficulties arise, as the data (may) live on the entire, unbounded boundary of the surface, which calls for weighted Sobolev spaces for an appropriate theoretical analysis. In the sequel we restrict our attention to three space dimensions ($n = 3$), as the two dimensional case needs some additional attention, cf. [56], and at the same time appears to be less interesting from a practical point of view.

We consider the half space $\Omega = \{x \in \mathbb{R}^3 : \nu \cdot x < 0\}$, where $\nu \in \mathbb{R}^3$ is a fixed unit vector, which coincides with the outer normal on the hyperplane $\{x : \nu \cdot x = 0\}$, which is the boundary of Ω. The main difficulty in the analysis of this problem is that solutions of the corresponding conductivity problem

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0 \quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial \nu} u = f \quad \text{on } \partial\Omega\,, \tag{1.2.38}$$

need no longer belong to $L^2(\Omega)$; instead one has to resort to weighted Sobolev spaces, such as

$$\mathcal{U} \;=\; \{\, u \in \mathcal{D}'(\Omega) \,:\, (1 + |\cdot|^2)^{-1/2} u \in L^2(\Omega),\ |\operatorname{grad} u| \in L^2(\Omega) \,\}\,,$$

to search for a unique solution of (1.2.38). If σ is given by (1.2.19), then a weak solution $u \in \mathcal{U}$ can be shown to exist provided that f belongs to

$$L^{2,-1}(\partial\Omega) \,=\, \{\, f \,:\, (1+|\cdot|^2)^{1/2} f \in L^2(\partial\Omega)\,\}\,,$$

in which case the trace of u belongs to the dual space $L^{2,1}(\partial\Omega)$ of $L^{2,-1}(\partial\Omega)$. Note that no normalization of u is required in (1.2.38) because solutions in $\mathcal{U}$ are implicitly normalized to vanish at infinity. We refer to [56] for further details about the forward problem.

Within this function space setting the Neumann-Dirichlet operator is defined in a natural way as an operator $\Lambda : L^{2,-1}(\partial\Omega) \to L^{2,1}(\partial\Omega)$, and the difference between Λ and $\Lambda_{\mathbb{1}}$ (the latter corresponding to the homogeneous half space) admits a factorization (1.2.22) as before, where now

$$A \,:\, \left\{ \begin{array}{ccc} H_*^{-1/2}(\partial D) & \to & L^{2,1}(\partial\Omega)\,, \\ \varphi & \mapsto & v|_{\partial\Omega}\,, \end{array} \right.$$

and v solves the same boundary value problem as in (1.2.23), except for the missing normalization over the boundary $\partial\Omega$. Furthermore, the self adjoint operator G is defined as before (with the appropriate definition of a weak solution of (1.2.28)), and is coercive again.

We emphasize that the dipole potential (1.2.7) for the half space is explicitly known, i.e., we have (up to a negligible multiplicative constant)

$$\phi_z(x) \,=\, \frac{(x-z)\cdot p}{|x-z|^3}\,, \qquad x \in \partial\Omega\,. \tag{1.2.39}$$

With these notations the characterization of the inclusions can be established in much the same way as before, see [56]:

Theorem 1.2.11 *A point $z \in \Omega$ belongs to D, if and only if ϕ_z of* (1.2.39) *belongs to $\mathcal{R}(A)$.*

For real applications the measuring device will only cover a bounded region $\Gamma \subset \partial\Omega$. The corresponding local Neumann-Dirichlet operator Λ^ℓ can then be embedded in the standard L^2 framework from the previous section, and the usual Picard series can be used to implement the range test. For the ease of completeness we briefly mention that for such local data the test dipole ϕ_z can be replaced by the function

$$\tilde{\phi}_z(x) \,=\, \frac{1}{|x-z|}\,, \qquad x \in \Gamma\,,$$

which is the trace of the corresponding Neumann function (again, up to a multiplicative constant), as the latter has a vanishing normal derivative on the boundary of the half space. We hasten to add, though, that $\tilde{\phi}_z$ must not be used for full data, as it does not belong to $L^{2,1}(\partial\Omega)$. Numerically, however, this modification of the method has no significant benefit.

The crack problem

Another case of interest are cracks, i.e., lower dimensional manifolds of codimension one, that are insulating, say. This setting has important applications in nondestructive testing of materials. Consider a domain $\Omega \subset \mathbb{R}^n$, with $n = 2$ or $n = 3$ again, and the union $\Sigma = \bigcup_{i=1}^m \Sigma_i \subset \Omega$ of m smooth, bounded manifolds (the insulating cracks), such that $\Sigma_i \cap \Sigma_j = \emptyset$ and $\Omega \setminus \Sigma$ is

connected. Given a boundary current $f \in L^2_\diamond(\partial\Omega)$ the induced potential satisfies the model equations

$$\Delta u_0 \,=\, 0 \quad \text{in } \Omega \setminus \Sigma\,, \qquad \frac{\partial}{\partial\nu} u_0 \,=\, 0 \quad \text{on } \Sigma\,, \qquad \frac{\partial}{\partial\nu} u_0 \,=\, f \quad \text{on } \partial\Omega\,, \tag{1.2.40}$$

and the corresponding Neumann-Dirichlet operator is the map that takes f onto the trace of u on $\partial\Omega$:

$$\Lambda \,:\, \left\{ \begin{array}{ccc} L^2_\diamond(\partial\Omega) & \to & L^2_\diamond(\partial\Omega)\,, \\ f & \mapsto & u|_{\partial\Omega}\,. \end{array} \right.$$

The crack case can be analyzed in a similar way as in Section 1.2.1, cf. [23], using a factorization $\Lambda - \Lambda_{\mathbb{I}} = K^* K$, where K is almost identical to the operator in (1.2.11), except that it maps into $H^1(\Omega \setminus \Sigma)$. There is a more important difference, though. As the crack has no interior points, the range test will always fail with the hitherto used test function ϕ_z, as the dipole singularity of U_z is too strong to belong to $H^1(\Omega \setminus \Sigma)$, even when $z \in \Sigma$. To detect a crack we therefore need to construct a new test function by integrating the function ϕ_z over z along some "test arc" (in $\mathbb{R}^2$) or some "test surface" (in $\mathbb{R}^3$).

The range test can then be implemented by placing linear (planar) test cracks in different sampling points with various orientations, see [23] for numerical reconstructions in two space dimensions. The amount of work thus grows substantially, as we now have 2 degrees of freedom to sample (a test point and a normal direction) instead of only one in the previous cases. Also, in a numerical realization, test cracks will – at best – only touch the crack tangentially, but in theory this already suffices to ruin the range test. It turns out that in practice the usual implementation with the test function ϕ_z performs as good as the more elaborate but expensive variant described above. As said before, in theory ϕ_z will never belong to the range of K; in practice, however, it will "almost" do so, i.e., the Picard series (1.2.18) will grow much more slowly in the close neighborhood of the crack.

One-dimensional cracks in three-dimensional objects cannot be reconstructed in this way, because the potential does not "see" inhomogeneities of this size. However, one can use an asymptotic analysis similar to the derivation of MUSIC type algorithms that are discussed in Section 1.4.2 below. Here we give a brief sketch of an argument provided in [48], and refer to this paper for further details. The basic idea is that realistic "one-dimensional" cracks in a 3D world are not exactly one-dimensional, but better modelled as extremely thin tubular inclusions of small diameter $\delta > 0$. The corresponding relative data $\Lambda_\delta - \Lambda_{\mathbb{I}}$, where Λ_δ is the Neumann-Dirichlet operator associated with the tubular inclusion and $\Lambda_{\mathbb{I}}$ is as usual, turn out to satisfy an asymptotic expansion in δ,

$$\Lambda_\delta - \Lambda_{\mathbb{I}} \,=\, \delta^2 \hat{M} \,+\, o(\delta^2)\,,$$

possibly after selecting an appropriate (sub)sequence $\delta_k \to 0$. The operator $\hat{M}$ that constitutes the dominating term of this expansion admits a factorization similar to (1.2.22). In contrast to the MUSIC framework below, this operator has infinite dimensional range. Although the operators of the corresponding factorization are somewhat different from the ones that we have encountered above, the bottom line is the same as for one-dimensional cracks in two space dimensions: The same integrated test function belongs to the range of the operator A of this factorization, if and only if the corresponding test arc is part of the crack. The singular value decomposition of $\hat{M}$ can be used to evaluate this test, and in practice this singular value decomposition can be approximated by the one of $\Lambda_\delta - \Lambda_{\mathbb{I}}$, i.e., by the given data.

1.3 The Factorization Method in Inverse Scattering Theory

The second part of this chapter is devoted to the Factorization Method for problems in inverse scattering theory for time-harmonic waves. The scattering of an incident plane wave by a medium gives rise to a scattered field which is measured "far away" from the medium. The Factorization Method characterizes the shape of the scattering medium from this far field information. The measurement operator will be the far field operator F which maps the density of the incident Herglotz-field to the corresponding far field pattern of the scattered field.

The far field operator F allows a factorization of the form (1.1.1) where the operators A and G depend on the specific situation. We will discuss two typical cases and start with the scattering by a sound-soft obstacle D in Section 1.3.1. This is an example of a non-absorbing medium which is mathematically reflected by the fact that the far field operator is normal - though not self adjoint as for the corresponding problem in impedance tomography. It was this example for which the Factorization Method was developed for the first time in [67]. In Section 1.3.2 we will study the scattering of time-harmonic electromagnetic plane waves by an absorbing medium. In this case the corresponding far field operator fails to be normal.

Each case study will start with a short repetition of the corresponding direct problem. Then the inverse problem will be stated and a factorization of the form (1.1.1) will be derived. As in impedance tomography a crucial point is to establish in each case a certain coercivity condition for G. In addition, one needs to prove a range identity which describes the range of A via the known – possibly non-normal – data operator F.

Here and throughout the following sections, $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^3$.

1.3.1 Inverse Acoustic Scattering by a Sound-Soft Obstacle

This section is devoted to the analysis of the Factorization Method for the most simplest case in scattering theory. We consider the scattering of time-harmonic plane waves by an impenetrable obstacle $D \subset \mathbb{R}^3$ which we model by assuming Dirichlet boundary conditions on the boundary ∂D of D. As before we assume that D is a finite union $D = \bigcup_{i=1}^m D_i$ of bounded domains D_i such that $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$. Furthermore, we assume that the boundaries ∂D_i are Lipschitz continuous, and that the exterior $\mathbb{R}^3 \setminus \overline{D}$ of $\overline{D}$ is connected. Finally, let $k > 0$ be the wave number and

$$u^i(x) \;=\; \exp(\mathrm{i}k\, x \cdot \hat{\theta})\,, \quad x \in \mathbb{R}^3\,, \tag{1.3.1}$$

be the incident plane wave of direction $\hat{\theta} \in S^2$. The obstacle D gives rise to a scattered field $u^s \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ which superposes u^i and results in the total field $u = u^i + u^s$ which satisfies the *Helmholtz equation*

$$\Delta u + k^2 u \;=\; 0 \quad \text{outside } \overline{D}\,, \tag{1.3.2}$$

and the Dirichlet boundary condition

$$u \;=\; 0 \quad \text{on } \partial D\,. \tag{1.3.3}$$

The scattered field u^s satisfies the *Sommerfeld radiation condition*

$$\frac{\partial u^s}{\partial r} - \mathrm{i}k\, u^s \;=\; \mathcal{O}\big(r^{-2}\big) \quad \text{for } r = |x| \to \infty \tag{1.3.4}$$

uniformly with respect to $\hat{x} = x/|x| \in S^2$.

The **direct scattering problem** is to determine the scattered field u^s for a given obstacle $D \subset \mathbb{R}^3$, some $\hat{\theta} \in S^2$ and $k > 0$.

