On the existence of weak solution to the coupled fluid-structure interaction problem for non-Newtonian shear-dependent fluid

A. Hundertmark-Zaušková, M. Lukáčová-Medviďová, Š. Nečasová

Abstract

We study the existence of weak solution for unsteady fluid-structure interaction problem for shear-thickening flow. The time dependent domain has at one part a flexible elastic wall. The evolution of fluid domain is governed by the generalized string equation with action of the fluid forces. The power law viscosity model is applied to describe shear-dependent non-Newtonian fluids.

keywords: non-Newtonian fluids, fluid-structure interaction, shear-thinning fluids, shear-thickening fluids, hemodynamics, existence of weak solution

1 Mathematical model

Consider a two-dimensional fluid motion governed by the momentum and the continuity equation

$$\begin{aligned} \rho\partial_t \boldsymbol{v} + \rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} - \operatorname{div}[2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})] + \nabla\pi &= 0 \\ \operatorname{div}\boldsymbol{v} &= 0 \end{aligned} \tag{1.1}$$

with ρ denoting the constant density of fluid, $\boldsymbol{v} = (v_1, v_2)$ the velocity vector, π the pressure, $e(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$ the symmetric deformation tensor and μ the viscosity of the fluid. We assume that fluid is obeying the non-Newtonian shear-dependent model, cf. [23, 28, 29, 39]. A typical example is the following power-law model

$$\mu(|e(\boldsymbol{v})|) = \mu(1 + |e(\boldsymbol{v})|^2)^{\frac{p-2}{2}} \quad p > 1, \tag{1.2}$$

see also Section 3.1 for a more general description of the considered non-Newtonian model. Note that according to the parameter p, the non-Newtonian fluid is either shear-thinning ($p < 2$) or shear-thickening ($p \geq 2$). Models for fluids with the shear-dependent viscosity are used in many areas of engineering science such as geophysics, glaciology, polymer mechanics, blood or food rheology. For $p > 2$ this model is an analogy of the so-called Ladyzhenskaja's fluid, for $p = 3$ it yields the Smagorinskij model of turbulence. In our recent article [23], where numerical simulations of blood flow has been presented, the shear-thinning model of Carreau has been used in order to model blood flow.

Let us refer to several previous works on the existence of weak solution to the power law-viscosity models. Ladyzenskaya and Lions proved in the late sixties the existence of non-steady weak solution with the use of classical compactness theory and theory of monotonous operators for $p \geq 2$ in two dimensions and $p \geq 11/5$ in three dimensions, see [25, 26]. This result is valid for power law models for space periodic as well as the Dirichlet boundary value problem. The most difficult part of proof of the existence of weak solution is the limiting process in the non-linear viscous term having p-structure arising from the power law for viscosity (1.2). There are several approaches to overcome this difficulty. Málek, Nečas, Růžička [28] proved the existence of unsteady weak solution in d dimensions for $p > 3d/(d+2)$ for space periodic case using fractional higher differentiability, see also [27, 24] for related results for the Dirichlet problem. Further results were obtained by Frehse, Málek, Steinhauer [17] or by Wolf [40] using the L^∞-truncation method and the Lipschitz truncation method [12, 16]. Diening, Růžička and Wolf used in [11] the Lipschitz truncation method and the local pressure method to prove the existence of weak solution in $L^p(0,T;W^{1,p}(\Omega))$ for $p > 2d/(d+2)$.

We follow with the description of the mathematical model. The two dimensional computational domain

$$\Omega(\eta(t)) \equiv \{(x_1,x_2);\ 0 < x_1 < L,\ 0 < x_2 < R_0(x_1) + \eta(x_1,t)\}\,,\ 0 < t < T$$

is given by a reference radius function $R_0(x_1)$ and the unknown free boundary function $\eta(x_1,t)$ describing the domain deformation. The fluid and the geometry of the computational domain are coupled through the following Dirichlet boundary condition on the deformable part of the boundary $\Gamma_w(t)$

$$\boldsymbol{v}(x_1, R_0(x_1) + \eta(x_1,t), t) = \left(0, \frac{\partial \eta(x_1,t)}{\partial t}\right), \tag{1.3}$$

where $\Gamma_w(t) = \{(x_1,x_2);\ x_2 = R_0(x_1) + \eta(x_1,t),\ x_1 \in (0,L)\}$. The normal component of the fluid stress tensor $\mathbf{T}_f\boldsymbol{n}$ and the outside pressure P_w provide the forcing term for the deformation equation of the free boundary η, that is modeled by the generalized string equation.

$$\begin{aligned}\tilde{E}\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right] = \\ g\left(-\mathbf{T}_f^{ref} - P_w^{ref}\mathbf{I}\right)\boldsymbol{n}^{ref}\cdot\boldsymbol{e}_2 \quad \text{on } \Gamma_w^0.\end{aligned} \tag{1.4}$$

Here $[(\mathbf{T}_f^{ref} - P_w^{ref}\mathbf{I})\boldsymbol{n}^{ref}](x^{ref}) = [(\mathbf{T}_f - P_w\mathbf{I})\boldsymbol{n}](x)$, $x \in \Gamma_w(t)$, $x^{ref} \in \Gamma_w^0$, $\mathbf{T}_f = -\pi\mathbf{I} + 2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})$, $\boldsymbol{n}$ is the unit outward normal on $\Gamma_w(t)$, $\boldsymbol{n}|\boldsymbol{n}| = (-\partial_{x_1}(R_0+\eta), 1)^T$ and $\Gamma_w^0 := \Gamma_w(t)|_{t=0}$ is the initial position of the deformable part of the boundary. The coefficient $g = \frac{(R_0+\eta)\sqrt{1+(\partial_{x_1}(R_0+\eta))^2}}{R_0\sqrt{1+(\partial_{x_1}R_0)^2}}$ arrises from the transformation from the Eulerian frame of the fluid forces into the Lagrangian formulation of the string. Equation (1.4) is equipped with the following boundary and initial conditions

$$\begin{aligned} &\eta(0,t) = \eta(L,t) = 0 \quad \text{and} \quad \eta(x_1,0) = \frac{\partial \eta}{\partial t}(x_1,0) = 0, \\ &\eta_{x_1}(0,t) = \eta_{x_1}(L,t) = 0. \end{aligned} \tag{1.5}$$

Positive coefficients $\tilde{E}$, a, b, c appearing in (1.4) are given as follows [23],

$$\tilde{E} = \rho_w \hbar, \qquad a = \frac{|\sigma_z|}{\left(1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right)^2}, \quad b = \frac{\mathcal{E}}{(R_0+\eta)R_0}, \qquad c > 0,$$

where $\mathcal{E}$ is the Young modulus, $\hbar$ the wall thickness, ρ_w the density of the vessel wall tissue, the coefficient $c = \gamma/(\rho_w \hbar)$, γ positive constant. $|\sigma_z| = G\kappa$ is the longitudinal stress, $\kappa = 1$ is the Timoshenko's shear correction factor and G is the shear modulus, equal to $G = \mathcal{E}/2(1+\sigma)$ with $\sigma = 1/2$ for incompressible materials. Note that the coefficients a, b are non-constant, however, according to the assumption (2.1) below they are upper- and down-bounded. In what follows, we linearize the term $b = \frac{\mathcal{E}}{(R_0+\eta)R_0}$ by $\frac{\mathcal{E}}{\rho_w R_0^2}$ and for the sake of simplicity we work with constant coefficients a, b, c.

The equation (1.4) can be transformed as follows.

$$\begin{aligned} E\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \\ \Big[-\mathbf{T}_f\boldsymbol{n}|\boldsymbol{n}|\cdot\boldsymbol{e}_2 - P_w\Big](x_1, R_0(x_1)+\eta(x_1,t),t), \end{aligned} \tag{1.6}$$

$x_1 \in (0,L)$. Here $E = \tilde{E}\sqrt{1+(\partial_{x_1}R_0)^2}$. We assume that E is bounded.

We complete the system (1.1) with the following boundary and initial conditions: on the inflow part of the boundary, which we denote Γ_{in}, we set

$$v_2(0,x_2,t) = 0, \tag{1.7}$$

$$\left(2\mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_1} - \pi + P_{in} - \frac{\rho}{2}|v_1|^2\right)(0,x_2,t) = 0 \tag{1.8}$$

for any $0 < x_2 < R_0(0)$, $0 < t < T$ and for a given function $P_{in} = P_{in}(x_2,t)$. On the opposite, outflow part of the boundary Γ_{out}, we set

$$v_2(L,x_2,t) = 0\ , \tag{1.9}$$

$$\left(2\mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_1} - \pi + P_{out} - \frac{\rho}{2}|v_1|^2\right)(L,x_2,t) = 0 \tag{1.10}$$

for any $0 < x_2 < R_0(L)$, $0 < t < T$ and for a given function $P_{out} = P_{out}(x_2, t)$. Note that we require here that the so-called kinematic pressure is prescribed on the inflow and outflow boundary. This implies that the fluxes of kinetic energy on inflow and outflow boundary will disappear in the weak formulation. Finally, on the remaining part of the boundary, Γ_c, we set the flow symmetry condition

$$v_2(x_1, 0, t) = 0 \; , \qquad \mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_2}(x_1, 0, t) = 0 \tag{1.11}$$

for any $0 < x_1 < L$, $0 < t < T$. The initial conditions read

$$\boldsymbol{v}(x_1, x_2, 0) = \boldsymbol{0} \quad \text{ for any } 0 < x_1 < L,\ 0 < x_2 < R_0(x_1). \tag{1.12}$$

Our main goal in this paper is to show global existence in time of weak solution of fully unsteady fluid-structure interaction problem, see Theorem 1.1. In fact we will be able to show that a weak solution of the coupled fluid-structure interaction problem exists until a contact of the elastic boundary with a fixed boundary part. For the simplicity of presentation we will consider here only the case of shear-thickening fluids, i.e. $p \geq 2$. The generalization for shear-thinning fluids is a goal of our future research. It may be done in an analogous way as here, but using an appropriate techniques for shear-thinning fluids, e.g. technique of Wolf [40] by using the local pressure method and the Minty theorem for monotone operators as well as the results of Diening, Růžička and Wolf [11].

The problem (1.1)–(1.12) is also a generalization of the problem studied in [15] or [41], where the Newtonian flow was considered, see also [5, 6, 7, 9, 10, 19, 20, 32, 36, 37] for other theoretical results on fluid-structure interaction problems or related problems. Note, however, that in the previous works of one of the author [15, 41] the third order term η_{txx} has been used in order to regularize string model, see also [35, 34]. In this paper we were inspired by work of Grandmont, Desjardin, Esteban, Chambolle [9], where the authors used a different model for structure equation having regularization of the form η_{txxxx}. As far as we know, the question of existence of weak solution of fully unsteady fluid-structure interaction problem with the original generalized string model of Quarteroni [35, 34], i.e. using a regularization of the form η_{txx} for generalized Newtonian fluids is still an open problem.

The proof of the main result formulated in Theorem 1.1 will be realized in several steps:

- approximation of the solenoidal spaces on a moving domain by the artificial compressibility approach: ε - approximation (2.7)
- splitting of the boundary conditions (1.3)–(1.4) by introducing the semi-pervious boundary: κ - approximation (2.5), (2.6)

- assuming a given, sufficiently smooth free boundary deformation $\delta(x_1, t)$ and actual radius $h(t) := R_0 + \delta(t)$ we transform the weak formulation on a time dependent domain $\Omega(h(t)) := \Omega(\delta(t))$ to a fixed reference domain $D = (0, L) \times (0, 1)$, cf. (2.8): h - approximation
- limiting process for $\varepsilon \to 0,\ \kappa \to \infty$
- fixed point procedure for the domain deformation $\eta(x_1, t)$.

The present paper is organized as follows: In the next section we will define weak solution of the fully coupled unsteady fluid-structure interaction and introduce suitable functional spaces. In the Section 2 we will formulate (κ, ε, h) - approximate problem, transform it to a fixed domain and present its weak formulation. The Sections 3 and 4 deal with the existence of a weak solution to our approximate problem. Here we firstly show the existence of weak solutions of stationary problems obtained by time discretization, cf. Section 3. Furthermore, we derive suitable a priori estimates for piecewise approximations in time. By using the compactness arguments due to the Lions-Aubin lemma and the theory of monotone operators we finally show the convergence of time approximations to its weak unsteady solution. Thus we obtain the existence of a weak solution to the (κ, ε, h) - approximate problem. The Section 5 deals with the limiting processes for κ, ε in (2.13). First of all we show the limiting process in $\varepsilon \to 0$ since necessary a priori estimates obtained in Section 4 are independent on ε. In order to realize the limiting process in κ; $\kappa \to \infty$, we however need new a priori estimates and show the equicontinuity in time, cf. Section 5.1. Thus, letting $\varepsilon \to 0$ and $\kappa \to \infty$ we obtain the h - approximate problem depending only on the approximation of the domain deformation $h(x_1, t) = R_0(x_1) + \delta(x_1, t)$. The final step regardning the geometric nonlinearity of the fluid-structure interaction problem will be realized by the Schauder fixed point arguments in Section 6. We will show, that the weak solution of the generalized string equation η is associated with the deformation of the free boundary of the moving domain. This finally yields the existence of at least one weak solution of the fully coupled unsteady fluid-structure interaction between the non-Newtonian shear-dependent fluid and the elastic string.