For the treatment of this direct problem we refer to [36] (see also [Handbook:Inverse Scattering, Section 2.2]). There it is also shown that the scattered field u^s has the asymptotic behavior

$$u^s(x) \;=\; \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}\, u^\infty(\hat{x}) \;+\; \mathcal{O}(|x|^{-2})\,, \quad |x| \to \infty\,, \tag{1.3.5}$$

uniformly with respect to $\hat{x} = x/|x| \in S^2$. The function $u^\infty : S^2 \to \mathbb{C}$ is analytic and is called the *far field pattern* of u^s. It depends on the wave number k, the direction $\hat{\theta} \in S^2$, and on the domain D. Since we will keep $k > 0$ fixed, only the dependence on $\hat{\theta}$ is indicated: $u^\infty = u^\infty(\hat{x}; \hat{\theta})$ for $\hat{x}, \hat{\theta} \in S^2$.

In the **inverse scattering problem** the far field pattern $u^\infty(\hat{x}; \hat{\theta})$ is known for all $\hat{x}, \hat{\theta} \in S^2$ and some fixed $k > 0$ and the domain D has to be determined. We refer again to [36] or [Handbook:Inverse Scattering] for the presentation of the most important properties of this inverse scattering problem. The knowledge of $u^\infty(\hat{x}; \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^2$ determines the integral kernel of the *far field operator* F from $L^2(S^2)$ into itself, which is defined by

$$\big(Fg\big)\big(\hat{x}\big) \;=\; \int_{S^2} u^\infty(\hat{x}; \hat{\theta})\, g(\hat{\theta})\, \mathrm{d}s(\hat{\theta}) \quad \text{for } \hat{x} \in S^2\,. \tag{1.3.6}$$

The far field operator F is compact, normal[9], and the so-called *scattering operator* $I + \frac{\mathrm{i}k}{8\pi^2} F$ is unitary in $L^2(S^2)$.

As in Section 1.2.2 the first step is to derive a factorization of F in the form (1.1.1).

The operator A is the *data to pattern operator* which maps $f \in H^{1/2}(\partial D)$ to the far field pattern v^∞ of the radiating[10] solution $v \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$ of

$$\Delta v + k^2 v = 0 \text{ in the exterior of } \overline{D}\,, \quad v = f \text{ on } \partial D\,. \tag{1.3.7}$$

Here, $H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$ is the space of functions v with $v|_{B \setminus \overline{D}} \in H^1(B \setminus \overline{D})$ for all balls $B \subset \mathbb{R}^3$. Existence and uniqueness is assured (see, e.g., Chapter 9 of [81]).

Theorem 1.3.1 *Define the operator $A : H^{1/2}(\partial D) \to L^2(S^2)$ by $Af = v^\infty$ where v^∞ is the far field pattern of the unique radiating solution $v \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$ of (1.3.7). Then A is one-to-one with dense range, and the following factorization holds.*

$$F \;=\; -A\, S^* A^* \tag{1.3.8}$$

where $A^* : L^2(S^2) \to H^{-1/2}(\partial D)$ is the dual of A and $S^* : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is the dual of the single layer boundary operator $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ defined by[11]

$$(S\varphi)(x) \;=\; \int_{\partial D} \varphi(y)\, \Phi(x, y)\, \mathrm{d}s(y)\,, \quad x \in \partial D\,. \tag{1.3.9}$$

Here, Φ denotes the fundamental solution *of the Helmholtz equation, i.e.*

$$\Phi(x, y) \;=\; \frac{\exp(\mathrm{i}k|x - y|)}{4\pi|x - y|}\,, \quad x, y \in \mathbb{R}^3\,,\; x \neq y\,. \tag{1.3.10}$$

[9] i.e. F commutes with its adjoint F^*

[10] i.e. v satisfies the Sommerfeld radiation condition (1.3.4)

[11] Actually, the explicit definition (1.3.9) of this operator makes only sense for smooth functions φ. It has to be extended to functionals $\varphi \in H^{-1/2}(\partial D)$ by a density or duality argument.

Proof: The injectivity of A follows immediately from Rellich's Lemma (see [36] or [Handbook:Inverse Scattering, Lemma 2.2]). The denseness of the range of A can be shown by approximating any $g \in L^2(S^2)$ by a finite sum of spherical harmonics to which the corresponding field can be written down explicitely.
To derive the factorization define the auxiliary operator $\mathcal{H} : L^2(S^2) \to H^{1/2}(\partial D)$ by

$$(\mathcal{H}g)(x) \;=\; \int_{S^2} g(\hat{\theta})\, \exp(\mathrm{i}k\, x \cdot \hat{\theta})\, \mathrm{d}s(\hat{\theta}) \;=\; \int_{S^2} g(\hat{\theta})\, u^i(x;\hat{\theta})\, \mathrm{d}s(\hat{\theta})\,, \quad x \in \partial D\,.$$

First we note that $u^\infty(\cdot;\hat{\theta}) = -Au^i(\cdot;\hat{\theta})$ by the definition of A and thus, by the superposition principle, $Fg = -A\mathcal{H}g$ for all $g \in L^2(S^2)$, i.e. $F = -A\mathcal{H}$. We compute the dual $\mathcal{H}^* : H^{-1/2}(\partial D) \to L^2(S^2)$ as

$$(\mathcal{H}^*\varphi)(\hat{x}) \;=\; \int_{\partial D} \varphi(y)\, \exp(-\mathrm{i}k\, \hat{x} \cdot y)\, \mathrm{d}s(y)\,, \quad \hat{x} \in S^2\,.$$

The fundamental solution Φ has the asymptotic behaviour

$$\Phi(x,y) \;=\; \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}\, \exp(-\mathrm{i}k\, \hat{x} \cdot y) \;+\; \mathcal{O}(|x|^{-2})\,, \quad |x| \to \infty\,, \tag{1.3.11}$$

uniformly w.r.t. $\hat{x} \in S^2$ and $y \in \partial D$, and thus has the far field pattern $\Phi^\infty(\hat{x}, y) = \exp(-\mathrm{i}k\, \hat{x} \cdot y)$. Therefore, again by superposition, $\mathcal{H}^*\varphi = AS\varphi$, i.e. $\mathcal{H} = S^*A^*$. Substituting this into $F = -A\mathcal{H}$ yields (1.3.8). □

Therefore, F allows a factorization in the form (1.1.1) with $G = -S^*$. The most important properties of this operator are collected in the following theorem. (For a proof see, e.g., [81, 74].)

Theorem 1.3.2 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the following holds.*

(a) S is an isomorphism from the Sobolev space $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$.

(b) $\mathrm{Im}\langle \varphi, S\varphi\rangle < 0$ for all $\varphi \in H^{-1/2}(\partial D)$ with $\varphi \neq 0$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing in $\langle H^{-1/2}(\partial D), H^{1/2}(\partial D)\rangle$.

(c) Let S_i be the single layer boundary operator (1.3.9) corresponding to the wave number $k = \mathrm{i}$. The operator S_i is self adjoint and coercive as an operator from $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$, i.e. there exists $c_0 > 0$ with

$$\langle \varphi, S_\mathrm{i}\varphi\rangle \;\geq\; c_0 \|\varphi\|^2_{H^{-1/2}(\partial D)} \quad \text{for all } \varphi \in H^{-1/2}(\partial D)\,. \tag{1.3.12}$$

(d) The difference $S - S_\mathrm{i}$ is compact from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$.

From this theorem the following coercivity result can be derived.
Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then there exists $c_1 > 0$ with

$$\big|\langle \varphi, S\varphi\rangle\big| \;\geq\; c_1 \|\varphi\|^2_{H^{-1/2}(\partial D)} \quad \text{for all } \varphi \in H^{-1/2}(\partial D)\,. \tag{1.3.13}$$

This establishes the first step of the Factorization Method. In the second step the domain D is characterized by the range of the operator A.

Theorem 1.3.3 *For any $z \in \mathbb{R}^3$ define the function $\phi_z \in L^2(S^2)$ by*

$$\phi_z(\hat{x}) \;=\; \exp(-\mathrm{i}k\,\hat{x}\cdot z)\,, \quad \hat{x} \in S^2\,. \tag{1.3.14}$$

Then z belongs to D, if and only if $\phi_z \in \mathcal{R}(A)$

Proof: Let first $z \in D$. From (1.3.11) we conclude that ϕ_z is the far field pattern of $\Phi(\cdot, z)$, thus $\phi_z = Af$ where $f = \Phi(\cdot, z)|_{\partial D} \in H^{1/2}(\partial D)$.
Let now $z \notin D$ and assume, on the contrary, that $\phi_z = Af$ for some $f \in H^{1/2}(\partial D)$. Let v be as in the definition of Af. Then $\phi_z = v^\infty$. From Rellich's Lemma and unique continuation we conclude that $\Phi(\cdot, z)$ and v coincide in $\mathbb{R}^3 \setminus \big(\overline{D} \cup \{z\}\big)$. By the same arguments as in the proof of Theorem 1.2.1 this is a contradiction since v is regular and $\Phi(\cdot, z)$ is singular at z. □

From the factorization (1.3.8) we conclude that $\mathcal{R}(F) \subset \mathcal{R}(A)$ and thus

$$\phi_z \in \mathcal{R}(F) \quad \Longrightarrow \quad z \in D\,.$$

Therefore, the condition on the left hand side determines only a subset of D. One can show, cf. [35], that for the case of D being a ball the left hand side is only satisfied for the center of this ball. Nevertheless, the (regularized version) of the test $\phi_z \in \mathcal{R}(F)$ leads to the *Linear Sampling Method*, cf. Section 1.4.1.

In the third step of the Factorization Method the range $\mathcal{R}(A)$ of A has to be expressed by the known data operator F. This is achieved by a second factorization of F based on the spectral decomposition of the normal operator F. From now on we make the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the far field operator is one-to-one as it follows directly from the factorization (1.3.8) and part (a) of Theorem 1.3.2.

Since F is compact, normal and one-to-one there exists a complete set of orthonormal eigenfunctions $\psi_j \in L^2(S^2)$ with corresponding eigenvalues $\lambda_j \in \mathbb{C}$, $j = 1, 2, 3, \ldots$ (see, e.g., [89]). Furthermore, since the operator $I + \mathrm{i}k/(8\pi^2)\,F$ is unitary the eigenvalues λ_j of F lie on the circle of radius $1/r$ and center i/r where $r = k/(8\pi^2)$. The spectral theorem for normal operators yields that F has the form

$$F\psi \;=\; \sum_{j=1}^{\infty} \lambda_j (\psi, \psi_j)_{L^2(S^2)}\, \psi_j\,, \quad \psi \in L^2(S^2)\,. \tag{1.3.15}$$

Therefore, F has a second factorization in the form

$$F \;=\; (F^*F)^{1/4}\, G_2\, (F^*F)^{1/4}\,, \tag{1.3.16}$$

where the self adjoint operator $(F^*F)^{1/4} : L^2(S^2) \to L^2(S^2)$ and the signum $G_2 : L^2(S^2) \to L^2(S^2)$ of F are given by

$$(F^*F)^{1/4}\psi \;=\; \sum_{j=1}^{\infty} \sqrt{|\lambda_j|}\, (\psi, \psi_j)_{L^2(S^2)}\, \psi_j\,, \quad \psi \in L^2(S^2)\,, \tag{1.3.17}$$

$$G_2\psi \;=\; \sum_{j=1}^{\infty} \frac{\lambda_j}{|\lambda_j|}\, (\psi, \psi_j)_{L^2(S^2)}\, \psi_j\,, \quad \psi \in L^2(S^2)\,. \tag{1.3.18}$$

Also this operator G_2 satisfies a coercivity condition of the form (1.3.13).

Theorem 1.3.4 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then there exists $c_2 > 0$ with*

$$\left|(\psi, G_2\psi)_{L^2(S^2)}\right| \;\geq\; c_2\|\psi\|^2_{L^2(S^2)} \quad \textit{for all } \psi \in L^2(S^2)\,. \tag{1.3.19}$$

Proof: It is sufficient to prove (1.3.19) for $\psi \in L^2(S^2)$ of the form $\psi = \sum_j c_j\psi_j$ with $\|\psi\|^2_{L^2(S^2)} = \sum_j |c_j|^2 = 1$. With the abbreviation $s_j = \lambda_j/|\lambda_j|$ it is

$$\left|(G_2\psi, \psi)_{L^2(S^2)}\right| \;=\; \left|\left(\sum_{j=1}^{\infty} s_j\, c_j\, \psi_j\,,\, \sum_{j=1}^{\infty} c_j\, \psi_j\right)_{L^2(S^2)}\right| \;=\; \left|\sum_{j=1}^{\infty} s_j|c_j|^2\right|\,.$$

The complex number $\sum_{j=1}^{\infty} s_j|c_j|^2$ belongs to the closure of the convex hull $\mathcal{C} = \mathrm{conv}\{s_j : j \in \mathbb{N}\} \subset \mathbb{C}$ of the complex numbers s_j. We conclude that

$$\left|(G_2\psi, \psi)_{L^2(S^2)}\right| \;\geq\; \inf\{|z| : z \in \mathcal{C}\}$$

for all $\psi \in L^2(S^2)$ with $\|\psi\|_{L^2(S^2)} = 1$. From the facts that λ_j lie on the circle with center i/r passing through the origin and that λ_j tends to zero as j tends to infinity we conclude that the only accumulation points of the sequence $\{s_j\}$ can be $+1$ or -1. From the factorization (1.3.8) and Theorem 1.3.2 it can be shown (see the proof of Theorem 1.23 of [74]) that indeed 1 is the only accumulation point, i.e. $s_j \to 1$ as j tends to infty. Therefore, the set $\mathcal{C}$ is contained in the part of the upper half-disk which is above the line $\ell = \left\{t\hat{s} + (1-t)1 : t \in \mathbb{R}\right\}$ passing through $\hat{s}$ and 1. Here, $\hat{s}$ is the point in $\{s_j : j \in \mathbb{N}\}$ with the smallest real part. Therefore, the distance of the origin to this convex hull $\mathcal{C}$ is positive, i.e., there exists c_2 with (1.3.19). □

From Theorem 1.3.1 and equation (1.3.16) the scattering operator F can be written as

$$F \;=\; A\, G_1\, A^* \;=\; (F^*F)^{1/4}\, G_2\, (F^*F)^{1/4} \tag{1.3.20}$$

where we have set $G_1 = -S^*$. Both of the operators G_j, $j = 1, 2$, are coercive in the sense of (1.3.13) and (1.3.19), respectively. By the range identity of Corollary 1.5.4 the ranges of A and $(F^*F)^{1/4}$ coincide. The combination of this result and Theorem 1.3.3 yields the main result of this section.[12]

Theorem 1.3.5 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. For any $z \in \mathbb{R}^3$ define again $\phi_z \in L^2(S^2)$ by (1.3.14), i.e.*

$$\phi_z(\hat{x}) \;:=\; \exp(-\mathrm{i}k\,\hat{x}\cdot z)\,, \quad \hat{x} \in S^2\,.$$

Then

$$z \in D \iff \phi_z \in \mathcal{R}((F^*F)^{1/4}) \iff \sum_j \frac{\left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2}{|\lambda_j|} \;<\; \infty\,. \tag{1.3.21}$$

Here, $\lambda_j \in \mathbb{C}$ are the eigenvalues of the normal operator F with corresponding normalized eigenfunctions $\psi_j \in L^2(S^2)$.

Formula (1.3.21) provides a simple and fast technique to visualize the object D by plotting the inverse of the series on the right hand side. In practise, this will be a finite sum instead of a series, but the value of the finite sum is much larger for points z outside than for points inside of D. We refer to the original paper [67] for some typical plots.

[12] To derive the second equivalence of (1.3.21) Theorem 1.5.1 of Picard has been applied.

Remark 1.3.6 It is illuminating to compare the presentation in this section with the one for impedance tomography from Section 1.2.2. The relative potential $u - u_{\mathbb{1}}$ considered there corresponds to the scattered wave $u^s = u - u^i$, i.e., the total field minus the incoming field; the incoming field is the potential that is induced by the excitation if the background is homogeneous, whereas the total field is the corresponding solution in the presence of the scatterer.

In both cases, the operator that maps the excitation onto the associated "relative data" can be factorized in three operators: the one that is applied first, i.e. A^*, maps the excitation/the incoming field onto the boundary of the obstacle(s), the operator A that is applied last, maps appropriate boundary data on the obstacle onto the "outgoing field" and its measured data. Accordingly, the operator in the middle encodes the "refraction" at the obstacle(s).

As such, we can view the factorization from impedance tomography as a generalization of Huygen's principle to the diffusion problem (1.2.21), although the time causality from scattering theory has no apparent physical analog in stationary diffusion processes.

1.3.2 Inverse Electromagnetic Scattering by an Inhomogeneous Medium

This section is devoted to the analysis of the Factorization Method for the inverse scattering of electromagnetic time-harmonic plane waves by an inhomogeneous non-magnetic and conducting medium. Let $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$ be the *wave number* with angular frequency ω, electric permittivity ε_0, and magnetic permeability μ_0 in vacuum. The incident plane wave has the form

$$H^i(x) \;=\; p\,\exp(\mathrm{i}k\,\hat{\theta}\cdot x)\,, \quad E^i(x) \;=\; -\frac{1}{\mathrm{i}\omega\varepsilon_0}\,\operatorname{curl} H^i(x)\,, \tag{1.3.22}$$

for some polarization vector $p \in \mathbb{C}^3$ and some direction $\hat{\theta} \in S^2$ such that $p \cdot \hat{\theta} = 0$. This pair satisfies the time harmonic Maxwell system in vacuum, i.e.

$$\operatorname{curl} E^i - \mathrm{i}\omega\mu_0 H^i \;=\; 0 \quad \text{in } \mathbb{R}^3, \tag{1.3.23}$$

$$\operatorname{curl} H^i + \mathrm{i}\omega\varepsilon_0 E^i \;=\; 0 \quad \text{in } \mathbb{R}^3\,. \tag{1.3.24}$$

This incident wave is scattered by a medium with space dependent electric permittivity $\varepsilon = \varepsilon(x)$ and conductivity $\sigma = \sigma(x)$. We assume that the magnetic permeability μ is constant and equal to the permeability μ_0 of vacuum. Furthermore, we assume that $\varepsilon \equiv \varepsilon_0$ and $\sigma \equiv 0$ outside of some bounded domain. The total fields are superpositions of the incident and scattered fields, i.e. $E = E^i + E^s$ and $H = H^i + H^s$ and satisfy the Maxwell system

$$\operatorname{curl} E - \mathrm{i}\omega\mu_0 H \;=\; 0 \quad \text{in } \mathbb{R}^3, \tag{1.3.25}$$

$$\operatorname{curl} H + \mathrm{i}\omega\varepsilon E \;=\; \sigma E \quad \text{in } \mathbb{R}^3\,. \tag{1.3.26}$$

Also, the tangential components of E and H are continuous on interfaces where σ or ε are discontinuous. Finally, the scattered fields have to satisfy the *Silver-Müller radiation condition*

$$\sqrt{\mu_0}\,H^s(x)\times\hat{x} \;-\; \sqrt{\varepsilon_0}\,E^s(x) \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x|\to\infty \tag{1.3.27}$$

uniformly w.r.t. $\hat{x} = x/|x| \in S^2$. The complex-valued *relative electric permittivity* ε_r is defined by

$$\varepsilon_r(x) \;=\; \frac{\varepsilon(x)}{\varepsilon_0} \;+\; \mathrm{i}\,\frac{\sigma(x)}{\omega\,\varepsilon_0}\,. \tag{1.3.28}$$

Note that $\varepsilon_r \equiv 1$ outside of some bounded domain. The equation (1.3.26) can then be written in the form

$$\operatorname{curl} H + \mathrm{i}\omega\varepsilon_0\varepsilon_r E \;=\; 0 \quad \text{in } \mathbb{R}^3\,. \tag{1.3.29}$$

It is preferable to work with the magnetic field H only. This is motivated by the fact that the magnetic field is divergence free as seen from (1.3.25) and the fact that $\operatorname{div}\operatorname{curl} = 0$. In general, this is not the case for the electric field E. Eliminating the electric field E from the system (1.3.25), (1.3.29) leads to

$$\operatorname{curl}\left[\frac{1}{\varepsilon_r}\operatorname{curl} H\right] - k^2 H \;=\; 0 \quad \text{in } \mathbb{R}^3\,. \tag{1.3.30}$$

The incident field H^i satisfies

$$\operatorname{curl}^2 H^i - k^2 H^i \;=\; 0 \quad \text{in } \mathbb{R}^3. \tag{1.3.31}$$

Subtracting both equations yields

$$\operatorname{curl}\left[\frac{1}{\varepsilon_r}\operatorname{curl} H^s\right] - k^2 H^s \;=\; \operatorname{curl}\big[q\operatorname{curl} H^i\big] \quad \text{in } \mathbb{R}^3\,, \tag{1.3.32}$$

where the contrast q is defined by $q = 1 - 1/\varepsilon_r$. The Silver-Müller radiation condition turns into

$$\operatorname{curl} H^s(x) \times \hat{x} - \mathrm{i}k\, H^s(x) \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \tag{1.3.33}$$

The continuity of the tangential components of E and H translates into analogous requirements for H^s and $\operatorname{curl} H^s$.

It will be necessary to allow more general source terms on the right-hand side of (1.3.32). In particular, we will consider the problem to determine a radiating[13] solution $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ of

$$\operatorname{curl}\left[\frac{1}{\varepsilon_r}\operatorname{curl} v\right] - k^2 v \;=\; \operatorname{curl} f \quad \text{in } \mathbb{R}^3 \tag{1.3.34}$$

for given $f \in L^2(\mathbb{R}^3)^3$ with compact support[14]. The solutions v of (1.3.34) as well as of (1.3.30) and (1.3.32) have to be understood in the variational sense, i.e. $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ satisfies

$$\int_{\mathbb{R}^3}\left[\frac{1}{\varepsilon_r}\operatorname{curl} v \cdot \operatorname{curl}\psi - k^2\, v\cdot\psi\right]\mathrm{d}x \;=\; \int_{\mathbb{R}^3} f \cdot \operatorname{curl}\psi\;\mathrm{d}x \tag{1.3.35}$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support. For any domain Ω the Sobolev space $H(\operatorname{curl}, \Omega)$ is the space of all vector fields $v \in L^2(\Omega)^3$ such that also $\operatorname{curl} v \in L^2(\Omega)^3$. Furthermore, $H_{loc}(\operatorname{curl}, \mathbb{R}^3) = \{v : v|_B \in H(\operatorname{curl}, B) \text{ for all balls } B \subset \mathbb{R}^3\}$.

Outside of the supports of $\varepsilon_r - 1$ and f the solution satisfies $\operatorname{curl}^2 v - k^2 v = 0$. Taking the divergence of this equation and using the identities $\operatorname{div}\operatorname{curl} = 0$ and $\operatorname{curl}^2 = -\Delta + \operatorname{grad}\operatorname{div}$ this system is equivalent to the pair of equations

$$\Delta v + k^2 v \;=\; 0 \quad \text{and} \quad \operatorname{div} v \;=\; 0\,.$$

Classical interior regularity results (cf. [81] combined with [36]) yield that v is analytic outside of the supports of $\varepsilon_r - 1$ and f. In particular, the radiation condition (1.3.33) is well defined.

[13] i.e. v satisfies the Silver-Müller radiation condition (1.3.33)

[14] For any open set D the space $L^2(D)^3$ denotes the space of vector functions $v : D \to \mathbb{C}^3$ such that all components are in $L^2(D)$

There are several ways to show the Fredholm property of equation (1.3.34). We refer to [82] for the treatment by a variational equation with non-local boundary conditions or to [74] for a treatment by an integro-differential equation of Lippmann-Schwinger type.

The question of uniqueness of radiating solutions to (1.3.34) is closely related to the validity of the unique continuation principle. It is known to hold for piecewise Hölder-continuously differentiable functions ε_r (see [82]).