1.1 Weak formulation

In this subsection our aim is to present the weak formulation of the problem (1.1)–(1.12). Assuming that η is enough regular (see below) and taking into account the results from [9] we can define the functional spaces that gives sense to the trace of velocity from $W^{1,p}(\Omega(\eta(t)))$ and thus to define the weak solution of the problem. We assume that $R_0 \in C_0^2(0, L)$.

Definition 1.1 [Weak formulation]

We say that $(\boldsymbol{v},\eta)$ is a weak solution of (1.1)–(1.12) on $[0,T)$ if the following conditions hold

- $\boldsymbol{v}\in L^p(0,T;W^{1,p}(\Omega(\eta(t))))\cap L^\infty(0,T;L^2(\Omega(\eta(t))))$,
- $\eta\in W^{1,\infty}(0,T;L^2(0,L))\cap H^1(0,T;H_0^2(0,L))$,
- $\operatorname{div}\boldsymbol{v}=0$ a.e. on $\Omega(\eta(t))$,
- $\boldsymbol{v}\big|_{\Gamma_w(t)}=(0,\eta_t)$ for a.e. $x\in\Gamma_w(t),\ t\in(0,T),\ v_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c}=0$,

$$\begin{aligned}
&\int_0^T\int_{\Omega(\eta(t))}\Big\{-\rho\boldsymbol{v}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}+2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})e(\boldsymbol{\varphi})+\rho\sum_{i,j=1}^{2}v_i\frac{\partial v_j}{\partial x_i}\boldsymbol{\varphi}_j\Big\}dx\,dt\\
&+\int_0^T\int_0^{R_0(L)}\Big(P_{out}-\frac{\rho}{2}|v_1|^2\Big)\varphi_1(L,x_2,t)\,dx_2\,dt\\
&-\int_0^T\int_0^{R_0(0)}\Big(P_{in}-\frac{\rho}{2}|v_1|^2\Big)\varphi_1(0,x_2,t)\,dx_2\,dt\\
&+\int_0^T\int_0^L P_w\varphi_2(x_1,R_0(x_1)+\eta(x_1,t),t)-a\frac{\partial^2R_0}{\partial x_1^2}\xi\,dx_1\,dt\\
&+\int_0^T\int_0^L-\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial t}+c\frac{\partial^3\eta}{\partial x_1^2\partial t}\frac{\partial^2\xi}{\partial x_1^2}+a\frac{\partial\eta}{\partial x_1}\frac{\partial\xi}{\partial x_1}+b\eta\,\xi\,dx_1\,dt\;=\;0
\end{aligned}\tag{1.13}$$

for every test functions

$$\begin{aligned}
&\boldsymbol{\varphi}(x_1,x_2,t)\in H^1(0,T;W^{1,p}(\Omega(\eta(t))))\text{ such that}\\
&\operatorname{div}\boldsymbol{\varphi}=0\text{ a.e on }\Omega(\eta(t)),\\
&\varphi_2\big|_{\Gamma_w(t)}\in H^1(0,T;H_0^2(\Gamma_w(t))),\quad \varphi_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c}=\varphi_1\big|_{\Gamma_w(t)}=0\quad\text{and}\\
&\xi(x_1,t)=E\rho\,\varphi_2(x_1,R_0(x_1)+\eta(x_1,t),t).
\end{aligned}\tag{1.14}$$

Theorem 1.1 (Main result: existence of a weak solution)**.**
Let $p\geq 2$. Assume that the boundary data fulfill $P_{in}\in L^{p'}(0,T;L^2(0,R_0(0)))$, $P_{out}\in L^{p'}(0,T;L^2(0,R_0(L)))$, $P_w\in L^{p'}(0,T;L^2(0,L))$, $\frac{1}{p}+\frac{1}{p'}=1$. Furthermore, assume that the properties (3.1)–(3.4) for the viscous stress tensor hold. Then there exists a weak solution $(\boldsymbol{v},\eta)$ of the problem (1.1)-(1.12) such that
i) $\boldsymbol{v}\in L^p(0,T;W^{1,p}(\Omega(\eta(t))))\cap L^\infty(0,T;L^2(\Omega(\eta(t))))$,
$\eta\in W^{1,\infty}(0,T;L^2(0,L))\cap H^1(0,T;H_0^2(0,L))$,
ii) $\boldsymbol{v}\big|_{\Gamma_w(t)}=(0,\eta_t)$ for a.e. $x\in\Gamma_w(t),\ t\in(0,T),\ v_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c}=0$,
iii) $\boldsymbol{v}$ satisfies the condition $\operatorname{div}\boldsymbol{v}=0$ a.e on $\Omega(\eta(t))$ and (1.13) holds.

2 Formulation of the (κ, ε, h) - problem

In what follows we will formulate a suitable approximation of the original problem (1.1)–(1.12). We will call this approximation the (κ, ε, h) - problem.

First of all we approximate the deformable boundary Γ_w by a given function $h = R_0 + \delta$, $\delta \in H^1(0,T; H_0^2(0,L)) \cap W^{1,\infty}(0,T; L^2(0,L))$, $R_0(x_1) \in C^2[0,L]$ satisfying for all $x_1 \in [0,L]$

$$0 < \alpha \leq h(x_1,t) \leq \alpha^{-1}, \quad \left|\frac{\partial h(x_1,t)}{\partial x_1}\right| + \int_0^T \left|\frac{\partial h(x_1,t)}{\partial t}\right|^2 dt \leq K \tag{2.1}$$

$h(0,t) = R_0(0), \quad h(L,t) = R_0(L)$.

We look for a solution $(\boldsymbol{v}, \pi, \eta)$ of the following problem

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \operatorname{div}[2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})] - \nabla \pi \quad \text{in} \quad \Omega(h(t)), \tag{2.2}$$

and for all $x_1 \in (0,L)$, see (1.6), $0 < t < T$

$$-E\rho \left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta + c\frac{\partial^5 \eta}{\partial t \partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \tag{2.3}$$

$$\left[\mu(|e(\boldsymbol{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial h}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - \pi + P_w\right](\bar{x},t),$$

$$\boldsymbol{v}(\bar{x},t) = \left(0, \frac{\partial \eta}{\partial t}(x_1,t)\right), \tag{2.4}$$

$\bar{x} = (x_1, h(x_1,t))$.

Furthermore, in the analysis of problem (1.1)–(1.12) the boundary condition (1.3)-(1.4), cf. (2.3)-(2.4), is splitted in the following way, see [15]

$$\left[\mu(|e(\boldsymbol{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial h}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - \pi + P_w\right](\bar{x},t) \tag{2.5}$$
$$-\frac{\rho}{2}v_2\left(v_2(\bar{x},t) - \frac{\partial h}{\partial t}(x_1,t)\right) = \rho\kappa\left[\frac{\partial \eta}{\partial t}(x_1,t) - v_2(\bar{x},t)\right]$$

and

$$-E\left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta + c\frac{\partial^5 \eta}{\partial t \partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \kappa\left[\frac{\partial \eta}{\partial t}(x_1,t) - v_2(\bar{x},t)\right] \tag{2.6}$$

with $\kappa \gg 1$.

We will show later, that the approximation with κ is reasonable. One of the possible physical interpretations for introducing finite κ comes from the mathematical modeling of semi-pervious boundary, where this type of boundary condition occurs. In our case, the boundary Γ_w seems to be partly permeable for finite κ, but letting $\kappa \to \infty$ it becomes impervious. In fact, we prove the existence of solution if $\kappa \to \infty$ and thus we get the original boundary condition (2.3)-(2.4).

Furthermore, we overcome the difficulties with solenoidal spaces by means of the artificial compressibility. We approximate the continuity equation similarly as in [15] with

$$\begin{aligned}
&\varepsilon\left(\frac{\partial \pi_\varepsilon}{\partial t} - \Delta\pi_\varepsilon\right) + \operatorname{div} \boldsymbol{v}_\varepsilon = 0 \quad \text{in } \Omega(h(t)),\ t \in (0,T), \qquad (2.7)\\
&\frac{\partial \pi_\varepsilon}{\partial \boldsymbol{n}} = 0, \quad \text{on } \partial\Omega(h(t)),\ t \in (0,T), \quad \pi_\varepsilon(0) = 0 \text{ in } \Omega(h(0)), \quad \varepsilon > 0.
\end{aligned}$$

By letting $\varepsilon \to 0$ we show that $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$, where $\boldsymbol{v}$ is the weak solution of (1.1). For fixed ε, due to the lack of solenoidal property for velocity, we have the additional term in momentum equation $(1.1)_1$ $\frac{\varrho}{2} v_i \operatorname{div} \boldsymbol{v}$, see also [38].

Our approximated problem is defined on a moving domain depending on the function $h = R_0 + \delta$, cf. (2.1). Now we will reformulate it to a fixed rectangular domain. Set

$$\begin{aligned}
\boldsymbol{u}(y_1, y_2, t) &\overset{\text{def}}{=} \boldsymbol{v}(y_1, h(y_1,t)y_2, t)\\
q(y_1, y_2, t) &\overset{\text{def}}{=} \rho^{-1}\pi(y_1, h(y_1,t)y_2, t) \qquad (2.8)\\
\sigma(y_1, t) &\overset{\text{def}}{=} \frac{\partial \eta}{\partial t}(y_1, t)
\end{aligned}$$

for $y \in D = \{(y_1, y_2);\ 0 < y_1 < L,\ 0 < y_2 < 1\}$, $0 < t < T$.

We define the following space

$$\begin{aligned}
\boldsymbol{V} &\equiv \left\{\boldsymbol{w} \in W^{1,p}(D) :\ w_1 = 0 \text{ on } S_w \text{ and } w_2 = 0 \text{ on } S_{in} \cup S_{out} \cup S_c\right\},\\
S_w &= \{(y_1, 1) : 0 < y_1 < L\}, \qquad S_{in} = \{(0, y_2) : 0 < y_2 < 1\}, \qquad (2.9)\\
S_{out} &= \{(L, y_2) : 0 < y_2 < 1\}, \qquad S_c = \{(y_1, 0) : 0 < y_1 < L\}.
\end{aligned}$$

Let us introduce the following notations

$$\operatorname{div}_h \boldsymbol{u} \overset{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2},$$

$$\begin{aligned}
a_1(q,\phi) = \int_D \Bigg\{ &\left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_1} \qquad (2.10)\\
&+ \left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_2}\Bigg\} dy,
\end{aligned}$$

viscous term

$$\begin{aligned}
&((\boldsymbol{u}, \boldsymbol{\psi})) = \int_D h\tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{\psi}) dy, \qquad (2.11)\\
&\tau_{ij}(\hat{e}(\boldsymbol{u})) = 2\rho^{-1}\mu(|\hat{e}(\boldsymbol{u})|)\hat{e}_{ij}(\boldsymbol{u}), \qquad \hat{e}_{ij}(\boldsymbol{u}) = \frac{1}{2}(\hat{\partial}_i(u_j) + \hat{\partial}_j(u_i)),\\
&\hat{\partial}_1 = \left(\frac{\partial}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_2}\right), \quad \hat{\partial}_2 = \frac{1}{h}\frac{\partial}{\partial y_2},
\end{aligned}$$

convective term

$$
\begin{aligned}
b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) \;=\; & \int_D \left(hu_1 \left(\frac{\partial \boldsymbol{z}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{z}}{\partial y_2} \right) \cdot \boldsymbol{\psi} + \frac{h}{2}\, \boldsymbol{z}\cdot\boldsymbol{\psi}\, \mathrm{div}_h \boldsymbol{u}\, dy \\
& -\frac{1}{2}\int_0^1 R_0 u_1 z_1 \psi_1\,(L,y_2)\,dy_2 + \frac{1}{2}\int_0^1 R_0 u_1 z_1 \psi_1\,(0,y_2)\,dy_2 \\
& -\frac{1}{2}\int_0^L u_2 z_2 \psi_2\,(y_1,1)\,dy_1. \qquad (2.12)
\end{aligned}
$$

Remark: Since the terms defined in (2.10), (2.11) and (2.12) are dependent on the domain deformation h, it will be sometimes useful to denote this explicitely, e.g., $b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) = b_h(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi})$.