As in the case of the Helmholtz equation every radiating vector field v satisfying $\operatorname{curl}^2 v - k^2 v = 0$ outside of some ball has the asymptotic behavior

$$v(x) \;=\; \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|}\, v^\infty(\hat{x}) \;+\; \mathcal{O}(|x|^{-2}), \quad |x| \to \infty,$$

uniformly with respect to $\hat{x} = x/|x| \in S^2$ (see again [36]). The vector field v^∞ is uniquely determined and again called the *far field pattern* of v. It is a tangential vector field, i.e. $v^\infty \in L_t^2(S^2)$ where

$$L_t^2(S^2) \;=\; \left\{ w \in L^2(S^2)^3 : w(\hat{x}) \cdot \hat{x} = 0,\ \hat{x} \in S^2 \right\}.$$

The **inverse problem** is to determine the shape D of the contrast q from the far field pattern $H^\infty(\hat{x}; \hat{\theta}, p)$ for all $\hat{x}, \hat{\theta} \in S^2$ and $p \in \mathbb{C}^3$ with $p \cdot \hat{\theta} = 0$. Because of the linear dependence of H^∞ on p it is sufficient to know H^∞ only for a basis of two vectors for p. As in impedance tomography the task of determining only D is rather modest since it is well known that one can even reconstruct q uniquely from this set of data, see [38]. However, the proof of uniqueness is non-constructive while the Factorization Method will provide an explicit characterization of the characteristic function of D which can, e.g., be used for numerical purposes. Also, the Factorization Method can – with only minor modifications – be carried over for anisotropic media (as in Section 1.2.2) where it is well known that ε_r can only be determined up to smooth change of coordinates.

For the remaining part of this section we make the following assumption.

Assumption 1.3.7 *Let $D \subset \mathbb{R}^3$ be a finite union $D = \bigcup_{i=1}^m D_i$ of bounded domains D_i such that $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$. Furthermore, we assume that the boundaries ∂D_i are Lipschitz continuous and the exterior $\mathbb{R}^3 \setminus \overline{D}$ of $\overline{D}$ is connected. Let $\varepsilon_r \in L^\infty(D)$ satisfy*

(1) $\operatorname{Im} \varepsilon_r \geq 0$ *in* D.

(2) There exists $c_2 > 0$ *with* $\operatorname{Re} \varepsilon_r \geq 1 + c_2$ *on* D.

(3) For every $f \in L^2(\mathbb{R}^3)^3$ *with compact support there exists a unique radiating solution of (1.3.34).*

We extend ε_r *by one outside of* D *and define the contrast by* $q = 1 - 1/\varepsilon_r$, *thus* $\operatorname{Im} q \geq 0$ *and* $\operatorname{Re} q \geq (1 + c_2) c_2 / \|\varepsilon_r\|_\infty^2 > 0$ *on* D.

Condition (3) is, e.g., satisfied for Hölder-continuously differentiable parameters ε and σ (see [82]).

The *far field operator* $F : L_t^2(S^2) \to L_t^2(S^2)$ is defined as

$$(Fp)(\hat{x}) \;:=\; \int_{S^2} H^\infty\big(\hat{x}; \theta, p(\theta)\big)\, ds(\theta), \quad \hat{x} \in S^2. \tag{1.3.36}$$

F is a linear operator since H^∞ depends linearly on the polarization p.

The first step in the Factorization Method is to derive a factorization of F in the form $F = A\,T^*\,A^*$ where the operators $A : L^2(D)^3 \to L^2_t(S^2)$ and $T : L^2(D)^3 \to L^2(D)^3$ are defined as follows.

The *data-to-pattern operator* $A : L^2(D)^3 \to L^2_t(S^2)$ is defined by $Af := v^\infty$ where v^∞ denotes the far field pattern corresponding to the radiating (variational) solution $v \in H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ of

$$\mathrm{curl}\left[\frac{1}{\varepsilon_r}\,\mathrm{curl}\,v\right] - k^2 v \;=\; \mathrm{curl}\left[\frac{q}{\sqrt{|q|}}\,f\right] \quad \text{in } \mathbb{R}^3\,. \tag{1.3.37}$$

Again, the contrast is given by $q = 1 - 1/\varepsilon_r$. We note that the solution exists by part (3) of Assumption 1.3.7.

The operator $T : L^2(D)^3 \to L^2(D)^3$ is defined by $Tf = (\mathrm{sign}\,\overline{q})\,f - \sqrt{|q|}\,\mathrm{curl}\,w\big|_D$, where $w \in H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ is the radiating solution of

$$\mathrm{curl}^2\,w \;-\; k^2 w \;=\; \mathrm{curl}\big[\sqrt{|q|}\,f\big] \quad \text{in } \mathbb{R}^3\,. \tag{1.3.38}$$

The solution exists and is unique (see, e.g. [74]).

Theorem 1.3.8 *Let Assumption 1.3.7 hold. Then F from (1.3.36) can be factorized as*

$$F \;=\; A\,T^*\,A^* \tag{1.3.39}$$

where $A^* : L^2_t(S^2) \to L^2(D)^3$ and $T^* : L^2(D)^3 \to L^2(D)^3$ denote the adjoints of A and T, respectively. Furthermore, A^* is injective.

For a proof of this and the following result we refer to [74].

Remark: The solution w of (1.3.38) can be expressed in the form (see [74])

$$w(x) \;=\; \mathrm{curl}\int_D \sqrt{|q(y)|}\,f(y)\,\Phi(x,y)\;\mathrm{d}y\,, \quad x \in \mathbb{R}^3\,,$$

which yields an explicit expression of T.

The following theorem corresponds to Theorem 1.3.2 and collects properties of the operator T needed for the analysis of the Factorization Method.

Theorem 1.3.9 *Let the conditions of Assumption 1.3.7 hold and let $T : L^2(D)^3 \to L^2(D)^3$ be defined above. Then the following holds:*

(a) The imaginary part $\mathrm{Im}\,T = \frac{1}{2\mathrm{i}}(T - T^*)$ is non-positive, i.e.

$$\mathrm{Im}(Tf, f)_{L^2(D)^3} \;\le\; 0 \quad \textit{for all } f \in L^2(D)^3\,.$$

(b) Define the operator T_0 from $L^2(D)^3$ into itself by $T_0 f = (\mathrm{sign}\,\overline{q})\,f$ for $f \in L^2(D)^3$. Then $T - T_0$ is compact in $L^2(D)^3$.

(c) T is an isomorphism from $L^2(D)^3$ onto itself.

As in Section 1.3.1 we first characterize the domain D by the range $\mathcal{R}(A)$ of A. The proof of the following result can again be found in [74].

Theorem 1.3.10 *Let the conditions of Assumption 1.3.7 hold. For any $z \in \mathbb{R}^3$ and fixed $p \in \mathbb{C}^3$ we define $\phi_z \in L_t^2(S^2)$ as the far field pattern of the electric dipole at z with moment p, i.e.*

$$\phi_z(\hat{x}) \;=\; -\mathrm{i}k\,(\hat{x}\times p)\,\exp(-\mathrm{i}k\,\hat{x}\cdot z)\,, \quad \hat{x}\in S^2\,. \tag{1.3.40}$$

Then z belongs to D, if and only if, $\phi_z \in \mathcal{R}(A)$.

In contrast to the data operators $\Lambda_0 - \Lambda_{\mathbb{I}}$ or $\Lambda - \Lambda_{\mathbb{I}}$ of Section 1.2, or the far field operator F of Section 1.3.1, the far field operator for absorbing media – as in the present case – fails to be normal or even self adjoint. Therefore, the approaches of the previous sections – i.e. the application of the range identities of Corollaries 1.5.6 and 1.5.4 – are not applicable. However, application of Theorem 1.5.5 to the far field operator F from $L_t^2(S^2)$ into itself and the operator $G = T^* : L^2(D)^3 \to L^2(D)^3$ yields the characterization of D via an auxiliary operator

$$F_\# \;=\; |\operatorname{Re} F| + \operatorname{Im} F\,, \tag{1.3.41}$$

cf. (1.5.5), which is easily obtained from the given far field data.

Theorem 1.3.11 *Let the conditions of Assumption 1.3.7 hold. For any $z \in \mathbb{R}^3$ define again $\phi_z \in L_t^2(S^2)$ by* (1.3.40)*. Then, with $F_\#$ of* (1.3.41) *there holds*

$$z\in D \iff \phi_z \in \mathcal{R}(F_\#^{1/2}) \iff \sum_j \frac{\left|(\phi_z,\psi_j)_{L^2(S^2)}\right|^2}{|\lambda_j|} \;<\; \infty\,. \tag{1.3.42}$$

Here, $\lambda_j \in \mathbb{C}$ are the eigenvalues of the self adjoint and positive compact operator $F_\#$ with corresponding normalized eigenfunctions $\psi_j \in L_t^2(S^2)$.

1.3.3 Historical Remarks and Open Questions

Historically, the Factorization Method originated from the Linear Sampling Method which will be explained in Section 1.4.1 below (see also [Handbook:Inverse Scattering, Section 5.2]). The Linear Sampling Method studies the *far field equation* $Fg = \phi_z$ in contrast to the Factorization Method which characterizes the domain D by *exactly* those points z for which the modified far field equation $F_\#^{1/2} g = \phi_z$ is solvable where $F_\# = (F^*F)^{1/2}$ in the case of Section 1.3.1 and $F_\# = |\operatorname{Re} F| + \operatorname{Im} F$ in the case of Section 1.3.2. It is easily seen that the points for which the far field equation $Fg = \phi_z$ is solvable determines only a subset of D – which can consist of a single point only, as the example of a ball shows.

The implementation of the Factorization Method is as simple and universal as of the Linear Sampling Method. Only the far field operator F – i.e., in practice a finite dimensional approximation – has to be known. No other a priori information on the unknown domain D such as the number of components or the kind of boundary condition has to be known in advance. The mathematical justification, however, has to be proven for every single situation. Since their first presentations, the Factorization Method has been justified for several problems in inverse acoustic and electromagnetic scattering theory such as the scattering by inhomogeneous media ([68, 70, 73, 74]), scattering by periodic structures ([11, 12]), and scattering by obstacles under different kinds of boundary conditions ([50, 74]). The Factorization Method can also be adapted for scattering problems for a crack ([75]) with certain modifications; we refer to the remarks concerning the crack problem in Section 1.2.4. The Factorization Method for elastic scattering problems and wave guides is studied in [9] and [30], respectively.

In many situations near field measurements on some surface Γ for point sources on the same surface Γ as incident fields rather than far field measurements for plane waves as incident fields are available. The corresponding "near field operator" $M : L^2(\Gamma) \to L^2(\Gamma)$ allows a factorization in the form $M = B\,G\,B'$ where B' is the adjoint with respect to the bilinear form $\int_\Gamma u\,v\,\mathrm{d}s$ rather than the (sesquilinear) inner product $\int_\Gamma u\,\overline{v}\,\mathrm{d}s$. The validity of the range identity for these kind of factorizations is not known so far and is one of the open problems in this field. For certain situations (see [74]) the corresponding far field operator F can be computed from M and the Factorization Method can then be applied to F.

Also the cases where the background medium is more complicated than the free space can be treated, see [49, 74] for scattering problems in a half space and [72] for scattering problems in layered media.

The justification of the Factorization Method for arbitrary elliptic boundary value problems or even more general problems is treated in [45, 71, 83].

1.4 Related Sampling Methods

This section is devoted to some alternate examples of sampling methods which were developed during the last decade: the *Linear Sampling Method*, first introduced by Colton and Kirsch in [35], the closely related *MUSIC*, the *Singular Sources Method* by Potthast (see [86]), and Ikehata's *Probe Method* (see [63]). However, it is not the aim of this section to report on all sampling methods. In particular, we do not discuss the *enclosure method* or the *no-response test* but refer to the monograph [87] and the survey article [88].

1.4.1 The Linear Sampling Method

Here we reconsider the inverse scattering problem for time-harmonic plane acoustic waves of Section 1.3.1, i.e. the problem to determine the shape of an acoustically soft obstacle D from the knowledge of the far field pattern $u^\infty(\hat{x};\hat{\theta})$ for all $\hat{x},\hat{\theta} \in S^2$. We refer to (1.3.1)–(1.3.6) for the mathematical model and the definition of the far field operator F from $L^2(S^2)$ into itself.