Definition 2.1 [Weak solution of (κ,ε,h) - approximate problem]
Let $\boldsymbol{u} \in L^p(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D))$, $q \in L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ and $\sigma \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$. A triple $\boldsymbol{w} = (\boldsymbol{u},q,\sigma)$ is called a weak solution of the regularized problem (1.1)–(1.12) if the following equation holds

$$
\begin{aligned}
& -\int_0^T \left\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \;= \\
& \int_0^T \int_D \left(-\frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}) - h\, q\, \mathrm{div}_h \boldsymbol{\psi} \right) dy \,+ ((\boldsymbol{u},\boldsymbol{\psi}))\, dt \\
& + \int_0^T \int_0^1 h(L,t) q_{out} \psi_1\;(L,y_2,t) - h(0,t) q_{in} \psi_1\;(0,y_2,t)\; dy_2 dt \\
& + \int_0^T \int_0^L \left(q_w + \frac{1}{2} u_2 \frac{\partial h}{\partial t} + \kappa\,(u_2 - \sigma) \right) \psi_2\;(y_1,1,t)\,dy_1 dt \\
& + \varepsilon \int_0^T \left\langle \frac{\partial (hq)}{\partial t}, \phi \right\rangle dt \qquad (2.13) \\
& + \int_0^T \int_D \left(-\varepsilon \frac{\partial h}{\partial t}\frac{\partial (y_2 q)}{\partial y_2}\phi + \varepsilon a_1(q,\phi) + \; h\, \mathrm{div}_h \boldsymbol{u}\; \phi \right) dy\, dt \\
& + \frac{\varepsilon}{2} \int_0^T \int_0^L \frac{\partial h}{\partial t}(y_1,t) q \phi(y_1,1,t)\,dy_1 dt \,+ \\
& + \int_0^T \int_0^L \left(\frac{\partial \sigma}{\partial t}\xi + c\,\frac{\partial^2 \sigma}{\partial y_1^2}\frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1}\int_0^t \sigma(y_1,s)ds \frac{\partial \xi}{\partial y_1} \right. \\
& \qquad \left. + b\int_0^t \sigma(y_1,s)ds\; \xi - a\frac{\partial^2 R_0}{\partial y_1^2}\xi + \frac{\kappa}{E}\,(\sigma - u_2)\,\xi \right) (y_1,t)\,dy_1 dt
\end{aligned}
$$

for every $(\boldsymbol{\psi},\phi,\xi) \in H_0^1(0,T;\boldsymbol{V}) \times L^2(0,T;H^1(D)) \times L^2(0,T;H_0^2(0,L))$. Here we remind $E = \tilde{E}\sqrt{1+(\partial_{y_1}R_0)^2}$. For simplicity and without lost of generality we assume in what follows that E, a, b, c are constant, cf. (2.1).

3 Existence of stationary solution

3.1 Preliminary properties for the shear-dependent model

Let us first specify the shear-dependent fluids that will be considered in this paper. We assume that there exists a potential $\mathcal{U} \in C^2(\mathbb{R}^{2\times 2})$ of shear stress tensor τ, such that for some $1 < p < \infty$, $C_1,\ C_2 > 0$ we have for all $\eta,\ \xi \in \mathbb{R}^{2\times 2}_{sym}$ and $i,\ j,\ k,\ l \in \{1, 2\}$, cf. [28]

$$\begin{aligned} \frac{\partial \mathcal{U}(\eta)}{\partial \eta_{ij}} &= \tau_{ij}(\eta) && (3.1)\\ \mathcal{U}(\mathbf{0}) = \frac{\partial \mathcal{U}(\mathbf{0})}{\partial \eta_{ij}} &= 0 && (3.2)\\ \frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{mn}\partial \eta_{rs}}\xi_{mn}\xi_{rs} &\geq C_1\ (1+|\eta|)^{p-2}|\xi|^2 && (3.3)\\ \left|\frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{ij}\partial \eta_{kl}}\right| &\leq C_2(1+|\eta|)^{p-2}. && (3.4) \end{aligned}$$

Note, that the stress tensor $\tau_{ij} = 2\rho^{-1}\mu(|\hat{e}(\boldsymbol{u})|)\hat{e}_{ij}(\boldsymbol{u})$ with $\mu(|\hat{e}(\boldsymbol{u})|)$ defined in (1.2) satisfies (3.1)–(3.4).

In what follows we show some suitable properties, that will be needed in order to obtain a priori estimates. We use notations $\|\cdot\|_p := \|\cdot\|_{L^p(D)}$, $\|\cdot\|_{1,p} := \|\cdot\|_{W^{1,p}(D)}$.

Lemma 3.1 (Interpolation inequality)**.**
Let φ be any function in $H^1(D)$ such that $\varphi = 0$ on S_w or S_c. Then there exists a constant $C = C(p, \theta)$ such that

$$\|\varphi\|_p \leq c\|\nabla\varphi\|_2^{\theta}\|\varphi\|_2^{1-\theta} \quad \textit{for} \qquad \frac{p-2}{p} \leq \theta \leq 1, \quad p \geq 2, \qquad (3.5)$$

Proof. See the Nirenberg-Gagliardo inequality [22] and [28].

Lemma 3.2.
Denote $S \equiv S_{in} \cup S_{out} \cup S_w \cup S_c$. Let φ be any function in $W^{1,p}(D)$ such that $\varphi = 0$ on S_w or S_c. Then for any $1 < r < \infty$ we have

$$\|\varphi\|_{L^r(S)} \leq c(r)\|\nabla\varphi\|_{L^2(D)}^{1-\frac{1}{r}}\|\varphi\|_{L^2(D)}^{\frac{1}{r}}. \qquad (3.6)$$

Proof. Analogous to the proof of Proposition 3.2 in [41].

Lemma 3.3 (Ellipticity of the form $a_1(\cdot,\cdot)$)**.**
Let the assumptions (2.1) on $h(x_1, t)$ be satisfied. Then

$$a_1(q, q) \geq \frac{\alpha}{2+K^2}\int_D |\nabla q|^2\, dy \qquad (3.7)$$

for any $q \in H^1(D)$, the form $a_1(\cdot,\cdot)$ is given by (2.10).

Proof. The proof can be found in [15, 41].

Lemma 3.4 (Coercivity of the viscous form)**.**
The viscous form defined in (2.11) satisfies for any $2 \le p < \infty$ the following estimates. There exists $\tilde{\delta} = \tilde{\delta}(K, \alpha) > 0$ such that

$$1)\ ((\boldsymbol{u}, \boldsymbol{u})) \ge \tilde{\delta}\|\boldsymbol{u}\|_{1,p}^p + \tilde{\delta}\|\boldsymbol{u}\|_{1,2}^2$$

$$2)\ ((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)) \ge \tilde{\delta}\int_D |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^2 + |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^p$$

$$3)\ ((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)) \ge 0.$$

Proof. Assertion 1). We have

$$\begin{aligned}
((\boldsymbol{u}, \boldsymbol{u})) &= \int_D h\tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{u}) = \int_D h\int_0^1 \frac{d}{ds}\frac{\partial \mathcal{U}(s\hat{e}(\boldsymbol{u}))}{\partial \hat{e}_{ij}}\, ds\, \hat{e}_{ij}(\boldsymbol{u}) \\
&= \int_D h\int_0^1 \frac{\partial^2 \mathcal{U}(s\hat{e}(\boldsymbol{u}))}{\partial \hat{e}_{ij}\partial \hat{e}_{kl}}\, ds\, \hat{e}_{kl}(\boldsymbol{u})\hat{e}_{ij}(\boldsymbol{u}) \overset{(3.3)}{\ge} C_1\alpha\int_D\int_0^1 (1 + s|\hat{e}(\boldsymbol{u})|)^{p-2} ds\, |\hat{e}(\boldsymbol{u})|^2 \\
&\overset{(1+s|\hat{e}|)^{p-2}\ge\frac{1}{2}(1+(s|\hat{e}|)^{p-2})}{\ge} \frac{C_1\alpha}{2}\int_D\int_0^1 1 + (s|\hat{e}(\boldsymbol{u})|)^{p-2} ds\, |\hat{e}(\boldsymbol{u})|^2 \\
&= \frac{C_1\alpha}{2}\int_D |\hat{e}(\boldsymbol{u})|^2 + \frac{C_1\alpha}{2(p-1)}\int_D s^{p-1}|\hat{e}(\boldsymbol{u})|^p\Big|_{s=0}^{s=1} |\hat{e}(\boldsymbol{u})| \\
&= \frac{C_1\alpha}{2}\int_D |\hat{e}(\boldsymbol{u})|^2 + \frac{1}{p-1}|\hat{e}(\boldsymbol{u})|^p.
\end{aligned}$$

Now we apply the generalized Korn's inequality, see [21, 30, 31, 33]. Indeed, $\hat{e}(\boldsymbol{u})$ could be written in the form:

$$\begin{aligned}
&\hat{e}(\boldsymbol{u}) = \nabla\boldsymbol{u}F(y_1) + (\nabla\boldsymbol{u}F(y_1))^T \in \mathbb{R}^{2\times 2}_{sym}, \qquad \text{where} \\
&F(y) = F(h, y_1) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -\frac{y_2}{h(y_1,t)}\frac{\partial h(y_1,t)}{\partial y_1} & \frac{1}{h(y_1,t)} \end{pmatrix}.
\end{aligned} \tag{3.8}$$

Since $F : \bar{D} \mapsto \mathbb{R}^{2\times 2}$ has a bounded inverse mapping, $\det F(y) = \frac{1}{h}$ and $\boldsymbol{u} \in \boldsymbol{V}$ (vanishing on some open subset of ∂D), according to Neff [31, Theorem 6] we get

$$\int_D |\hat{e}(\boldsymbol{u})|^p \ge c(K, \alpha)\int_D |\nabla\boldsymbol{u}|^p. \tag{3.9}$$

We should point out that the proof of this generalization of Korn's inequality with variable coefficient in [33, 31] could be performed also for $\boldsymbol{u}$ vanishing on S component-wisely, a.e. $u_1 = 0$ on S_w, $u_2 = 0$ on $S_{out} \cup S_c \cup S_{in}$.

Assertions 2), 3) are proven in [28, Lemma 1.19]. Note that applying (3.9) we obtain norms $\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,2}^2$, $\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p}^p$ on the right hand sides of assertion 2). ■

Lemma 3.5 (Boundedness of the viscous form)**.**
Let $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{V}$, then for $2 \le p < \infty$ it holds

$$((\boldsymbol{u},\boldsymbol{v})) \le C\|\boldsymbol{u}\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p}+C_0\|\boldsymbol{u}\|_{1,p}\|\boldsymbol{v}\|_{1,p}, \quad C_0 > 0. \tag{3.10}$$

Proof. We have

$$((\boldsymbol{u},\boldsymbol{v})) = \int_D h\tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}(\boldsymbol{v}) = \int_D h\int_0^1 \frac{d}{ds}\frac{\partial\mathcal{U}(s\hat{e}(\boldsymbol{u}))}{\partial\hat{e}_{ij}}\,ds\ \hat{e}_{ij}(\boldsymbol{v})$$
$$= \int_D h\int_0^1 \frac{\partial^2\mathcal{U}\left(s\hat{e}(\boldsymbol{u})\right)}{\partial\hat{e}_{ij}\partial\hat{e}_{kl}}\hat{e}_{kl}(\boldsymbol{u})\ ds\ \hat{e}_{ij}(\boldsymbol{v})$$
$$\overset{(3.4)}{\le} C_2\int_D\int_0^1 (1+s|\hat{e}(\boldsymbol{u})|)^{p-2}|\hat{e}(\boldsymbol{u})|\ ds|\hat{e}(\boldsymbol{v})|.$$

Now, we can estimate the right hand side of the above inequality as follows.

$$\int_D\int_0^1 (1+s|\hat{e}(\boldsymbol{u})|)^{p-2}|\hat{e}(\boldsymbol{u})|\ ds|\hat{e}(\boldsymbol{v})| = \frac{c}{p-1}\int_D [(1+|\hat{e}(\boldsymbol{u})|)^{p-1}-1]|\hat{e}(\boldsymbol{v})|$$
$$\le \frac{c}{p-1}\int_D \left(|\hat{e}(\boldsymbol{u})|^{p-1}+|\hat{e}(\boldsymbol{u})|\right)|\hat{e}(\boldsymbol{v})|$$
$$\le c\left(\int_D|\hat{e}(\boldsymbol{u})|^p\right)^{\frac{p-1}{p}}\left(\int_D|\hat{e}(\boldsymbol{v})|^p\right)^{\frac{1}{p}} + c\left(\int_D|\hat{e}(\boldsymbol{u})|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_D|\hat{e}(\boldsymbol{v})|^p\right)^{\frac{1}{p}}$$
$$\le c\|\boldsymbol{u}\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p} + c\|\boldsymbol{u}\|_{1,p}\|\boldsymbol{v}\|_{1,p}.$$

Here we have used the fact, that for $x \ge 0$ it holds $(1+x)^{p-1}-1 \le c_1x^{p-1}+c_2x$, which can be proven easily, see also [28, Chapter 5]. The last inequality follows from the imbedding $W^{1,p}(D) \hookrightarrow L^r(D)$ for any $\infty > r \ge 1$ if $p \ge 2$. ■

Lemma 3.6 (Continuity of the viscous form)**.**
For $\boldsymbol{u}_1,\boldsymbol{u}_2 \in \boldsymbol{V},\ \boldsymbol{v} \in C^1$ following estimate holds

$$((\boldsymbol{u}^1,\boldsymbol{v})) - ((\boldsymbol{u}^2,\boldsymbol{v})) \le C\|I_p\|_{p/(p-1)}\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,p}\|\boldsymbol{v}\|_{C^1},$$

where $I_p := \int_0^1(1+|s\hat{e}(\boldsymbol{u}^1-\boldsymbol{u}^2)+\hat{e}(\boldsymbol{u}^2)|)^{p-2}ds$ is bounded in $L^{p/p-1}(D)$.