The Factorization Method for inverse scattering problems studies solvability of the equation $F_\#^{1/2} g = \phi_z$ in $L^2(S^2)$ where $F_\# = (F^*F)^{1/2}$ in the case where F is normal (as, e.g., in Section 1.3.1) and $F_\# = |\operatorname{Re} F| + \operatorname{Im} F$ in the general case with absorbtion, see Theorems 1.3.5 and 1.3.11, respectively. In contrast to this equation, the *Linear Sampling Method* considers the *far field equation*

$$F g \;=\; \phi_z \quad \text{in } L^2(S^2)\,. \tag{1.4.1}$$

We mention again that in general no solution of this equation exists. However, one can compute "approximate solutions" $g = g_{z,\varepsilon}$ of (1.4.1) such that $\|g\|_{L^2(S^2)}$ behaves differently for z being inside or outside of D. We refer to [Handbook:Inverse Scattering, Theorem 5.3] for a more precise formulation of this behaviour.

The drawback of this result – and all the other attempts to justify the Linear Sampling Method rigorously – is that there is no guarantee that the solution of a regularized version of (1.4.1), e.g. by Tikhonov regularization, will actually pick the density $g = g_{z,\varepsilon}$ with the properties of the aforementioned "approximate solution". We refer to [54] for a discussion of this fact. However, numerically the method has proven to be very effective for a large class of inverse scattering problems, see e.g. [26] for the scattering by cracks, [27] for inverse scattering problems for anisotropic media, [19] for wave guide scattering problems, [33, 34, 52] for electromagnetic

scattering problems, and [29, 31, 41] for elastic scattering problems. Modifications of the Linear Sampling Method and combinations with other methods can be found in [8, 20, 80].

For the cases in which the Factorization Method in the form $(F^*F)^{1/4}g = \phi_z$ is applicable a complete characterization of the unknown obstacle D by a modification of the Linear Sampling Method can be derived by replacing the indicator value $\|g\|_{L^2(S^2)}$ by $(g, \phi_z)_{L^2(S^2)}$. This is summarized in the following theorem (see [10, 13] and, for the following presentation, [74]).

Theorem 1.4.1 *Let $u^\infty = u^\infty(\hat{x}; \hat{\theta})$ be the far field pattern corresponding to the scattering problem (1.3.1) – (1.3.4) with associated far field operator F, and assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Furthermore, for every $z \in D$ let $g_z \in L^2(S^2)$ denote the solution of $(F^*F)^{1/4}g_z = \phi_z$, i.e. the solution obtained by the Factorization Method, and for every $z \in \mathbb{R}^3$ and $\varepsilon > 0$ let $g = g_{z,\varepsilon} \in L^2(S^2)$ be the Tikhonov approximation of (1.4.1), i.e. the unique solution of*

$$(\varepsilon I + F^*F)g \;=\; F^*\phi_z \tag{1.4.2}$$

which is computed by the Linear Sampling Method (if Tikhonov's regularization technique is chosen). Here, $\phi_z \in L^2(S^2)$ is defined in (1.3.14). Furthermore, let $v_{g_{z,\varepsilon}}(z) = \big(g_{z,\varepsilon}, \phi_z\big)_{L^2(S^2)} = \int_{S^2} g_{z,\varepsilon}(\hat{\theta}) \exp(\mathrm{i}k\,\hat{\theta}\cdot z)\,\mathrm{d}s(\hat{\theta})$ denote the corresponding Herglotz wave function evaluated at z.

(a) For every $z \in D$ the limit $\lim_{\varepsilon\to 0} v_{g_{z,\varepsilon}}(z)$ exists. Furthermore, there exists $c > 0$, depending on F only, such that for all $z \in D$ the following estimates hold:

$$c\,\|g_z\|^2_{L^2(S^2)} \;\le\; \lim_{\varepsilon\to 0}\big|v_{g_{z,\varepsilon}}(z)\big| \;\le\; \|g_z\|^2_{L^2(S^2)}\,. \tag{1.4.3}$$

(b) For $z \notin D$ the absolute values $\big|v_{g_{z,\varepsilon}}(z)\big|$ tend to infinity as ε tends to zero.

Proof: Using an orthonormal system $\big\{\psi_j : j \in \mathbb{N}\big\}$ of eigenfunctions ψ_j corresponding to eigenvalues $\lambda_j \in \mathbb{C}$ of F one computes the Tikhonov approximation $g_{z,\varepsilon}$ from (1.4.2) as

$$g_{z,\varepsilon} \;=\; \sum_{j=1}^{\infty} \frac{\overline{\lambda_j}}{|\lambda_j|^2 + \varepsilon}\,(\phi_z, \psi_j)_{L^2(S^2)}\,\psi_j\,.$$

From $v_g(z) = (g, \phi_z)_{L^2(S^2)}$ for any $g \in L^2(S^2)$ we conclude that

$$v_{g_{z,\varepsilon}}(z) \;=\; \sum_{j=1}^{\infty} \frac{\overline{\lambda_j}}{|\lambda_j|^2 + \varepsilon}\,\big|(\phi_z, \psi_j)_{L^2(S^2)}\big|^2\,. \tag{1.4.4}$$

(a) Let now $z \in D$. Then $(F^*F)^{1/4}g_z = \phi_z$ is solvable in $L^2(S^2)$ by Theorem 1.3.5 and thus $(\phi_z, \psi_j)_{L^2(S^2)} = ((F^*F)^{1/4}g_z, \psi_j)_{L^2(S^2)} = (g_z, (F^*F)^{1/4}\psi_j)_{L^2(S^2)} = \sqrt{|\lambda_j|}\,(g_z, \psi_j)_{L^2(S^2)}$. Therefore, we can express $v_{g_{z,\varepsilon}}(z)$ as

$$v_{g_{z,\varepsilon}}(z) \;=\; \sum_{j=1}^{\infty} \frac{\overline{\lambda_j}\,|\lambda_j|}{|\lambda_j|^2 + \varepsilon}\,\big|(g_z, \psi_j)_{L^2(S^2)}\big|^2 \;=\; \|g_z\|^2_{L^2(S^2)} \sum_{j=1}^{\infty} \rho_j \frac{\overline{\lambda_j}\,|\lambda_j|}{|\lambda_j|^2 + \varepsilon}\,, \tag{1.4.5}$$

where $\rho_j = \big|(g_z, \psi_j)_{L^2(S^2)}\big|^2 / \|g_z\|^2_{L^2(S^2)}$ is non-negative with $\sum_j \rho_j = 1$. An elementary argument (theorem of dominated convergence) yields convergence

$$\sum_{j=1}^{\infty} \rho_j \frac{\overline{\lambda_j}\,|\lambda_j|}{|\lambda_j|^2 + \varepsilon} \;\longrightarrow\; \sum_{j=1}^{\infty} \rho_j\,\frac{\overline{\lambda_j}}{|\lambda_j|} \;=\; \sum_{j=1}^{\infty} \rho_j\,\overline{s_j}$$

as ε tends to zero where again $s_j = \lambda_j/|\lambda_j|$. The properties of ρ_j imply that the limit belongs to the closure $\mathcal{C}$ of the convex hull of the complex numbers $\{s_j : j \in \mathbb{N}\}$. The same argument as in the proof of Theorem 1.3.4 yields that $\mathcal{C}$ has a positive distance c from the origin, i.e. $\left|\sum_{j=1}^{\infty} \rho_j \, \overline{s_j}\right| \geq c$ which proves the lower bound. The upper estimate is seen directly from (1.4.5).

(b) Let now $z \notin D$ and assume on the contrary that there exists a sequence $\{\varepsilon_n\}$ which tends to zero and such that $|v_n(z)|$ is bounded. Here we have set $v_n = v_{g_{z,\varepsilon_n}}$ for abbreviation. Since s_j converge to 1 there exists $j_0 \in \mathbb{N}$ with $\operatorname{Re} \lambda_j > 0$ for $j \geq j_0$. From (1.4.4) for $\varepsilon = \varepsilon_n$ we get

$$v_n(z) \;=\; \sum_{j=1}^{j_0-1} \frac{\overline{\lambda_j}}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 \;+\; \sum_{j=j_0}^{\infty} \frac{\overline{\lambda_j}}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 .$$

Since the finite sum is certainly bounded for $n \in \mathbb{N}$ there exists $c_1 > 0$ such that

$$\left| \sum_{j=j_0}^{\infty} \frac{\lambda_j}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 \right| \;\leq\; c_1 \quad \text{for all } n \in \mathbb{N} .$$

Observing that for any complex number $w \in \mathbb{C}$ with $\operatorname{Re} w \geq 0$ and $\operatorname{Im} w \geq 0$ we have that $\operatorname{Re} w + \operatorname{Im} w \geq |w|$ we conclude (note that also $\operatorname{Im} \lambda_j > 0$)

$$\begin{aligned} 2\,c_1 \;&\geq\; 2 \left| \sum_{j=j_0}^{\infty} \frac{\lambda_j}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 \right| \;\geq\; \sum_{j=j_0}^{\infty} \frac{\operatorname{Re} \lambda_j + \operatorname{Im} \lambda_j}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 \\ &\geq\; \sum_{j=j_0}^{\infty} \frac{|\lambda_j|}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 \;\geq\; \sum_{j=j_0}^{J} \frac{|\lambda_j|}{|\lambda_j|^2 + \varepsilon_n} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2 \end{aligned}$$

for all $n \in \mathbb{N}$ and all $J \geq j_0$. Letting n tend to infinity yields boundedness of the finite sum uniformly w.r.t. J and thus convergence of the series $\sum_{j=j_0}^{\infty} \frac{1}{|\lambda_j|} \left|(\phi_z, \psi_j)_{L^2(S^2)}\right|^2$. From (1.3.21) therefore follows that $z \in D$, which is the desired contradiction. □

Obviously, this kind of modification of the original Linear Sampling Method can be done for all inverse scattering problems for which Theorem 1.3.5 holds. This includes scattering by acoustically hard obstacles or inhomogeneous non-absorbing media or, with appropriate modifications, scattering by open arcs.

1.4.2 MUSIC

The Linear Sampling Method investigates "to what extent" the far field equation

$$Fg \;=\; \phi_z$$

is solvable for a number of sampling points z within some region of interest. As we have mentioned before, this equation has a solution in very rare cases only, and usually not for every $z \in D$.

However, if the obstacle is very small then it turns out that the far field operator almost degenerates to a finite rank operator, in which case the "numerical range" of F and $(F^*F)^{1/4}$ would be the same finite dimensional subspace, where the latter is known to contain ϕ_z for every

$z \in D$ – under appropriate assumptions on the particular problem setting (see Sections 1.2 and 1.3).

To investigate this observation in more detail we embed the real scene in a parameterized family of problems, where the parameter $\delta > 0$ reflects the scale of the problem. Assume that the scatterer $D = \bigcup_{i=1}^m D_i$ consists of m obstacles given as

$$D_i \;=\; z_i \;+\; \delta U_i \qquad i = 1, \ldots, m\,, \tag{1.4.6}$$

where each domain U_i contains the origin, has Lipschitz continuous boundary, and the closure of U_i has a connected complement. We shall call z_i the *location* of D_i and U_i its *shape*. We focus our presentation on an inhomogeneous medium setting for acoustic scattering, i.e., the Helmholtz equation, to provide analogies to both settings from Section 1.3. Let ρ_0 and c_0 be the density and the speed of sound in vacuum, $k = \omega/c_0$ be the associated wave number with frequency ω, and $u^i(x) = \exp(\mathrm{i}kx \cdot \hat{\theta})$ be an incoming plane wave. Then, if we assume that the density ρ_i and the sound of speed c_i in each object D_i are real and constant, then the total field $u_\delta = u^i + u^s_\delta$ solves the Helmholtz equation (see, e.g., [36])

$$\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} u_\delta\right) \;+\; \omega^2 \eta\, u_\delta \;=\; 0 \quad \text{in } \mathbb{R}^3\,, \tag{1.4.7}$$

with the radiation condition

$$\frac{\partial u^s_\delta}{\partial r} - \mathrm{i}k\, u^s_\delta \;=\; \mathcal{O}\big(r^{-2}\big) \qquad \text{for } r = |x| \to \infty\,, \tag{1.4.8}$$

uniformly with respect to $\hat{x} = x/|x|$, and the parameter η equals $\eta_0 = 1/\rho_0$ in $\mathbb{R}^3 \setminus \overline{D}$, and $\eta_i = c_0^2/(c_i^2\rho_i)$ in D_i, $i = 1, \ldots, m$, respectively. We mention that for constant $\eta = 1/\rho_0$ it has been shown in [73] that the standard Factorization Method (with $F_\# = (F^*F)^{1/2}$) applies for this setting with fixed scaling parameter δ. We know of no result, however, where the Factorization Method is used to reconstruct the supports of $\rho - \rho_0$ *and* $\eta - \eta_0$ in this setting simultaneously, although there are partial results for a similar problem (in a bounded domain, and with a different sign of η) arising in optical tomography, cf. [44, 60].