Proof. We have using (3.4) and the Hölder inequality

$$((\boldsymbol{u}^1,\boldsymbol{v})) - ((\boldsymbol{u}^2,\boldsymbol{v})) = \int_D h\left[\tau_{ij}(\hat{e}(\boldsymbol{u}^1)) - \tau_{ij}(\hat{e}(\boldsymbol{u}^2))\right]\hat{e}_{ij}(\boldsymbol{v})$$
$$= \int_D h\int_0^1 \frac{d}{ds}\frac{\partial U\left(s\hat{e}(\boldsymbol{u}^1)+(1-s)\hat{e}(\boldsymbol{u}^2)\right)}{\partial\hat{e}_{ij}}\ ds\ \hat{e}_{ij}(\boldsymbol{v})$$
$$= \int_D h\int_0^1 \frac{\partial^2 U\left(s\hat{e}(\boldsymbol{u}^1)+(1-s)\hat{e}(\boldsymbol{u}^2)\right)}{\partial\hat{e}_{ij}\partial\hat{e}_{kl}}\left(\hat{e}_{kl}(\boldsymbol{u}^1)-\hat{e}_{kl}(\boldsymbol{u}^2)\right)\ ds\ \hat{e}_{ij}(\boldsymbol{v})$$
$$\overset{(3.4)}{\le} C_2\int_D\int_0^1 (1+|s\hat{e}(\boldsymbol{u}^1-\boldsymbol{u}^2)+\hat{e}(\boldsymbol{u}^2)|)^{p-2}\ ds|\hat{e}(\boldsymbol{u}^1)-\hat{e}(\boldsymbol{u}^2)||\hat{e}(\boldsymbol{v})|$$
$$\le C\|I_p\|_{p/(p-1)}\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,p}\|\boldsymbol{v}\|_{C^1}.$$
■

Lemma 3.7 (Nonlinear convective term $b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi})$).
For the trilinear form $b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi})$, defined in (2.12) the following properties hold

$$
\begin{aligned}
b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) &= \frac{1}{2}B(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) - \frac{1}{2}B(\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{z}), \qquad (3.11)\\
\textit{where} \quad B(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) &\equiv \int_D \left(hu_1 \left(\frac{\partial \boldsymbol{z}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{z}}{\partial y_2} \right) \cdot \boldsymbol{\psi}\, dy.
\end{aligned}
$$

Moreover for $p \geq 2$ we have

$$
|B(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi})| \leq c\|\boldsymbol{u}\|_{1,p}\|\boldsymbol{z}\|_{1,p}\|\boldsymbol{\psi}\|_{1,p}.
$$

Proof. The assertion (3.11) is obtained by integration by parts in the first integral term of (2.12) (in the term $\frac{1}{2}B(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi})$), see also [15, 41]. The last property follows easily from the Hölder inequality and the imbedding $W^{1,p}(D) \hookrightarrow L^r(D)$ for any $1 \leq r < \infty$ and $2 \leq p < \infty$. ■

3.2 Stationary problem

In this section we approximate the problem (2.13) by a sequence of stationary problems obtained by the implicit time discretization and show the existence of weak solution for one discrete time step. Thus let us approximate time derivatives by means of first order backward finite differences

$$
\frac{\partial(h\boldsymbol{u})}{\partial t} \approx \frac{h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}}{\Delta t}, \; \frac{\partial(hq)}{\partial t} \approx \frac{h^i q^i - h^{i-1}q^{i-1}}{\Delta t}, \; \frac{\partial \sigma}{\partial t} \approx \frac{\sigma^i - \sigma^{i-1}}{\Delta t},
$$

where $\boldsymbol{u}^i$, q^i and σ^i denote approximations of unknown $\boldsymbol{u}$, q and σ at time instance $i\Delta t$, e.g., $\boldsymbol{u}^i(y) = \boldsymbol{u}(y, i\Delta t)$. We replace $\int_0^t \sigma(s)\, ds$ by $\sum_{k=1}^i \sigma^k \Delta t$. Moreover, for given functions we use the following notations

$$
h^i(y_1) = h(y_1, i\Delta t), \quad \text{and} \quad q^i_{in}(y_2) = \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} q_{in}(y_2, s)\, ds,
$$

similarly q^i_{out}, q^i_w.
Let us introduce the following space

$$
V \equiv \boldsymbol{V} \times H^1(D) \times H^2_0(0,L). \qquad (3.12)
$$

After the time discretization the following variational problem is obtained from (2.13). Find $\boldsymbol{w}^i = (\boldsymbol{u}^i, q^i, \sigma^i) \in V$ such that

$$
a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) + \mathcal{B}^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) = L^i(\boldsymbol{\varpi}) \qquad \forall \boldsymbol{\varpi} \in V, \qquad (3.13)
$$

where $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta)$ and $\mathcal{B}^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) := b_{h^i}(\boldsymbol{u}^i, \boldsymbol{u}^i, \boldsymbol{\omega})$, see also (2.12). Further

1. $a^i(\cdot,\cdot): V \times V \mapsto \mathbb{R}$ is the following form on V

$$
\begin{aligned}
a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) &= ((\boldsymbol{u}^i, \boldsymbol{\omega})) + \varepsilon a_1(q^i, v) + \frac{1}{\Delta t}\int_D h^i\,(\boldsymbol{u}^i\boldsymbol{\omega} + q^i\,v)\,dy \\
&+ \int_0^L \left(c\frac{\partial^2\sigma^i}{\partial y_1^2}\frac{\partial^2\vartheta}{\partial y_1^2} + a\Delta t\frac{\partial\sigma^i}{\partial y_1}\frac{\partial\vartheta}{\partial y_1} + \left(\frac{1}{\Delta t} + b\Delta t\right)\sigma^i\vartheta \right) dy_1 \\
&- \int_D \frac{h^i - h^{i-1}}{\Delta t}\,\frac{\partial(y_2\boldsymbol{u}^i)}{\partial y_2}\boldsymbol{\omega}\,dy + \int_0^L \frac{1}{2}u_2^i\frac{h^i - h^{i-1}}{\Delta t}\omega_2\,(y_1,1)\,dy_1 \\
&- \varepsilon\int_D \frac{h^i - h^{i-1}}{\Delta t}\,\frac{\partial(y_2 q^i)}{\partial y_2}v\,dy + \frac{\varepsilon}{2}\int_0^L \frac{h^i - h^{i-1}}{\Delta t}\,q^i\,v\,(y_1,1)\,dy_1 \\
&+ \kappa\int_0^L (\sigma^i - u_2^i)\left(\frac{\vartheta}{E} - \omega_2\right)(y_1)\,dy_1 \\
&+ \int_D \left(h^i\,v\,\mathrm{div}_{h^i}\boldsymbol{u}^i - h^i q^i\,\mathrm{div}_{h^i}\boldsymbol{\omega}\right)dy,
\end{aligned}
$$

see also (2.10) and (2.11).
2. The trilinear form $\mathcal{B}^i(\cdot,\cdot,\cdot): V \times V \times V \to \mathbb{R}$ is defined by (2.12). Note that $\mathcal{B}^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{w}^i) = b_{h^i}(\boldsymbol{u}^i, \boldsymbol{u}^i, \boldsymbol{u}^i) = 0$, see Lemma 3.7, (3.11).
3. Finally, $L^i(\cdot)$ is the linear functional on V, such that

$$
\begin{aligned}
L^i(\boldsymbol{\varpi}) &= \frac{1}{\Delta t}\int_D h^{i-1}\left(\boldsymbol{u}^{i-1}\boldsymbol{\omega} + \varepsilon\,q^{i-1}\,v\right)\,dy + \frac{1}{\Delta t}\int_0^L \sigma^{i-1}\vartheta\,dy_1 \\
&\quad + \int_0^1 \left(h^i(0)q_{in}^i\,\omega_1\,(0,y_2) - h^i(L)q_{out}^i\,\omega_1\,(L,y_2)\right)dy_2 \\
&+ \int_0^L \left(-q_w^i\,\omega_2\,(y_1,1) - \sum_{k=1}^{i-1}\left(a\frac{\partial\sigma^k}{\partial y_1}\frac{\partial\vartheta}{\partial y_1} + b\sigma^k\vartheta \right)\Delta t + a\frac{\partial^2 R_0}{\partial y_1^2}\vartheta \right) dy_1.
\end{aligned}
$$

3.2.1 Existence of finite-dimensional solution

The existence of stationary solution is the consequence of coercivity of the viscosity form $((\cdot,\cdot))$ and of $a_1(\cdot,\cdot)$, see Lemma 3.4, Lemma 3.3, the continuity of these forms, see Lemma 3.6, and of the following lemma.

Lemma 3.8. *Let Y be a finite-dimensional Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let P be a continuous mapping from Y into itself, such that for a sufficiently large $\varrho > 0$,*

$$
(P(\zeta), \zeta) \geq 0 \quad \forall \zeta \in Y \text{ such that } \|\zeta\| = \varrho. \tag{3.14}
$$

Then there exists $\zeta \in Y$, $\|\zeta\| \leq \varrho$ such that $P(\zeta) = 0$.

Proof. See [38, Lemma 2.1.4]. ∎

The proof of existence of the finite-dimensional solution to (3.13) is analogous to the proof given in [15] or [41, Theorem 4.1]. In our case the finite-dimensional Hilbert space $Y = \mathscr{V}^m = \mathrm{span}\{\xi_1, \ldots \xi_m\}$, $\xi_k \in C^2$ is equipped with the norm $\|\cdot\|_{1,2}$ and P is a continuous mapping from Y into itself given by

$$(P(\boldsymbol{\zeta}), \boldsymbol{z}) = a^i(\boldsymbol{\zeta}, \boldsymbol{z}) + \mathcal{B}^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{z}) - L^i(\boldsymbol{z}) \quad \forall \boldsymbol{z} \in Y.$$

From Lemmas 3.5, 3.6 and 3.7 it is easy to see, that the assumption of continuity of P is fulfilled. By means of Lemmas 3.3 and 3.4 we obtain the property (3.14) as follows. For $\boldsymbol{\zeta} = (\boldsymbol{u}^i, q^i, E\sigma^i)$ one can verify that

$$\begin{aligned} a^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}) &\geq ((\boldsymbol{u}^i, \boldsymbol{u}^i)) + \varepsilon a_1(q^i, q^i) \\ &+ \int_D \left[\frac{h^i}{\Delta t} + \frac{1}{2} \frac{h^i - h^{i-1}}{\Delta t} \right] \left(|\boldsymbol{u}^i|^2 + \varepsilon |q^i|^2 \right) \, dy \\ &+ E \int_0^L c \left| \frac{\partial^2 \sigma^i}{\partial y_1^2} \right|^2 + a\Delta t \left| \frac{\partial \sigma^i}{\partial y_1} \right|^2 + \left(\frac{1}{\Delta t} + b\Delta t \right) |\sigma^i|^2 dy_1. \end{aligned}$$

Using coercivity of the forms $((\cdot, \cdot))$ and $a_1(\cdot, \cdot)$, we have for sufficiently small $\Delta t < \alpha/K$, cf. (2.1)

$$\begin{aligned} (P(\boldsymbol{\zeta}), \boldsymbol{\zeta}) &\geq \tilde{\delta} \|\boldsymbol{u}^i\|_{1,2}^2 + \frac{\varepsilon \alpha}{2 + K^2} \|q^i\|_{1,2}^2 + cE \|\sigma^i\|_{2,2}^2 - L^i(\boldsymbol{\zeta}) \\ &\geq C_{coerc} \|\boldsymbol{\zeta}\|_{1,2}^2 - C_L \|\boldsymbol{\zeta}\|_{1,2}. \end{aligned} \tag{3.15}$$

Thus $(P(\boldsymbol{\zeta}), \boldsymbol{\zeta}) \geq 0$ for e.g., $\boldsymbol{\zeta}$ such that $\|\boldsymbol{\zeta}\|_{1,2} = \varrho = \frac{C_L}{C_{coerc}}$.

Now we use Lemma 3.8 and obtain the existence of stationary weak solution to problem (3.13) $\boldsymbol{w}^m = (\boldsymbol{u}^m, q^m, \sigma^m)$, (written without temporal index i)

$$\boldsymbol{w}^m = \sum_{k=1}^m c_k^m \xi_k \in \mathscr{V}^m, \text{ such that } \|\boldsymbol{w}^m\|_{1,2} \leq \varrho = \frac{C_L}{C_{coerc}}. \tag{3.16}$$

In order to get further a priori estimates in $W^{1,p}(D)$ for $\boldsymbol{u}$ and in $H^2(0, L)$ for σ, we test (3.13) by $\boldsymbol{w}^m = (\boldsymbol{u}^m, q^m, E\sigma^m)$ and come to

$$a^i(\boldsymbol{w}^m, \boldsymbol{w}^m) = L^i(\boldsymbol{w}^m) \quad \forall\, \boldsymbol{w}^m \in \mathscr{V}^m. \tag{3.17}$$

Similarly as above using the coercivity property we obtain from (3.17) for sufficiently small Δt

$$C_L \|\boldsymbol{w}^m\|_{1,2} \geq \tilde{\delta} \|\boldsymbol{u}^m\|_{1,p}^p + \tilde{\delta} \|\boldsymbol{u}^m\|_{1,2}^2 + \frac{\varepsilon \alpha}{2 + K^2} \|q^m\|_{1,2}^2 + cE \|\sigma^m\|_{2,2}^2.$$

Considering (3.16) we get consequently

$$\|\boldsymbol{u}^m\|_{1,p}^p + \|\sigma^m\|_{2,2}^2 \leq C. \tag{3.18}$$

The boundedness in the reflexive Banach space $W^{1,p}(D)$, $W^{1,2}(D)$, $H^2(0, L)$ and the compact imbedding arguments, see [38, Theorem 1.1],

$$W^{1,p}(D) \quad \Subset \quad L^r(D), \quad \infty > r \geq 1, \tag{3.19}$$

imply the following convergences

$$\begin{array}{ll} \boldsymbol{u}^m \to \boldsymbol{u} \ \text{ in } L^r(D), \quad \infty > r \geq 1, & \\ q^m \to q \ \text{ in } L^r(D), \quad \infty > r \geq 1, & \boldsymbol{u}^m \rightharpoonup \boldsymbol{u} \ \text{ in } W^{1,p}(D), \\ \sigma^m \to \sigma \ \text{ in } L^r(0,L), \quad \infty > r \geq 1, & q^m \rightharpoonup q \ \text{ in } W^{1,2}(D), \\ \nabla\sigma^m \to \nabla\sigma \ \text{ in } L^2(0,L), & \sigma^m \rightharpoonup \sigma \ \text{ in } H^2(0,L). \end{array} \tag{3.20}$$

In view of the results (3.20) we pass to the limit for $m \to \infty$ and obtain the solution of infinitely dimensional stationary problem. The details of the limiting process are omitted here, cf. [15]. In order to pass to the limit in the nonlinear viscous term the technique of monotone operators is used, cf. also Section 4.