The idea to approach this problem is based on an asymptotic expansion of the far field u^∞_δ of the scattered wave with respect to the parameter δ in (1.4.6). We quote the following result from [4].

Theorem 1.4.2 *The far field of the scattering problem* (1.4.7) – (1.4.8) *for the scatterers given in* (1.4.6) *satisfies*

$$u^\infty_\delta(\hat{x};\hat{\theta}) \;=\; \delta^3 k^2 \sum_{i=1}^m \left(\Big(\frac{\rho_i}{\rho_0} - 1\Big)\hat{x} \cdot M_i \hat{\theta} - \Big(\frac{\eta_i}{\eta_0} - 1\Big)|U_i| \right) \exp\big(\mathrm{i}k\,(\hat{\theta} - \hat{x}) \cdot z_i\big) \;+\; o(\delta^3)\,, \tag{1.4.9}$$

and the associated far field operator can be rewritten as

$$F \;=\; \delta^3 \hat{F} + o(\delta^3) \tag{1.4.10}$$

in the norm of $\mathcal{L}(L^2(S^2))$, where the rank of the operator $\hat{F}$ is at most $4m$. Here, $|U_i|$ is the Lebesgue measure of U_i, and $M_i \in \mathbb{R}^{3\times 3}$ are symmetric positive definite matrices that depend on the shape U_i, the so-called polarization tensors.

As is obvious, the scattered field and its far field vanish as $\delta \to 0$. The corresponding rate δ^3 reflects the space dimension; in $\mathbb{R}^2$ the corresponding field decays like δ^2 as $\delta \to 0$.

The importance of Theorem 1.4.2 stems from the fact that the leading order approximation $\hat{F}$ of the far field operator F has finite rank, whereas F has infinite dimensional range. The rank of $\hat{F}$ is $4m$, unless some of the scatterers have the same material parameters as the background vacuum. Note that the dominating term of u_δ^∞ consists of two parts: The first contribution stems from the change in the density ρ and corresponds to the far field of a dipole (point source) in z_i; likewise, the second term corresponds to the far field of a monopole in z_i, and this is the result of a change in the parameter η.

It is easy to deduce from Theorem 1.4.2 that we can factorize $\hat{F}$ quite naturally in three factors.

Theorem 1.4.3 *The operator $\hat{F} : L^2(S^2) \to L^2(S^2)$ admits a factorization of the form*

$$\hat{F} \;=\; -BMB' \tag{1.4.11}$$

where $B : \mathbb{C}^{4m} \to L^2(S^2)$ maps a vector $[p_1, \ldots, p_m, a_1, \ldots, a_m]^T \in \mathbb{C}^{4m}$ with $p_i \in \mathbb{C}^3$ and $a_i \in \mathbb{C}$, $i = 1, \ldots, m$, to the far field of

$$u(x) \;=\; \sum_{i=1}^{m} \big(p_i \cdot \operatorname{grad}_z \Phi(x, z_i) \;+\; a_i \Phi(x, z_i) \big) \,,$$

where Φ is as in (1.3.10)*, $M \in \mathbb{R}^{4m \times 4m}$ is a real block diagonal matrix with m blocks of size 3×3 and m single elements on its diagonal, and M is nonsingular, if and only if $\rho_i \neq \rho_0$ and $\eta_i \neq \eta_0$ for all $i = 1, \ldots, m$. The operator B' is the dual operator of B with respect to the bilinear forms of $\mathbb{C}^{4m}$ and $L^2(S^2)$, i.e., $B'g$ consists of the gradients and point values of the Herglotz wave function*

$$v_g(x) \;=\; \int_{S^2} g(\hat{\theta}) \exp(\mathrm{i}kx \cdot \hat{\theta}) \, \mathrm{d}s(\hat{\theta}) \,, \qquad x \in \mathbb{R}^3 \,,$$

evaluated at the points z_i, $i = 1, \ldots, m$.

As M in (1.4.11) is invertible, the range of $\hat{F}$ and the range of B coincide, and it consists of the far fields of the monopoles and all possible dipoles emanating from the locations z_i of D_i, $i = 1, \ldots, m$. Using the unique continuation principle we can thus conclude the following result:

Corollary 1.4.4 *If each scatterer has a different permittivity than the background medium, then a point $z \in \mathbb{R}^3$ is the location z_i of one of the scatterers, if and only if ϕ_z belongs to the range of $\hat{F}$.*

When δ is small, it follows from (1.4.10) that numerically, the range of F and the range of $\hat{F}$ are the same, essentially. By this we mean that the dominating $4m$ singular values of F are small perturbations of the nonzero singular values of $\hat{F}$, and the corresponding singular subspaces are also close to each other. Moreover, we expect to see a sharp gap between the $4m$th and the $4m + 1$st singular value of F. We can search for this gap to determine the number m of the scatterers, and then determine the angle between the test function ϕ_z and the $4m$-dimensional dominating singular subspace of F. When z is close to the location of one of the scatterers then this angle will be small, otherwise this angle will be larger. This way images can be produced that enable one to visualize the approximate locations of the scatteres, but not their shape.

This approach applies for all problem settings that have been discussed in Sections 1.2 and 1.3, and many more. In impedance tomography, for example, the corresponding asymptotic expansion of the boundary potential has the form

$$u_\delta(x) - u_{\mathbb{1}}(x) \;=\; \delta^n \sum_{i=1}^{m} \frac{1-\kappa_i}{\kappa_i} \operatorname{grad}_z N(x,z_i) \cdot M_i \operatorname{grad} u_{\mathbb{1}}(z_i) \;+\; o(\delta^n)\,, \quad x \in \partial\Omega\,, \qquad (1.4.12)$$

where n is again the space dimension, N the Neumann function (1.2.6), and M_i the associated polarization tensor; cf. [28]. The leading order approximation of the difference between the associated Neumann-Dirichlet operators, $\Lambda_\delta - \Lambda_{\mathbb{1}}$, can be factorized in a similar way as in Theorem 1.4.3, and has an nm-dimensional range that is spanned by dipole potentials sitting in the locations z_i of the obstacles D_i, $i = 1, \ldots, m$; recall that n is the space dimension.

For the full Maxwell's equations considered in Section 1.3.2 the range space of the corresponding far field operator F of (1.3.36) consists of the magnetic far fields corresponding to electric dipoles at the infinitesimal scatterers; if the scatterers also differ in their magnetic permeability then the range space also contains the far fields of the magnetic dipoles in z_i, $i = 1, \ldots, m$.

The method described above for reconstructing the locations of small scatterers is often called *MUSIC* in the inverse problems community. Originally, the MUSIC algorithm is a signal processing tool for frequency estimation from the noisy spectrum of some signal[15], cf., e.g., [91]. In a seminal report [39] this algorithm was suggested to detect "point scatterers" on the basis of the Born approximation, which led to an algorithm that is not exactly the same, but related to the one we have sketched above. The relation between this algorithm and the Factorization Method has subsequently been recognized in [32, 70]. However, although the form of the factorization (1.4.11) is similar to the ones for the Factorization Method derived in Sections 1.2 and 1.3, it is slightly different in its precise interpretation; this has been exemplified in [2] by taking the limit of each of the factors from Theorem 1.2.6 as $\delta \to 0$.

The derivation of asymptotic formulas as in Theorem 1.4.2 goes back to the landmark paper [42]. In [24], formula (1.4.12) from [28] was used to provide the rigorous foundation of the MUSIC type algorithm from above. Important extensions and generalizations to other problem settings include [1, 4, 7, 47, 92]; for a more detailed survey and further references we refer to the monographs [5, 6].

Numerical illustrations of this approach can be found in various papers; see, for example, [3, 24, 47].

1.4.3 The Singular Sources Method

As in Section 1.3.1 we reconsider the simple inverse scattering problem for the Helmholtz equation in $\mathbb{R}^3$ to determine the shape of an acoustically soft obstacle D from the knowledge of the far field pattern $u^\infty(\hat{x};\hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^2$. We refer again to (1.3.1)–(1.3.6) for the mathematical model and the definition of the far field operator F from $L^2(S^2)$ into itself. Note that again $u^s = u^s(x;\hat{\theta})$ and $u^\infty = u^\infty(\hat{x};\hat{\theta})$ denote the scattered field and far field pattern, respectively, corresponding to the incident plane wave of direction $\hat{\theta} \in S^2$.

The basic tool in the *Singular Sources Method* is to consider also the scattered field $v^s = v^s(x;z)$ which corresponds to the incident field $v^i(x) = \Phi(x,z)$ of (1.3.10) of a point source, where $z \notin \overline{D}$ is a given point. The scattered field $v^s(z;z)$ evaluated at the source point blows up when z

[15] MUSIC stands for MUltiple SIgnal Classification.

tends to a boundary point. One can prove (see [74, 87]) that there exists a constant $c > 0$ (depending on D and k only) such that

$$\left|v^s(z;z)\right| \;\geq\; \frac{c}{d(z,\partial D)} \quad \text{for all } z \notin \overline{D}\,. \tag{1.4.13}$$

Here, $d(z,\partial D) = \inf\{|z-y| : y \in \partial D\}$ denotes the distance of z to the boundary of D.

The idea of the Singular Sources Method is to fix $z \notin \overline{D}$ and $\varepsilon > 0$ and a bounded domain $G_z \subset \mathbb{R}^3$ such that its exterior is connected and $z \notin \overline{G_z}$ and $\overline{D} \subset G_z$. Runge's Approximation Theorem (see, e.g., [74]) yields the existence of $g \in L^2(S^2)$ depending on z, G_z, and ε such that

$$\|v_g - \Phi(\cdot,z)\|_{C(\overline{G_z})} \;\leq\; \varepsilon \tag{1.4.14}$$

where v_g denotes the Herglotz wave function, defined by

$$v_g(x) \;=\; \int_{S^2} g(\hat{\theta})\, \exp(\mathrm{i}k\, x\cdot\hat{\theta})\, \mathrm{d}s(\hat{\theta})\,, \quad x \in \mathbb{R}^3\,.$$

In the following only the dependence on ε is indicated by writing g_ε. The following convergence result for the Singular Sources Method is known (see [74, 87]).

Theorem 1.4.5 *Let $u^\infty = u^\infty(\hat{x};\hat{\theta})$, $\hat{x},\hat{\theta} \in S^2$, be the far field pattern of the scattering problem (1.3.2), (1.3.3), (1.3.4). Fix $z \notin \overline{D}$ and a bounded domain $G_z \subset \mathbb{R}^3$ such that its exterior is connected and $z \notin \overline{G_z}$ and $\overline{D} \subset G_z$. For any $\varepsilon > 0$ choose $g = g_\varepsilon \in L^2(S^2)$ with (1.4.14). Then*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_{S^2} (Fg_\varepsilon)(-\hat{\theta})\, g_\delta(\hat{\theta})\, ds(\hat{\theta}) \;=\; v^s(z;z)\,,$$

i.e. by substituting the form of F,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_{S^2}\int_{S^2} u^\infty(-\hat{\theta};\hat{\eta})\, g_\varepsilon(\hat{\eta})\, g_\delta(\hat{\theta})\, \mathrm{d}s(\hat{\eta})\, \mathrm{d}s(\hat{\theta}) \;=\; v^s(z;z)\,.$$

Note that the limits are *iterated*, i.e. first the limit w.r.t. ε has to be taken and then the limit w.r.t. δ.