Let us summarize the main result of the Section 3 in the following theorem.

Theorem 3.1 (Stationary solution)**.**
Let $i \in \{1, 2, \ldots, n\}$ and $\boldsymbol{w}^j \in V$ for $j \leq i-1$ be given. Assume (3.1)–(3.4) and (2.1) hold; i.e. there are non-negative constants α, K, independent on i, such that

$$0 < \alpha \leq h^i(y_1) \leq \alpha^{-1} \quad \textit{and} \quad \left|\frac{\partial h^i}{\partial y_1}(y_1)\right| + \sum_{i=0}^{n} \left|\frac{h^i - h^{i-1}}{\Delta t}(y_1)\right|^2 \Delta t \leq K$$

for all $0 \leq y_1 \leq L$ and $i = 1, 2, \ldots n$. Moreover, assume that $q^i_{in}, q^i_{out} \in L^2(0,1)$, $q^i_w \in L^2(0,L)$ Then the problem (3.13) has at least one solution.

4 Existence of unsteady solution

4.1 A priori estimates

In this section we derive suitable a priori estimates for the sequence of piecewise constant and piecewise linear approximations in time of the weak solution. Since our ultimate goal is to let the parameter $\kappa \to \infty$, we would like to obtain estimates independent on κ.

We first rewrite (2.13) for piecewise constant $\boldsymbol{u}$, q, σ, replace time derivative in (2.13) with backward difference and replace integration in time by sum over $i = 1, 2, \ldots r$, $r \leq n$. This yields

$$\begin{aligned} \sum_{i=1}^{r} \Bigg[\int_D \Bigg\{ & \left(\frac{h^i \boldsymbol{u}^i - h^{i-1} \boldsymbol{u}^{i-1}}{\Delta t} - \frac{h^i - h^{i-1}}{\Delta t} \frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2} \right) \boldsymbol{\omega} \\ & + \left(h^i u_1^i \left(\frac{\partial \boldsymbol{u}^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) + u_2^i \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) \boldsymbol{\omega} + \frac{h^i}{2} \boldsymbol{u}^i \boldsymbol{\omega} \operatorname{div}_{h^i} \boldsymbol{u}^i \\ & + ((u^i, \boldsymbol{\omega})) + \varepsilon a_1(q^i, v) - h^i q^i \operatorname{div}_{h^i} \boldsymbol{\omega} \\ & + \varepsilon \left(\frac{h^i q^i - h^{i-1} q^{i-1}}{\Delta t} - \frac{h^i - h^{i-1}}{\Delta t} \frac{\partial (y_2 q^i)}{\partial y_2} \right) v + \operatorname{div}_{h^i} \boldsymbol{u}^i v \Bigg\} \, dy \end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&+\int_0^1 R_0(L)\Big(q^i_{out}-\frac{|u^i_1|^2}{2}\Big)\omega_1(L,y_2)-R_0(0)\Big(q^i_{in}-\frac{|u^i_1|^2}{2}\Big)\omega_1(0,y_2)\,dy_2\\
&+\int_0^L\left(q^i_w-\frac{1}{2}u^i_2\left(u^i_2-\frac{h^i-h^{i-1}}{\Delta t}\right)\right)\omega_2\,(y_1,1)\,dy_1\\
&+\int_0^L\left(\kappa\left(u^i_2-\sigma^i\right)\omega_2+\frac{\varepsilon}{2}\frac{h^i-h^{i-1}}{\Delta t}q^i v\right)(y_1,1)\,dy_1\\
&+\int_0^L\left\{\frac{\sigma^i-\sigma^{i-1}}{\Delta t}\vartheta+c\,\frac{\partial^2\sigma^i}{\partial y_1^2}\frac{\partial^2\vartheta}{\partial y_1^2}+a\Delta t\sum_{k=1}^i\frac{\partial\sigma^k}{\partial y_1}\frac{\partial\vartheta}{\partial y_1}\right.\\
&\left.\left.-a\frac{\partial^2 R^i_0}{\partial y_1^2}\vartheta+b\Delta t\left(\sum_{k=1}^i\sigma^k\right)\vartheta+\frac{\kappa}{E}\left(\sigma^i-u^i_2\right)\vartheta\right\}(y_1)\,dy_1\right]\Delta t\;=\;0
\end{aligned}
$$

for any $\boldsymbol{\varpi}=(\boldsymbol{\omega},v,\vartheta)\in V$.

We test the above identity with $(\boldsymbol{u}^i,\ q^i,E\sigma^i)$, find out that $b(\boldsymbol{u}^i,\boldsymbol{u}^i,\boldsymbol{u}^i)=0$ (Lemma 3.7), multiply with 2 and perform the following discrete calculus.

$$
\begin{aligned}
&2\sum_{i=1}^r\int_D\left(h^i\boldsymbol{u}^i-h^{i-1}\boldsymbol{u}^{i-1}\right)\boldsymbol{u}^i dy\;=\;\int_D h^r\,|\boldsymbol{u}^r|^2\,dy \qquad (4.2)\\
&\qquad+\sum_{i=1}^r\int_D\left\{\frac{1}{h^i}\left|h^i\boldsymbol{u}^i-h^{i-1}\boldsymbol{u}^{i-1}\right|^2+\frac{h^{i-1}}{h^i}\left(h^i-h^{i-1}\right)\left|\boldsymbol{u}^{i-1}\right|^2\right\}dy,\\
&2\sum_{i=1}^r\int_0^L\left(\sigma^i-\sigma^{i-1}\right)\sigma^i\,dy_1=\int_0^L|\sigma^r|^2\,dy_1+\sum_{i=1}^r\int_0^L\left|\sigma^i-\sigma^{i-1}\right|^2dy_1,\\
&a\Delta t\sum_{i=1}^r\int_0^L\frac{\partial U^i}{\partial y_1}\frac{\partial\sigma^i}{\partial y_1}dy_1=\frac{a}{2}\int_0^L\left\{\left|\frac{\partial U^r}{\partial y_1}\right|^2dy_1+\sum_{i=1}^r\left|\frac{\partial(U^i-U^{i-1})}{\partial y_1}\right|^2\right\}dy_1,\\
&b\Delta t\sum_{i=1}^r\int_0^L U^i\sigma^i\,dy_1\;=\frac{b}{2}\int_0^L\left\{|U^r|^2\,dy_1+\sum_{i=1}^r\left|U^i-U^{i-1}\right|^2\right\}dy_1.
\end{aligned}
$$

Here U^i denotes $U^i:=\sum_{k=1}^i\sigma^k\Delta t,\ U^0\equiv 0,$ and $\frac{U^i-U^{i-1}}{\Delta t}=\sigma^i$.

Using (4.2), the coercivity and ellipticity properties of the forms $((\cdot,\cdot))$ and $a_1(\cdot,\cdot)$ (Lemma 3.4, Lemma 3.3), the Hölder inequality, the boundary imbedding (3.6) and Young's inequality we get

$$
\begin{aligned}
&\int_D h^r(|\boldsymbol{u}^r|^2+\varepsilon|q^r|^2)dy+E\int_0^L|\sigma^r|^2dy_1 \qquad (4.3)\\
&+\Delta t\sum_{i=1}^r\int_D 2\tilde{\delta}|\nabla\boldsymbol{u}^i|^p+\frac{2\alpha\varepsilon}{2+K^2}|\nabla q^i|^2dy+2cE\Delta t\sum_{i=1}^r\int_0^L\left|\frac{\partial^2\sigma^i}{\partial y_1^2}\right|^2dy_1\\
&+\int_0^L+aE\left|\frac{\partial U^r}{\partial y_1}\right|^2+bE\,|U^r|^2+2\kappa\Delta t\sum_{i=1}^r[\sigma^i-u^i_2]^2\,dy_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Delta t \sum_{i=1}^{r} H^i \int_D h^i(|\boldsymbol{u}^i|^2 + \varepsilon |q^i|^2) dy + C_1 \Delta t \sum_{i=1}^{r} \mathcal{P}^i \|\nabla \boldsymbol{u}^i\|_p \\
&+\Delta t \sum_{i=1}^{r} \int_0^L 2aE \frac{\partial^2 R_0^i}{\partial y_1^2} \sigma^i dy_1, \quad \text{where}
\end{aligned}
$$

$$
H^i \equiv \max_{0\leq y_1 \leq L} \left[-\frac{\Upsilon h^{i+1}}{h^{i+1}} \right]_+ + \frac{1}{2\tilde{\delta} h^i} \left(\max_{0 \leq y_1 \leq L} \Upsilon h^i \right)^2, \quad \Upsilon h^i := \frac{h^i - h^{i-1}}{\Delta t},
$$

$$
\text{and} \qquad \mathcal{P}^i := \left\| q_{in}^i \right\|_{L^2(0,1)} + \left\| q_{out}^i \right\|_{L^2(0,1)} + \left\| q_w^i \right\|_{L^2(0,L)}. \tag{4.4}
$$

Constant C_1 comes from (3.6) and the compact imbeddings $W^{1,p}(D) \Subset L^2(D)$, cf. (3.19).

By applying Young's inequality in terms on the right hand side of (4.3) with appropriate constants $\tilde{\delta}, C(\tilde{\delta})$, $\tilde{\delta} = \tilde{\delta}(K, \alpha)$ from Lemma 3.4 we obtain

$$
\begin{aligned}
&\xi^r + \sum_{i=1}^{r} \int_D \tilde{\delta} |\nabla \boldsymbol{u}^i|^p + \frac{2\alpha\varepsilon}{2+K^2} |\nabla q^i|^2 dy + \frac{E}{2} \int_0^L c \left| \frac{\partial^2 \sigma^i}{\partial y_1^2} \right|^2 dy_1 \Delta t \\
&+ \int_0^L \frac{aE}{2} \left| \frac{\partial U^r}{\partial y_1} \right|^2 + \frac{bE}{2} |U^r|^2 + 2 \sum_{i=1}^{r} \kappa [\sigma^i - u_2^i]^2 dy_1 \Delta t \\
&\leq \Delta t \sum_{i=1}^{r} H^i \xi^i + \Delta t \sum_{i=1}^{r} f^i,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
\text{where} \quad \xi^r &= \int_D h^r(|\boldsymbol{u}^r|^2 + \varepsilon |q^r|^2) dy + \frac{E}{2} \int_0^L |\sigma^r|^2 dy_1, \\
f^i &= M \|q_{\partial D}\|_2^{p'} + \frac{2C_1 E}{c} \int_0^L a^2 \left| \frac{\partial^2 R_0^i}{\partial y_1^2} \right|^2 dy_1
\end{aligned}
$$

and $M = C_1 \frac{p-1}{p} \left(\frac{C_1}{p\tilde{\delta}} \right)^{\frac{1}{p-1}}$. After omitting positive terms on the left hand side of (4.5) we obtain $\xi^r \leq \Delta t \sum_{i=1}^{r} H^i \xi^i + \Delta t \sum_{i=1}^{r} f^i$ and applying the discrete Gronwall inequality [13] we get

$$
\xi^r \leq e^{\Delta t \sum_{i=1}^{r} H^i} \Delta t \sum_{i=1}^{r} f^i. \tag{4.6}
$$

Consequently the right hand side of inequality (4.5) can be estimated with use of (4.6) by $\left(1 + \mathcal{H} e^{\mathcal{H}}\right) \Delta t \sum_{i=1}^{n} f^i$, $\mathcal{H} := \Delta t \sum_{i=1}^{n} H^i$ and we obtain the **first a priori estimate:**

$$
\begin{aligned}
\mathbf{I.} \quad &\max_{1\leq r \leq n} \int_D h^r(|\boldsymbol{u}^r|^2 + \varepsilon |q^r|^2) dy + \frac{E}{2} \int_0^L |\sigma^r|^2 dy_1 \\
&+\Delta t \sum_{i=1}^{n} \int_D \tilde{\delta} |\nabla \boldsymbol{u}^i|^p + \frac{2\alpha\varepsilon}{2+K^2} |\nabla q^i|^2 dy + \frac{E}{2} \int_0^L c \left| \frac{\partial^2 \sigma^i}{\partial y_1^2} \right|^2 dy_1
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
&+\max_{1\le r\le n}\int_0^L \frac{aE}{2}\left|\frac{\partial U^r}{\partial y_1}\right|^2+\frac{bE}{2}\left|U^r\right|^2+\Delta t\sum_{i=1}^n 2\kappa\int_0^L\left|\sigma^i-u_2^i\right|^2 dy_1\\
&+\sum_{i=1}^n\int_D\frac{1}{h^i}|h^i\boldsymbol{u}^i-h^{i-1}\boldsymbol{u}^{i-1}|^2+\varepsilon|h^iq^i-h^{i-1}q^{i-1}|^2\,dy\\
&+\Delta t\sum_{i=1}^n\int_0^L E|\sigma^i-\sigma^{i-1}|^2dy_1 \quad\le\quad \tilde M\Delta t\sum_{i=1}^n f^i,
\end{aligned}
$$

where $\tilde M = (1+\mathcal{H}e^{\mathcal{H}}),\ \mathcal{H}\le C(\alpha)\sum_{i=1}^n\|\Upsilon h^{i+1}\|_{C[0,L]}+\|\Upsilon h^i\|^2_{C[0,L]}\Delta t$ is bounded and f^i depends only on the given data R_0, q_{in}, q_{out}, q_w. Note that constant $\tilde M$ does not depend on κ. We will see later in Section 5 that the continuous analogy of this estimates will be useful to prove convergence of the approximate solution for $\varepsilon\to 0$ and $\kappa\to\infty$.

Now we are ready to show suitable properties of time differences of the weak solution. We first show that the time difference of the domain deformation velocity is bounded in $L^2((0,T)\times D)$ with some constant dependent on κ. To prove it, we test (4.1) with $\boldsymbol{\psi}^i=(\mathbf{0},\ 0,\ E\Upsilon\sigma^i)$. This yields

$$
\begin{aligned}
&\Delta t\sum_{i=1}^r\int_0^L E\left|\Upsilon\sigma^i\right|^2+E\,c\frac{\partial^2\sigma^i}{\partial y_1^2}\frac{\partial^2\Upsilon\sigma^i}{\partial y_1^2}+\kappa(\sigma^i-u_2)\Upsilon\sigma^i \qquad (4.8)\\
&+E\,a\left(\sum_{k=1}^i\frac{\partial\sigma^k}{\partial y_1}\Delta t\right)\frac{\partial\Upsilon\sigma^i}{\partial y_1}+E\,b\left(\sum_{k=1}^i\sigma^k\Delta t\right)\Upsilon\sigma^i-E\,a\frac{\partial^2R_0^i}{\partial y_1^2}\Upsilon\sigma^i\,dy_1=0.
\end{aligned}
$$

Using the discrete integration by parts in time (4.2), Young's inequality and the previous estimate (4.7) lead to the **second a priori estimate**:

$$
\begin{aligned}
\textbf{II a}).\quad &\int_0^L\frac{E}{2}\sum_{i=1}^n\left|\frac{\sigma^i-\sigma^{i-1}}{\Delta t}\right|^2\Delta t+\max_{1\le r\le n}\frac{cE}{4}\left|\frac{\partial^2\sigma^r}{\partial y_1^2}\right|^2\\
&\le C\kappa\tilde M\sum_{i=1}^n f^i\Delta t. \qquad (4.9)
\end{aligned}
$$

Since the term $\sum_{i=1}^r\int_0^L\kappa^2(\sigma^i-u_2^i)^2dy_1\Delta t$ is bounded using (4.7) with $\kappa\tilde M\sum_{i=1}^n f^i\Delta t$, this a priori estimate depends on κ.

Let us define

$$\boldsymbol{\mathcal{U}}^i=h^i\boldsymbol{u}^i,\quad Q^i=h^iq^i.$$

Using the sequences $\{\boldsymbol{\mathcal{U}}^i\}_{i=1}^n$, $\{Q^i\}_{i=1}^n$, $\{\sigma^i\}_{i=1}^n$, $\{h^i\}_{i=1}^n$ we construct the piecewise constant step functions

$$\boldsymbol{u}_n^s(y,t),\ q_n^s(y,t),\ \boldsymbol{\mathcal{U}}_n^s(y,t),\ Q_n^s(y,t),\ \sigma_n^s(y_1,t),\ h_n^s(y_1,t)$$

and the piecewise linear approximations of the weak solution and of $h(y_1,t)$

$$\boldsymbol{u}_n(y,t),\ q_n(y,t),\ \boldsymbol{\mathcal{U}}_n(y,t),\ Q_n(y,t),\ \sigma_n(y_1,t),\ h_n(y_1,t).$$

We show now a priori estimate for the time derivative of piecewise linear approximation of the weak solution. To this goal we test (2.13) with $(\boldsymbol{\psi},0,0)$, $\boldsymbol{\psi}\in L^p(0,T;\boldsymbol{V})$. From (2.13) we have

$$-\int_0^T\left\langle\frac{\partial\boldsymbol{\mathcal{U}}_n}{\partial t},\boldsymbol{\psi}\right\rangle dt=\ldots\int_0^T((\boldsymbol{u}_n,\boldsymbol{\psi}))+b(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{\psi})\ldots dy\,dt.$$

We concentrate only on particular terms that yield some restrictions. Estimates for other terms do not lead to additional difficulties. According to Lemma 3.7 we have $2b(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{\psi})=B(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{\psi})-B(\boldsymbol{u}_n,\boldsymbol{\psi},\boldsymbol{u}_n)$. Now, using the Hölder inequality, imbedding of the space $W^{1,p}(D)$ into $L^{\frac{2p}{p-2}}(D)$ for $p>2$ we have

$$\begin{aligned}\int_0^T B(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{\psi})&\le C(K,\alpha)\int_0^T\|\boldsymbol{u}_n\|_{1,p}\|\boldsymbol{u}_n\|_2\|\boldsymbol{\psi}\|_{\frac{2p}{p-2}}\\&\le C(K,\alpha)\|\boldsymbol{u}_n\|_{L^\infty(0,T;L^2(D))}\int_0^T\|\boldsymbol{u}_n\|_{1,p}\|\boldsymbol{\psi}\|_{1,p}.\\&\le C(K,\alpha)\|\boldsymbol{u}_n\|_{L^\infty(0,T;L^2(D))}\|\boldsymbol{u}_n\|_{L^p(0,T;W^{1,p}(D))}\|\boldsymbol{\psi}\|_{L^{p'}(0,T;W^{1,p}(D))},\end{aligned}\tag{4.10}$$

which is bounded for all $p>2$ due to the a priori estimate (4.7) for all $\boldsymbol{\psi}\in L^p(0,T;\boldsymbol{V})$. Analogously the term $\int_0^T B(\boldsymbol{u}_n,\boldsymbol{\psi},\boldsymbol{u}_n)$ is bounded, which leads to

$$\int_0^T b(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{\psi})\le C(K,\alpha)\text{ for }p\in(2,\infty).\tag{4.11}$$

For $p=2$ this estimate is valid for $\boldsymbol{\psi}\in L^p(0,T;\boldsymbol{V})\cap L^4((0,T)\times D)$, cf. [15].

Further, using Lemma 3.5 we get

$$\begin{aligned}\int_0^T((\boldsymbol{u}_n,\boldsymbol{\psi}))&\le C(K,\alpha)\int_0^T\|\boldsymbol{\psi}\|_{1,p}\|\boldsymbol{u}_n\|_{1,p}^{p-1}+C_0\int_0^T\|\boldsymbol{\psi}\|_{1,p}\|\boldsymbol{u}_n\|_{1,p}\\&\le C(K,\alpha)\|\boldsymbol{\psi}\|_{L^p(0,T;W^{1,p}(D))}\left(\|\boldsymbol{u}_n\|_{L^p(0,T;W^{1,p}(D))}^{p-1}+\|\boldsymbol{u}_n\|_{L^p(0,T;W^{1,p}(D))}\right).\end{aligned}\tag{4.12}$$

Thus, the viscous term $\int_0^T((\boldsymbol{u}_n,\boldsymbol{\psi}))$ is bounded for any $\psi\in L^p(0,T;W^{1,p}(D))$. Consequently we have proved the **second a priori estimate** for velocity

$$\textbf{II b)}\qquad\begin{aligned}&\frac{\partial\boldsymbol{\mathcal{U}}_n}{\partial t}\in L^{p'}(0,T;\boldsymbol{V}^*)\text{ for }p\in(2,\infty),\\&\frac{\partial\boldsymbol{\mathcal{U}}_n}{\partial t}\in L^{p'}(0,T;\boldsymbol{V}^*)\oplus L^{4/3}((0,T)\times D)\text{ for }p=2.\end{aligned}\tag{4.13}$$

where p' is given by $\frac{1}{p'}+\frac{1}{p}=1$. This estimate is analogously as the estimate for $\partial_t\sigma_n$ dependent on κ.

By testing (2.13) with $(0,\phi,0)$ we obtain after standard calculation that $\int_0^T \left\langle \sqrt{\varepsilon}\frac{\partial Q_n}{\partial t},\phi\right\rangle_{H^1} \le C(\frac{1}{\sqrt{\varepsilon}})$, i.e., the **second a priori estimate** for pressure

$$\textbf{II c)} \qquad \frac{\partial Q_n}{\partial t} \in L^2(0,T;(H^1(D))^*), \tag{4.14}$$

which is dependent on ε.

Let us summarize the above results in the following lemma.

Lemma 4.1 (A priori estimates)**.**
Let us assume that $h \in W^{1,\infty}((0,T),L^2(0,L)) \cap H^1((0,T);H_0^2(0,L))$ *and the assumptions (2.1), (3.1)–(3.4) hold. Then we have for the approximate sequences of piecewise constant and piecewise linear functions the following results*

$$\{\boldsymbol{u}_n^s\}_{n=0}^\infty,\ \{\boldsymbol{\mathcal{U}}_n^s\}_{n=0}^\infty,\ \{\boldsymbol{\mathcal{U}}_n\}_{n=0}^\infty\ \in L^p(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D)), \tag{4.15}$$

$$\left.\begin{array}{l} \{\partial_t \boldsymbol{\mathcal{U}}_n\}_{n=0}^\infty\ \in L^{p'}(0,T;\boldsymbol{V}^*) \textit{ for } p \in (2,\infty), \\ \{\partial_t \boldsymbol{\mathcal{U}}_n\}_{n=0}^\infty\ \in L^{p'}(0,T;\boldsymbol{V}^*) \oplus L^{4/3}((0,T)\times D) \textit{ for } p=2, \end{array}\right\} \tag{4.16}$$

$$\{q_n^s\}_{n=0}^\infty, \{Q_n^s\}_{n=0}^\infty, \{Q_n\}_{n=0}^\infty \in L^2(0,T;W^{1,2}(D)) \cap L^\infty(0,T;L^2(D)), \tag{4.17}$$

$$\{\partial_t Q_n\}_{n=0}^\infty\ \in L^2(0,T;H^{-1}(D)), \tag{4.18}$$

$$\{\sigma_n^s\}_{n=0}^\infty,\ \{\sigma_n\}_{n=0}^\infty\ \in L^2(0,T;H_0^2(0,L)) \cap L^\infty(0,T;L^2(0,L)), \tag{4.19}$$

$$\left.\begin{array}{l} \{\partial_t \sigma_n\}_{n=0}^\infty\ \in L^2((0,T)\times(0,L)), \\ \{\sigma_n\}_{n=0}^\infty \in L^\infty(0,T;H_0^2(0,L)). \end{array}\right\} \tag{4.20}$$

The estimates (4.16), (4.20) depend on κ*, (4.18) depends on* ε*.*

Proof. These results follow from a priori estimates (4.7), (4.9), (4.13), (4.14). ■

Consequently we have following convergences.