Combining this result with (1.4.13) yields

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left| \int_{S^2}\int_{S^2} u^\infty(-\hat{\theta};\hat{\eta})\, g_\varepsilon(\hat{\eta})\, g_\delta(\hat{\theta})\, \mathrm{d}s(\hat{\eta})\, \mathrm{d}s(\hat{\theta}) \right| \;\geq\; \frac{c}{d(z,\partial D)}\,. \tag{1.4.15}$$

This result assures that for z sufficiently close to the boundary ∂D (and regions G_z chosen appropriately) the quantity

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left| \int_{S^2}\int_{S^2} u^\infty(-\hat{\theta};\hat{\eta})\, g_\varepsilon(\hat{\eta})\, g_\delta(\hat{\theta})\, \mathrm{d}s(\hat{\eta})\, \mathrm{d}s(\hat{\theta}) \right|$$

becomes large.

It is convenient to use domains G_z of the special form

$$G_{z,p} \;=\; (z+\rho\, p) \;+\; \left\{ x \in \mathbb{R}^3 : |x| < R\,,\; \frac{x}{|x|}\cdot p > -\cos\beta \right\}$$

for some (large) radius $R > 0$, opening angle $\beta \in [0, \pi/2)$, direction of opening $p \in S^2$, and $\rho > 0$. The dependence on β, ρ, and R is not indicated since they are kept fixed. This domain $G_{z,p}$ is a ball centered at $z + \rho p$ with radius R from which the cone of direction $-p$ and opening angle β has been removed. Obviously, it is chosen such that $z \notin \overline{G_{z,p}}$. These sets $G_{z,p}$ are translations and rotations of the reference set

$$\hat{G} \;=\; \left\{ x \in \mathbb{R}^3 : |x| < R\,,\; \frac{x}{|x|} \cdot \hat{p} > -\cos\beta \right\}$$

for $\hat{p} = (0, 0, 1)^\top$, i.e. $G_{z,p} = z + M\hat{G}$ for some orthogonal $M \in \mathbb{R}^{3\times 3}$.

With these transformations, we can consider the singular sources method as a sampling method with sampling objects z and M.

From the arguments used in the proof of Theorem 1.4.5 it is not clear whether or not the common limit $\lim_{\varepsilon,\delta\to 0}$ exists. However, if k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D then the following stronger result than (1.4.15) can be obtained by using the factorization (1.3.8).

Theorem 1.4.6 *Let $z \notin \overline{D}$ and $G_z \subset \mathbb{R}^3$ be a bounded domain such that its exterior is connected and $z \notin \overline{G_z}$ and $\overline{D} \subset G_z$. For any $\varepsilon > 0$ choose $g_\varepsilon \in L^2(S^2)$ with (1.4.14) with respect to the H^1-norm, i.e.*

$$\|v_{g_\varepsilon} - \Phi(\cdot, z)\|_{H^1(G_z)} \;\le\; \varepsilon\,.$$

Assume furthermore that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then there exists a constant $c > 0$ depending only on D and k such that

$$\left| \lim_{\varepsilon\to 0} \int_{S^2}\int_{S^2} u^\infty(\hat{\theta};\hat{\eta})\, g_\varepsilon(\hat{\eta})\, \overline{g_\varepsilon(\hat{\theta})}\, \mathrm{d}s(\hat{\eta})\, \mathrm{d}s(\hat{\theta}) \right| \;=\; \lim_{\varepsilon\to 0} \left| (Fg_\varepsilon, g_\varepsilon)_{L^2(S^2)} \right| \;\ge\; \frac{c}{d(z, \partial D)}\,.$$

For a proof we refer to [74]. Numerical reconstructions with the Singular Sources Methods are shown in [87].

1.4.4 The Probe Method

The *Probe Method* has originally been proposed in [63] for the inverse problem of impedance tomography of Section 1.2.2, and here we also restrict our attention to this setting. To be precise, let $\sigma \in L^\infty(\Omega)$ be a (complex valued) admittivity function, and define $u \in H^1_\diamond(\Omega)$ as the unique (weak) solution of the boundary value problem

$$\mathrm{div}(\sigma \,\mathrm{grad}\, u) \;=\; 0 \quad \text{in } \Omega\,, \qquad \sigma\,\frac{\partial}{\partial \nu} u \;=\; f \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} u\,\mathrm{d}s \;=\; 0\,, \tag{1.4.16}$$

where $f \in L^2_\diamond(\partial\Omega)$. For the spaces $H^1_\diamond(\Omega)$ and $L^2_\diamond(\partial\Omega)$ we refer to Section 1.2.1.

As in Section 1.2.2 we assume that $\sigma \in L^\infty(\Omega)$ is a perturbation of the constant background admittivity function $\sigma_{\mathbb{1}} = 1$. More precisely, let $D \subset \Omega$ be again the finite union of domains such that $\Omega \setminus \overline{D}$ is connected and $\sigma = 1$ in $\Omega \setminus D$, and let there be a constant $c_0 > 0$ such that

$$\mathrm{Im}\,\sigma(x) \;\le\; 0 \quad \text{and} \quad \mathrm{Re}\,\sigma(x) \;\ge\; 1 + c_0 \quad \text{on } D\,. \tag{1.4.17}$$

The case $0 < c_0 \le \mathrm{Re}\,\sigma(x) \le 1 - c_0$ can be treated in a similar way (see [74]). The unique solvability of the **direct problem**, i.e., the boundary value problem (1.4.16), guarantees existence of the *Neumann-to-Dirichlet* operators $\Lambda, \Lambda_{\mathbb{1}} : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega)$ corresponding to σ and $\sigma_{\mathbb{1}} = 1$, respectively.

As in Section 1.2.2 the goal of the **inverse problem** is to determine the support D of $\sigma - 1$ from the knowledge of the absolute data Λ, or the relative data $\Lambda - \Lambda_{\mathbb{1}}$. The difference to the setting in Section 1.2.2 is that σ is now a scalar and complex valued function.

In the probe method the sampling objects are curves in Ω starting at the boundary $\partial\Omega$ of Ω. In the original paper [63] these curves are called needles. We keep this notation but mention that - perhaps in contrast to the colloquial meaning - these needles don‘t need to be straight segments but can be curved in general. By choosing a family of needles the Probe Method determines the first point on the needle which intersects the boundary ∂D (see Theorem 1.4.8 below). Therefore, in contrast to the Factorization Method and the Linear Sampling Method the Probe Method tests on curves instead on points.

Definition 1.4.7 *A needle*[16] $\mathcal{C}$ *is the image of a continuously differentiable function* $\eta : [0,1] \to \overline{\Omega}$ *such that* $\eta(0) \in \partial\Omega$ *and* $\eta(t) \in \Omega$ *for all* $t \in (0,1]$ *and* $\eta'(t) \neq 0$ *for all* $t \in [0,1]$ *and* $\eta(t) \neq \eta(s)$ *for* $t \neq s$. *We call* η *a parameterization of the needle.*

The following monotonicity property is the basic ingredient for the Probe Method.

Under the above assumptions on $\sigma \in L^\infty(\Omega)$ there exists $c > 1$ such that

$$\frac{1}{c}\int_D |\operatorname{grad} u_{\mathbb{1}}|^2 \,\mathrm{d}x \;\leq\; \operatorname{Re}\langle f, (\Lambda_{\mathbb{1}} - \Lambda) f\rangle \;\leq\; c\int_D |\operatorname{grad} u_{\mathbb{1}}|^2 \,\mathrm{d}x \tag{1.4.18}$$

for every $f \in L^2_\diamond(\partial\Omega)$. Here, $u_{\mathbb{1}} \in H^1_\diamond(\Omega)$ denotes the unique solution of (1.4.16) for the constant background case $\sigma_{\mathbb{1}} = 1$.

Let $\eta : [0,1] \to \overline{\Omega}$ be the parameterization of a given needle, $t \in (0,1]$ a fixed parameter, and $\mathcal{C}_t = \{\eta(s) : 0 \leq s \leq t\}$ the part of the needle from $s = 0$ to $s = t$. Let $\Phi(x,y)$ denote the fundamental solution of the Laplace equation, e.g.

$$\Phi(x,y) \;=\; \frac{1}{4\pi|x-y|}\,, \quad x \neq y\,,$$

in $\mathbb{R}^3$. The Approximation Theorem of Runge (see, e.g., [74]) yields the existence of a sequence $w_n \in H^1(\Omega)$ of harmonic functions in Ω such that

$$\big\|w_n - \Phi\big(\cdot, \eta(t)\big)\big\|_{H^1(U)} \;\to\; 0\,, \quad n \to \infty\,, \tag{1.4.19}$$

for every subset U with $\overline{U} \subset \Omega \setminus \mathcal{C}_t$. We set $f_n = \partial w_n/\partial\nu$ on $\partial\Omega$ and note that f_n depends on $\mathcal{C}_t$ but not on the unknown domain D. The dependence on $\mathcal{C}_t$ is denoted by writing $f_n(\mathcal{C}_t)$. It can – at least in principle – be computed beforehand.

Theorem 1.4.8 *Let the above assumptions on* σ *hold and fix a needle with parameterization* $\eta : [0,1] \to \overline{\Omega}$. *Define the set* $\mathcal{T} \subset [0,1]$ *by*

$$\mathcal{T} \;=\; \left\{ t \in [0,1] : \sup_{n\in\mathbb{N}} \big\{ \big|\operatorname{Re}\big\langle f_n(\mathcal{C}_t), (\Lambda - \Lambda_{\mathbb{1}}) f_n(\mathcal{C}_t)\big\rangle\big| < \infty \right\}. \tag{1.4.20}$$

Here, $f_n(\mathcal{C}_t) = \partial w_n/\partial\nu \in H^{-1/2}(\partial\Omega)$ *is determined from (1.4.19)*[17]. *Then* $\mathcal{T} \neq \emptyset$, *and one can define* $t^* = \sup\{t \in [0,1] : [0,t] \in \mathcal{T}\}$, *which satisfies*

$$t^* \;=\; \begin{cases} \min\{t \in [0,1] : \eta(t) \in \partial D\}\,, & \text{if } \mathcal{C}_1 \cap \overline{D} \neq \emptyset\,, \\ 1\,, & \text{if } \mathcal{C}_1 \cap \overline{D} = \emptyset\,. \end{cases} \tag{1.4.21}$$

We recall that $\mathcal{C}_1 = \mathcal{C} = \{\eta(t) : t \in [0,1]\}$.

[16] This notation is taken from the original paper [63].

[17] So far, we have chosen the boundary current f in (1.4.16) from $L^2_\diamond(\partial\Omega)$ for convenience; however, the quadratic form in (1.4.20) extends as dual pairing $\langle H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)\rangle$ to $f \in H^{-1/2}(\partial\Omega)$ with vanishing mean.

For a proof we refer to [63, 74].

Note that for every needle the set $\mathcal{T}$ of the form (1.4.20) is determined by the given data: It depends on η and the approximating functions w_n. Formula (1.4.21) provides a constructive way to determine ∂D from $\Lambda - \Lambda_{\mathbb{1}}$: One has to choose a family of needles which cover the domain Ω, and for each needle one computes t^* as the largest point of $\mathcal{T}$; if $t^* < 1$ then $\eta(t^*) \in \partial D$. Obviously, this procedure is very expensive from a computational point of view. However, if one samples with "linear" needles only, i.e. rays of the form $\mathcal{C} = \{z + tp : t \geq 0\} \cap \Omega$ for $z \in \Omega$ and unit vectors $p \in S^2$ then the computational effort can be reduced considerably since the approximating sequence (1.4.19) has to be computed only once for a reference needle. However, by using only rays as needles one can not expect to detect the boundary of D completely. Only the "visible points" of ∂D can be detected, i.e., those which can be connected completely in $\Omega \setminus \overline{D}$ by straight lines to $\partial\Omega$.