Lemma 4.2.
There exists a subsequence of $\{h_n^s, h_n, \boldsymbol{u}_n^s, \boldsymbol{u}_n, \boldsymbol{\mathcal{U}}_n^s, \boldsymbol{\mathcal{U}}_n, q_n^s, q_n, \sigma_n^s, \sigma_n\}_{n=1}^\infty$ *and functions* $\boldsymbol{u} \in L^p(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D))$, $q \in L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ *and* $\sigma \in L^2(0,T;H_0^2(0,L)) \cap L^\infty(0,T;L^2(0,L))$ *(we denote the subsequence again by* $\{h_n^s, h_n, \boldsymbol{u}_n^s, \boldsymbol{u}_n, \boldsymbol{\mathcal{U}}_n^s, \boldsymbol{\mathcal{U}}_n, q_n^s, q_n, \sigma_n^s, \sigma_n\}_{n=1}^\infty$*), such that*

$$h_n \rightharpoonup h \qquad \textit{*-weakly in } \ W^{1,\infty}(0,T;L^2(0,L)), \tag{4.21}$$

$$h_n^s,\ h_n \to h \qquad \textit{strongly in } \ L^\infty(0,T;C^1[0,L]), \tag{4.22}$$

$$\begin{array}{l} \boldsymbol{\mathcal{U}}_n,\ \boldsymbol{\mathcal{U}}_n^s\ \rightharpoonup h\boldsymbol{u}\ ,\quad \boldsymbol{u}_n^s \rightharpoonup \boldsymbol{u} \textit{ weakly in } L^p(0,T;\boldsymbol{V}), \\ \boldsymbol{\mathcal{U}}_n,\ \boldsymbol{\mathcal{U}}_n^s\ \rightharpoonup h\boldsymbol{u}\ ,\quad \boldsymbol{u}_n^s \rightharpoonup \boldsymbol{u} \textit{ *-weakly in } L^\infty(0,T;L^2(D)), \end{array} \tag{4.23}$$

$$\left.\begin{array}{r} \boldsymbol{\mathcal{U}}_n^s,\ \boldsymbol{\mathcal{U}}_n \to\ h\boldsymbol{u} \\ \boldsymbol{u}_n^s,\ \boldsymbol{u}_n \to\ \boldsymbol{u} \end{array}\right\} \begin{array}{l} \textit{strongly in } L^p((0,T)\times D), \\ \textit{strongly in } L^r((0,T)\times S),\ \infty > r > 1, \end{array} \tag{4.24}$$

$$
\begin{aligned}
&q_n,\ q_n^s \rightharpoonup q \quad \textit{weakly in } L^2(0,T;H^1(D)),\\
&q_n,\ q_n^s \to q \quad \textit{strongly in } L^2((0,T)\times D), && (4.25)\\
&q_n,\ q_n^s \rightharpoonup q \quad \textit{*-weakly in } L^\infty(0,T;L^2(D)),\\
&Q_n \rightharpoonup hq \quad \textit{weakly in } \ H^1(0,T;H^{-1}(D)), && (4.26)
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_n \rightharpoonup \sigma \quad \textit{weakly in } \ L^2((0,T);H^2(0,L)),\\
&\sigma_n \rightharpoonup \sigma \quad \textit{*-weakly in } \ L^\infty(0,T;L^2(0,L)), && (4.27)\\
&\sigma_n \to \sigma \quad \textit{strongly in } \ L^2(0,T;H^1(0,L)),\\
&\partial_t\sigma_n \rightharpoonup \partial_t\sigma \quad \textit{weakly in } \ L^2((0,T)\times(0,L)) && (4.28)
\end{aligned}
$$

as $n \to \infty$*. The convergence (4.24), (4.27)*$_3$*, (4.28) is dependent on* κ*, (4.25)*$_2$ *and (4.26) depend on* ε*.*

Proof. The convergence (4.21) follows from the Taylor expansion of h_t, integration by parts in time and the boundedness of h_t in $L^\infty(0,T;L^2(0,L))$. The proof of (4.22), (4.26), (4.27), (4.28) can be found in [15] or [41, p. 47]. The assertion (4.27)$_3$ follows from the imbeddings $H^2(0,L) \Subset H^1(0,L) \subset L^2(0,L)$ and the Lions-Aubin lemma. This convergence depends on κ.

In the following we only prove the strong convergences of $\boldsymbol{\mathcal{U}}_n$, $\boldsymbol{\mathcal{U}}_n^s$, Q_n, Q_n^s in the corresponding spaces, cf. (4.24), (4.25). Consider $p > 2$, for proof of (4.24)$_1$ for $p = 2$ we refer to [15, Lemma 6.1]. Note that

$$
W^{1,p}(D) \Subset L^p(D) \subset (W^{1,p}(D))^*,
$$

where imbedding $W^{1,p}(D)$ into $L^p(D)$ is compact, imbedding $L^p(D)$ into $(W^{1,p}(D))^*$ is continuous and $W^{1,p}(D)$ and $(W^{1,p}(D))^*$ are reflexive spaces ($p \neq \infty$), see [1]. According to the Lions-Aubin lemma, the imbedding of the space $\mathcal{X} := \{\boldsymbol{\mathcal{U}}_n \in L^p(0,T;\boldsymbol{V}),\ \partial_t\boldsymbol{\mathcal{U}}_n \in L^{p'}(0,T;\boldsymbol{V}^*)\}$ into $L^p(0,T;L^p(D))$ is compact, where $\frac{1}{p'} + \frac{1}{p} = 1$. This, together with (4.22) implies that

$$
\boldsymbol{\mathcal{U}}_n \to h\boldsymbol{u} \text{ strongly in } L^p(0,T;L^p(D)), \tag{4.29}
$$

dependently on κ. The first part of the result in (4.24) is now proven.

It remains to show the strong convergence of piecewise constant sequence $\{\boldsymbol{\mathcal{U}}_n^s\}$. Since $|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s| \le |h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}|$ for $t \in \Big((i-1)\Delta t, i\Delta t\Big)$, we have from the first a priori estimate (4.7)

$$
\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|_{L^2((0,T)\times D)} = \sqrt{\Delta t}\left(\sum_{i=1}^n \int_D |h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}|^2\right)^{1/2} \le C(\alpha)\Delta t^{\frac{1}{2}}.
$$

Moreover, Lemma 3.1 (with $\theta = \frac{p-1}{p}$) and the Hölder inequality implies

$$
\begin{aligned}
\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|^p_{L^p((0,T)\times D)} &\le c_1 \int_0^T \|\nabla\boldsymbol{\mathcal{U}}_n - \nabla\boldsymbol{\mathcal{U}}_n^s\|_2^{p-1}\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|_2\\
&\le c_2\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|_{L^p(0,T;L^2(D))}.
\end{aligned}
$$

The constant c_2 depends on $\||\nabla \boldsymbol{\mathcal{U}}_n|+|\nabla \boldsymbol{\mathcal{U}}_n^s|\|_{L^p(0,T;L^2(D))}$ and it is bounded, see (4.15). The term $\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|_{L^p(0,T;L^2(D))}$ can be upper bounded with

$$c_3\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|^{1/p}_{L^2((0,T)\times D)}\||\boldsymbol{\mathcal{U}}_n| + |\boldsymbol{\mathcal{U}}_n^s|\|^{(p-1)/p}_{L^\infty(0,T;L^2(D))}.$$

Since $\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|_{L^2((0,T)\times D)} \leq C(\alpha)\Delta t^{\frac{1}{2}}$, we obtain from the previous estimate that

$$\|\boldsymbol{\mathcal{U}}_n - \boldsymbol{\mathcal{U}}_n^s\|^p_{L^p(0,T;L^p(D))} \leq c_4(\alpha)\Delta t^{\frac{1}{2p}}$$

and thus with use of (4.29) we get

$$\boldsymbol{\mathcal{U}}_n^s \to h\boldsymbol{u} \quad \text{strongly in } L^p((0,T)\times D). \tag{4.30}$$

To complete the proof of (4.24) we consider the boundary integrals. By means of Lemma 3.2 for $r = p$ and the Hölder inequality we get

$$\begin{aligned}\|\boldsymbol{\mathcal{U}}_n^s - h\boldsymbol{u}\|^p_{L^p((0,T)\times S)} &\leq c\int_0^T \|\nabla(\boldsymbol{\mathcal{U}}_n^s - h\boldsymbol{u})\|_2^{p-1}\|\boldsymbol{\mathcal{U}}_n^s - h\boldsymbol{u}\|_2 \\ &\leq c_1\|\boldsymbol{\mathcal{U}}_n^s - h\boldsymbol{u}\|_{L^p(0,T;W^{1,p}(D))}\|\boldsymbol{\mathcal{U}}_n^s - h\boldsymbol{u}\|^{p-1}_{L^p(D\times(0,T))}\end{aligned}$$

which tends to zero due to (4.30). This result together with (4.29), (4.30) implies the assertion (4.24).

Analogously as above we prove the strong convergence of q_n, $q_n^s \to q$ in $L^2(D\times(0,T))$ for fixed ε, cf. (4.25). In this case, we obtain from the Lions-Aubin lemma using the imbeddings $W^{1,2}(D) \Subset L^2(D) \subset (W^{1,2}(D))^*$ the strong convergence of q_n in $L^2(0,T;L^2(D))$. Since $|Q_n - Q_n^s| \leq |h^i q^i - h^{i-1}q^{i-1}|$ for $t \in ((i-1)\Delta t, i\Delta t)$ we have from (4.7) also

$$\|Q_n - Q_n^s\|_{L^2((0,T)\times D)} = \left(\Delta t\sum_{i=1}^n \int_D |h^i q^i - h^{i-1}q^{i-1}|^2\right)^{1/2} \leq C\left(\frac{\Delta t}{\varepsilon}\right)^{\frac{1}{2}}.$$

Letting $n \to \infty$ we get the strong convergence of q_n^s in $L^2(0,T;L^2(D))$. ∎

4.2 Limiting process

Now we are ready to let $n \to \infty$ and by means of Lemma 4.2 to prove the existence of unsteady weak solution to our problem defined in (2.13).

Consider first smooth test functions $\boldsymbol{\psi} \in C^1([0,T]\times\overline{D})$, $\phi \in C(0,T;H^1(D))$, $\xi \in C(0,T;H_0^2(0,L))$. Then construct piecewise constant and piecewise linear approximations in time $\boldsymbol{\psi}_n$, $\boldsymbol{\psi}_n^s$, ϕ_n, ϕ_n^s, ξ_n^s. It is easy to verify that

$$\begin{gathered}\boldsymbol{\psi}_n \to \boldsymbol{\psi} \text{ in } H^1(0,T;\boldsymbol{V}), \qquad \boldsymbol{\psi}_n^s \to \boldsymbol{\psi} \text{ in } L^\infty(0,T;C^1(\overline{D})), \\ \phi_n^s \to \phi \text{ in } L^2(0,T;H^1(D)) \text{ and } \xi_n^s \to \xi \text{ in } L^\infty(0,T;H_0^2(0,L))\end{gathered} \tag{4.31}$$

as $n \to \infty$.

In the identity (4.1) with $r = n$ we put $\boldsymbol{\omega} = \boldsymbol{\psi}^i = \boldsymbol{\psi}(y, i\Delta t) \in \boldsymbol{V}$, $v = \phi^i \in H^1(D), \vartheta = \xi^i \in H_0^2(0, L)$ and replace

$$\Delta t \sum_{i=1}^{n} \int_D \frac{\partial \boldsymbol{\mathcal{U}}_n}{\partial t} \boldsymbol{\psi}^i \, dy \quad \text{by} \quad - \int_{\Delta t}^{T} \int_D \boldsymbol{\mathcal{U}}_n^s (t - \Delta t) \frac{\partial \boldsymbol{\psi}_n}{\partial t}(t) \, dy \, dt$$

for all $\boldsymbol{\psi}_n \in H^1(0, T; \boldsymbol{V})$ such that $\boldsymbol{\psi}_n(T) = 0$. This yields

$$\begin{aligned}
&\int_{\Delta t}^{T} \int_D \boldsymbol{\mathcal{U}}_n^s (t - \Delta t) \cdot \frac{\partial \boldsymbol{\psi}_n}{\partial t} \, dy \, dt = \int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)) \, dt \qquad (4.32)\\
&+ \int_0^T \int_D \left\{ - \frac{\partial h_n}{\partial t} \frac{\partial (y_2 \boldsymbol{u}_n^s)}{\partial y_2} \cdot \boldsymbol{\psi}_n^s - h_n^s q_n^s \operatorname{div}_{h_n^s} \boldsymbol{\psi}_n^s \right\} dy + b(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s) \, dt\\
&+ \int_0^T \left\{ \varepsilon \left\langle \frac{\partial Q_n}{\partial t}, \phi_n^s \right\rangle + \int_D - \varepsilon \frac{\partial h_n}{\partial t} \frac{\partial (y_2 q_n^s)}{\partial y_2} \phi_n^s + \varepsilon a_1 (q_n^s, \phi_n^s) dy \right.\\
&\quad + \int_D \operatorname{div}_{h_n^s} \boldsymbol{u}_n^s \phi_n^s \, dy + \int_0^1 R_0(L) q_{out}^{n,s} \boldsymbol{\psi}_{1n}^{\ s} \ (L, y_2) \, dy_2\\
&\quad - \int_0^1 R_0(0) q_{in}^{n,s} \boldsymbol{\psi}_{1n}^{\ s} \ (0, y_2) \, dy_2 + \int_0^L \left(q_w^{n,s} - \frac{u_{n2}^s}{2} \frac{\partial h_n}{\partial t} \right) \boldsymbol{\psi}_{2n}^{\ s} \ (y_1, 1) \, dy_1\\
&\quad \left. + \int_0^L \left(\kappa \left(u_{2n}^s - \sigma_n^s \right) \boldsymbol{\psi}_{2n}^s + \frac{\varepsilon}{2} \frac{\partial h_n}{\partial t} q_n^s \phi_n^s \right) (y_1, 1) \, dy_1 \right\} dt\\
&+ \int_0^T \int_0^L \left\{ \frac{\partial \sigma_n}{\partial t} \xi_n^s + c \frac{\partial^2 \sigma_n^s}{\partial y_1^2} \frac{\partial^2 \xi_n^s}{\partial y_1^2} + a \left(\int_0^t \frac{\partial \sigma_n^s}{\partial y_1} (y_1, \tau) d\tau \right) \frac{\partial \xi_n^s}{\partial y_1} \right.\\
&\quad \left. - a \frac{\partial^2 R_0}{\partial y_1^2} \xi_n^s + b \left(\int_0^t \sigma_n^s (y_1, \tau) d\tau \right) \xi_n^s + \frac{\kappa}{E} \left(\sigma_n^s - u_{2n}^s \right) \xi_n^s \right\} (y_1) \, dy_1 \, dt.
\end{aligned}$$