In an implementation of the definition of $\mathcal{T}$ of (1.4.20) one has to decide whether a supremum is finite or infinite. Numerically, this is certainly not an easy task. In [63] it has been suggested to replace $\mathcal{T}$ of (1.4.20) by

$$\mathcal{T}_M \;=\; \left\{ t \in [0,1] : \sup_{n \in \mathbb{N}} \left\{ \left| \mathrm{Re} \left\langle f_n(\mathcal{C}_t), (\Lambda - \Lambda_{\mathbb{1}}) f_n(\mathcal{C}_t) \right\rangle \right| \leq M \right\} \right.$$

for some $M > 0$, for which a result analogously to the one in Theorem 1.4.8 can be established. We refer to [63] for more details.

Again, the probe method is general enough to have extensions to a number of related inverse problems in elasticity (see [64]) and scattering theory (see [62]). For numerical reconstructions we refer to [88].

1.5 Appendix

In this appendix we collect some functional analytic results on range identities. The Factorization Method makes use of the fact that the unknown domain D can be characterized by the range of some compact operator $A : X \to Y$, where A is related to the known operator $M : Y \to Y$ through the factorization

$$M \;=\; A\, G\, A^* \,. \tag{1.5.1}$$

Throughout this whole chapter we assume that Y is a Hilbert space and X a reflexive Banach space with dual X^*. We denote by $A^* : Y \to X^*$ the adjoint of A, where Y is identified with its dual.

For a computable characterization of D the range of the operator A has to be expressed by the operator M which is the goal of the *range identity*.

In the simplest case where also X is a Hilbert space and G is the identity I, the range identity is easily obtained via the singular system of A and the Theorem of Picard. We recall that $\{\sigma_j, x_j, y_j : j \in J\}$ is a singular system of a linear and compact operator $T : X \to Y$ between Hilbert spaces X and Y if $\{x_j : j \in J\}$ and $\{y_j : j \in J\}$ are complete countable orthonormal systems in the subspaces $\mathcal{N}(T)^\perp \subset X$ and $\mathcal{N}(T^*)^\perp \subset Y$, respectively, and $\sigma_j \in \mathbb{R}_{>0}$ such that $Tx_j = \sigma_j y_j$ and $T^* y_j = \sigma_j x_j$ for all $j \in J$.

We note that $\{\sigma_j^2, x_j : j \in J\}$, together with a basis of the null space $\mathcal{N}(T)$ of T and associated eigenvalue 0, is an eigensystem of the self adjoint and non-negative operator T^*T. Furthermore,

$$\begin{aligned} Tx &= \sum_{j\in J} \sigma_j (x, x_j)_X \, y_j \,, \quad x \in X \,, \\ T^*y &= \sum_{j\in J} \sigma_j (y, y_j)_Y \, x_j \,, \quad y \in Y \,. \end{aligned}$$

Theorem 1.5.1 *(Picard) Let X, Y be Hilbert spaces and $T : X \to Y$ be a compact operator with singular system $\{\sigma_j, x_j, y_j : j \in J\}$. Then there holds: An element $y \in Y$ belongs to the range $\mathcal{R}(T)$ of T, if and only if,*

$$y \in \mathcal{N}(T^*)^\perp \quad \textit{and} \quad \sum_{j\in J} \frac{\big|(y, y_j)_Y\big|^2}{\sigma_j^2} \;<\; \infty \,.$$

For a proof we refer to, e.g., [40]. Applying this theorem to the factorization (1.5.1) with $G = I$, and when X^* is identified with X, one obtains:

Corollary 1.5.2 *Let $A : X \to Y$ be a compact operator between Hilbert spaces X and Y with dense range and $M = A\,A^* : Y \to Y$. Then the ranges of A and $M^{1/2}$ coincide. Here, the self adjoint and non-negative operator $M^{1/2} : Y \to Y$ is given by*

$$M^{1/2} y \;=\; \sum_{j\in J} \sqrt{\lambda_j}\, (y, y_j)_Y \, y_j \,, \quad y \in Y \,,$$

where $\{y_j : j \in J\}$ are the orthonormal eigenelements of the self adjoint, compact, and non-negative operator M corresponding to the positive eigenvalues λ_j. It follows that

$$y \in \mathcal{R}(A) \quad \Longleftrightarrow \quad \sum_{j\in J} \frac{\big|(y, y_j)_Y\big|^2}{\lambda_j} \;<\; \infty \,.$$

For more general factorizations of the form $M = AGA^*$ the following (preliminary) characterization is useful (see [69]; for an equivalent formulation see Theorem 3 of [83]).

Theorem 1.5.3 *Let X be a reflexive Banach space with dual X^* and dual form $\langle\cdot,\cdot\rangle$ in $\langle X^*, X\rangle$. Furthermore, let Y be a Hilbert space and $M : Y \to Y$ and $A : X \to Y$ be linear bounded operators such that the factorization (1.5.1) holds for some linear and bounded operator $G : X^* \to X$, which satisfies a coercivity condition of the form: There exists $c > 0$ with*

$$\big|\langle \varphi, G\varphi\rangle\big| \;\geq\; c\|\varphi\|_{X^*}^2 \quad \textit{for all } \varphi \in \mathcal{R}(A^*) \subset X^* \,. \tag{1.5.2}$$

Then, for any $\phi \in Y$, $\phi \neq 0$,

$$\phi \in \mathcal{R}(A) \;\Longleftrightarrow\; \inf\big\{\big|(\psi, M\psi)_Y\big| : \psi \in Y,\; (\psi, \phi)_Y = 1\big\} \;>\; 0 \,. \tag{1.5.3}$$

Proof: The form $\big|(\psi, M\psi)_Y\big|$ can be estimated by

$$\big|(\psi, M\psi)_Y\big| \;=\; \big|\langle A^*\psi, GA^*\psi\rangle\big| \;\geq\; c\|A^*\psi\|_{X^*}^2 \quad \text{for all } \psi \in Y \,. \tag{1.5.4}$$

Let first $\phi = A\varphi_0$ for some $\varphi_0 \in X$. For $\psi \in Y$ with $(\psi, \phi)_Y = 1$ there holds that

$$\begin{aligned} \big|(\psi, M\psi)_Y\big| &\geq c\|A^*\psi\|_{X^*}^2 = \frac{c}{\|\varphi_0\|_X^2}\,\|A^*\psi\|_{X^*}^2\,\|\varphi_0\|_X^2 \\ &\geq \frac{c}{\|\varphi_0\|_X^2}\,\big|\langle A^*\psi, \varphi_0\rangle\big|^2 = \frac{c}{\|\varphi_0\|_X^2}\,\big|(\psi, \underbrace{A\varphi_0}_{=\phi})_Y\big|^2 = \frac{c}{\|\varphi_0\|_X^2}\,. \end{aligned}$$

This provides the lower bound of the infimum.

Second, assume that $\phi \notin \mathcal{R}(A)$. Define the closed subspace $V := \big\{\psi \in Y : (\psi, \phi)_Y = 0\big\}$. Then $A^*(V)$ is dense in $\mathcal{R}(A^*) \subset X^*$. Indeed, this is equivalent to the statement that the annihilators $\big[A^*(V)\big]^\perp$ and $\big[\mathcal{R}(A^*)\big]^\perp = \mathcal{N}(A)$ coincide. Therefore, let $\varphi \in \big[A^*(V)\big]^\perp$, i.e. $\langle A^*\psi, \varphi\rangle = 0$ for all $\psi \in V$, i.e. $(\psi, A\varphi)_Y = 0$ for all $\psi \in V$, i.e. $A\varphi \in V^\perp = \operatorname{span}\{\phi\}$. Since $\phi \notin \mathcal{R}(A)$ this implies $A\varphi = 0$, i.e. $\varphi \in \mathcal{N}(A)$. Therefore, $A^*(V)$ is dense in $\mathcal{R}(A^*)$.

Choose a sequence $\{\hat\psi_n\}$ in V such that $A^*\hat\psi_n \to -\frac{1}{\|\phi\|_Y^2}\,A^*\phi$ as n tends to infinity and set $\psi_n = \hat\psi_n + \phi/\|\phi\|_Y^2$. Then $(\psi_n, \phi)_Y = 1$ and $A^*\psi_n \to 0$. The first equation of (1.5.4) yields

$$\big|(\psi_n, M\psi_n)_Y\big| \leq \|G\|\,\|A^*\psi_n\|_{X^*}^2$$

and thus $(\psi_n, M\psi_n)_Y \to 0$, $n \to \infty$, which proves that $\inf\big\{\big|(\psi, M\psi)_Y\big| : \psi \in Y,\ (\psi, \phi)_Y = 1\big\} = 0$. □

We note that the inf-condition only depends on M and not on the factorization. Therefore, we have as a corollary:

Corollary 1.5.4 *Let Y be a Hilbert space and X_1 and X_2 be reflexive Banach spaces with duals X_1^* and X_2^*, respectively. Furthermore, let $M : Y \to Y$ have two factorizations of the form $M = A_1\,G_1\,A_1^* = A_2\,G_2\,A_2^*$ as in* (1.5.1) *with compact operators $A_j : X_j \to Y$ and bounded operators $G_j : X_j^* \to X_j$, which both satisfy the coercivity condition (1.5.2). Then the ranges of A_1 and A_2 coincide.*

Corollary 1.5.4 is useful for the analysis of the Factorization Method as long as M is normal. However, there are many scattering problems for which the corresponding far field operator fails to be normal, e.g. in the case of absorbing media. For these problems one can utilize the self adjoint operator

$$M_\# = |\operatorname{Re} M| + \operatorname{Im} M\,, \tag{1.5.5}$$

that can be computed from M. Note that $\operatorname{Re} M = \frac{1}{2}\,(M + M^*)$ and $\operatorname{Im} M = \frac{1}{2\mathrm{i}}\,(M - M^*)$ are again self adjoint and compact, and the absolute value $|\operatorname{Re} M|$ of $\operatorname{Re} M$ is defined to be

$$|\operatorname{Re} M|\psi = \sum_{j\in J} |\lambda_j|\,(\psi, \psi_j)_Y \psi_j\,, \quad \psi \in Y\,,$$

where $\{\lambda_j, \psi_j : j \in J\}$ denotes the spectral system of $\operatorname{Re} M$.

Now we can apply Corollary 1.5.4 to obtain the following result (see [74] for the lengthy proof, and [77] for a weaker form of assumption (d)).

Theorem 1.5.5 *Let X be a reflexive Banach space with dual X^* and dual form $\langle\cdot,\cdot\rangle$ in $\langle X^*, X\rangle$. Furthermore, let Y be a Hilbert space and $M : Y \to Y$ and $A : X \to Y$ be linear bounded operators such that the factorization* (1.5.1) *holds true for some linear and bounded operator $G : X^* \to X$. Furthermore, let the following conditions be satisfied:*

(a) The range of A is dense in Y.

(b) There holds $\operatorname{Re} G = G_0 + G_1$, where G_0 satisfies (1.5.2) and $G_1 : X^* \to X$ is compact.

(c) The imaginary part $\operatorname{Im} G$ of G is non-negative, i.e. $\operatorname{Im}\langle \varphi, G\varphi\rangle \geq 0$ for all $\varphi \in X^*$.

(d) G is injective or $\operatorname{Im} G$ is positive on the nullspace of $\operatorname{Re} G$.

Then the self adjoint operator $M_\#$ of (1.5.5) is positive and the ranges of A and $M_\#^{1/2}$ coincide.

As an immediate corollary we have

Corollary 1.5.6 *Let $M : Y \to Y$ and $A : X \to Y$ and $G : X^* \to X$ be as in Theorem 1.5.5, and let G be self adjoint, i.e. $G^* = G$, and satisfy (1.5.2). Then the ranges of A and $M^{1/2}$ coincide, and*

$$y \in \mathcal{R}(A) \quad \Longleftrightarrow \quad \sum_{j\in J} \frac{\left|(y, y_j)_Y\right|^2}{\lambda_j} \; < \; \infty\,,$$

where $\{\lambda_j, y_j \; : \; j \in J\}$ denotes a spectral system of the self adjoint and compact operator $M = AGA^*$.

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