Now we let $n \to \infty$ in (4.32). We will show only the convergence of some chosen terms. The limiting process other terms is analogous.
We have from (4.24), (4.26) and (4.31)

$$\int_{\Delta t}^{T} \int_D \boldsymbol{\mathcal{U}}_n^s (t - \Delta t) \cdot \frac{\partial \boldsymbol{\psi}_n}{\partial t}(t) \, dy \, dt \longrightarrow \int_0^T \int_D h \boldsymbol{u}(t) \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(t) \, dy \, dt,$$

$$\int_T \left\langle \frac{\partial Q_n}{\partial t}, \phi_n^s \right\rangle_{H^1} dt \longrightarrow \int_T \left\langle \frac{\partial (hq)}{\partial t}, \phi \right\rangle_{H^1} dt.$$

Next, we prove convergence in the nonlinear term $b(\cdot, \cdot, \cdot)$ defined in (2.12). Let us estimate

$$\begin{aligned}
&\left| \int_0^T b(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}) dt \right| \le \\
&\underbrace{\int_0^T |b(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s)| dt}_{[I]} + \underbrace{\int_0^T |b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})| dt}_{[II]}.
\end{aligned}$$

According to Lemma 3.7, the term $[I] = \frac{1}{2}\int_0^T B(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s,) - B(\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s, \boldsymbol{u}_n^s) - B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s) + B(\boldsymbol{u}, \boldsymbol{\psi}_n^s, \boldsymbol{u}) dt$ can be estimated as follows

$$
\begin{aligned}
&2\int_0^T |b(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s)| dt \\
&\leq \int_0^T |B(\boldsymbol{u}_n^s - \boldsymbol{u}, \boldsymbol{\psi}_n^s, \boldsymbol{u}_n^s)| + |B(\boldsymbol{u}, \boldsymbol{\psi}_n^s, \boldsymbol{u}_n^s - \boldsymbol{u})| \\
&\qquad + |B(\boldsymbol{u}_n^s - \boldsymbol{u}, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)| + |B(\boldsymbol{u}, \boldsymbol{u}_n^s - \boldsymbol{u}, \boldsymbol{\psi}_n^s)| dt \\
&\leq C(K,\alpha) \int_0^T \int_D |\boldsymbol{u}_n^s - \boldsymbol{u}| \left|\frac{\partial \boldsymbol{\psi}_n^s}{\partial y_1}\right| |\boldsymbol{u}_n^s + \boldsymbol{u}| + |\boldsymbol{\psi}_n^s| \left|\frac{\partial \boldsymbol{u}_n^s}{\partial y_1}\right| |\boldsymbol{u}_n^s - \boldsymbol{u}| \, dy\, dt \\
&\quad + \int_0^T \int_D \frac{\partial h_n^s}{\partial y_1} u_{1n}^s y_2 \left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_2} - \frac{\partial \boldsymbol{u}}{\partial y_2}\right) \cdot \boldsymbol{\psi}_n^s + \ldots dy\, dt \\
&\leq C(K,\alpha) \|\boldsymbol{\psi}_n^s\|_{L^\infty(0,T;C^1(\overline{D}))} \|\boldsymbol{u}_n^s - \boldsymbol{u}\|_{L^{p'}(Q_T)} \||\boldsymbol{u}_n^s| + |\nabla \boldsymbol{u}_n^s| + |\boldsymbol{u}|\|_{L^p(Q_T)} \\
&\quad + \int_0^T \int_D \frac{\partial h_n^s}{\partial y_1} u_{1n}^s y_2 \left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_2} - \frac{\partial \boldsymbol{u}}{\partial y_2}\right) \cdot \boldsymbol{\psi}_n^s + \ldots dy\, dt.
\end{aligned}
$$

Here $Q_T := ((0,T) \times D)$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

From (4.15), the weak convergences in $L^p(0,T;\boldsymbol{V})$, cf. (4.23) and the strong convergences cf. (4.24), (4.22) we deduce $[I] \to 0$. The second term $[II]$ can be estimated in the following way

$$
\begin{aligned}
&\int_0^T |b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})| dt \\
&\leq C(K,\alpha) \int_0^T \int_D |\boldsymbol{u}||\nabla \boldsymbol{u}||\boldsymbol{\psi}_n^s - \boldsymbol{\psi}| + |\boldsymbol{u}|^2 |\nabla \boldsymbol{\psi}_n^s - \nabla \boldsymbol{\psi}| dy\; dt \\
&\leq C(K,\alpha) \left(\|\boldsymbol{u}\|_{L^{p'}(Q_T)} \|\nabla \boldsymbol{u}\|_{L^p(Q_T)} + \|\boldsymbol{u}\|^2_{L^2(Q_T)} \right) \|\boldsymbol{\psi}_n^s - \boldsymbol{\psi}\|_{L^\infty(0,T;C^1(\overline{D}))}.
\end{aligned}
$$

Thus, from (4.31) we get also the convergence of the term $[II] \to 0$.

Now we show the convergence in the viscous term

$$
\int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)) \to \int_0^T ((\boldsymbol{u}, \boldsymbol{\psi})) \quad \forall \boldsymbol{\psi} \in C^1([0,T] \times \overline{D}). \tag{4.33}
$$

By means of the Minty-Browder argument [3] we prove the convergence $\int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi})) \to \int_0^T ((\boldsymbol{u}, \boldsymbol{\psi}))$, the limiting process $\int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)) \to \int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi}))$ is straightforward and follows from (4.31). We know that $\boldsymbol{u}_n^s \in L^p(0,T;\boldsymbol{V})$ and $\boldsymbol{u}_n^s \rightharpoonup \boldsymbol{u}$ in $L^p(0,T;\boldsymbol{V})$, $\hat{e}(\boldsymbol{u}_n^s) \rightharpoonup \hat{e}(\boldsymbol{u})$ in $L^p(0,T;L^p(D))$. Let us define the operator $\boldsymbol{\mathcal{A}} : L^p(0,T;L^p(D)) \to L^{p'}(0,T;L^{p'}(D))$, $\frac{1}{p} + \frac{1}{p'} = 1$ in the following way

$$
\Big\langle \mathcal{A}(\hat{e}(\boldsymbol{w})), \hat{e}(\boldsymbol{\psi}) \Big\rangle = \int_0^T \int_D h\tau_{ij}(\hat{e}(\boldsymbol{w}))\hat{e}_{ij}(\boldsymbol{\psi}) = \int_0^T ((\boldsymbol{w}, \boldsymbol{\psi}))
$$
$$
\forall \boldsymbol{w}, \boldsymbol{\psi} \in L^p(0,T;\boldsymbol{V}), \text{ see also (2.11).}
$$

From (4.12) it follows that $\mathcal{A}(\hat{e}(\boldsymbol{u}_n^s))$ is bounded. Thus, it converges weakly $\mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)) \rightharpoonup f$ in $L^{p'}(0,T;L^{p'}(D))$. Lemma 3.4, assertion 3, see also [28, Lemma 1.19], implies the monotonicity of operator $\mathcal{A}$. From the monotonicity of the operator $\mathcal{A}$ we have

$$0 \le \liminf_{n\to\infty} \Big\langle \mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)) - \mathcal{A}(\hat{e}(\boldsymbol{u})), \hat{e}(\boldsymbol{u}_n^s) - \hat{e}(\boldsymbol{u}) \Big\rangle =$$
$$\liminf_{n\to\infty} \Big\{ \Big\langle \mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)), \hat{e}(\boldsymbol{u}_n^s) \Big\rangle - \Big\langle \mathcal{A}(\hat{e}(\boldsymbol{u})), \hat{e}(\boldsymbol{u}_n^s) - \hat{e}(\boldsymbol{u}) \Big\rangle - \Big\langle \mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)), \hat{e}(\boldsymbol{u}) \Big\rangle \Big\}$$

and thus $\liminf_{n\to\infty} \big\langle \mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)), \hat{e}(\boldsymbol{u}_n^s) \big\rangle \ge \big\langle f, \hat{e}(\boldsymbol{u}) \big\rangle$. Limiting in the rest terms of the weak formulation (4.32) for test functions $\boldsymbol{\psi}_n^s = \boldsymbol{u}_n^s - \boldsymbol{u}$, $\phi_n^s = 0$, $\xi_n^s = 0$, using available weak and strong convergences we moreover get $\lim_{n\to\infty} \big\langle \mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)), \hat{e}(\boldsymbol{u}_n^s) - \hat{e}(\boldsymbol{u}) \big\rangle = 0$. This implies that $\lim_{n\to\infty} \big\langle \mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)), \hat{e}(\boldsymbol{u}_n^s) \big\rangle = \big\langle f, \hat{e}(\boldsymbol{u}) \big\rangle$. According to the Minty Trick we get $f = \mathcal{A}(\hat{e}(\boldsymbol{u}))$, i.e.

$$\mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)) \rightharpoonup \mathcal{A}(\hat{e}(\boldsymbol{u})) \ \text{ in } L^{p'}(0,T;(L^{p'}(D))),$$

which implies (4.33).

Letting $n \to \infty$ in (4.32) we obtain the weak formulation (2.13) with smooth test functions $\boldsymbol{\varpi} = (\boldsymbol{\psi}, \phi, \xi)$. Due to the standard approximation argument of the Sobolev functions by smooth functions we can conclude that (2.13) holds for any $\boldsymbol{\psi} \in H^1(0,T;\boldsymbol{V})$, $\phi \in L^2(0,T;H^1(D))$ and $\xi \in L^2(0,T;H_0^2(0,L))$.

The limiting process in (4.32) for $n \to \infty$ is now completed. Note that instead of the term $-\int_0^T \big\langle \partial_t(h\boldsymbol{u}),\, \boldsymbol{\psi} \big\rangle_{W^{1,p}}$ we have $\int_0^T \int_D h\boldsymbol{u}\partial_t\boldsymbol{\psi}$.

4.2.1 Weak time derivative

It remains to show that the limit of $\partial_t(\boldsymbol{\mathcal{U}}_n)$ is $\partial_t(h\boldsymbol{u})$. We show it for $p > 2$, for the case $p = 2$ see, e.g., [15]. From the previous section we obtained

$$-\int_0^T \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} = \int_0^T \Big\langle \chi,\, \boldsymbol{\psi} \Big\rangle_{\boldsymbol{V}} \qquad \forall \boldsymbol{\psi} \in H^1(0,T;\boldsymbol{V}),\ \boldsymbol{\psi}(T) = 0. \tag{4.34}$$

Here χ is the weak limit of $\partial_t \boldsymbol{\mathcal{U}}_n$, see (4.13),

$$\frac{\partial(h_n \boldsymbol{u}_n)}{\partial t} \rightharpoonup \chi \quad \text{weakly in} \quad L^{p'}(0,T;\boldsymbol{V}^*).$$

Since $h\boldsymbol{u} \in L^2(D)$, it can be identified with an element in $L^2(D)^*$ by Riesz' representation. Further, using the embedings $W^{1,p}(D) \subset L^2(D) \equiv L^2(D)^* \subset W^{1,p}(D)^*$, cf. [14, 38], it is possible to represent the duality between $W^{1,p}(D)$ and $(W^{1,p}(D))^*$ by means of the scalar product in $L^2(D)$. Thus, for the left hand side in (4.34) we can write

$$-\int_0^T \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} = -\int_0^T \Big\langle h\boldsymbol{u}, \frac{\partial \boldsymbol{\psi}}{\partial t} \Big\rangle_{W^{1,p}}. \tag{4.35}$$

Choose $\boldsymbol{\psi} = \boldsymbol{w}(x)\xi(t)$ such that $\boldsymbol{w} \in \boldsymbol{V},\ \xi \in C_0^1(0,T)$. Insert it in (4.34) and (4.35) and obtain

$$-\int_0^T \Big\langle h\boldsymbol{u},\, \boldsymbol{w}\Big\rangle_{\boldsymbol{V}} \xi'(t) = \int_0^T \Big\langle \chi,\, \boldsymbol{w}\Big\rangle_{\boldsymbol{V}} \xi(t).$$

Consequently we get that χ is the time derivative in distributive sense

$$\chi = \frac{\partial(h\boldsymbol{u})}{\partial t} \quad \text{in} \quad L^{p'}(0,T;\boldsymbol{V}^*)$$

and

$$-\int_0^T \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} = \int_0^T \Big\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi}\Big\rangle_{\boldsymbol{V}} \tag{4.36}$$

for every $\boldsymbol{\psi} \in H_0^1(0,T;\boldsymbol{V})$.

Moreover, for $0 < t < T$ the above distributive time derivative fulfill the equality

$$-\int_0^t \Big\langle \partial_t(h\boldsymbol{u}), \boldsymbol{\psi}\Big\rangle_{\boldsymbol{V}} ds = \int_0^t \int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} dyds - \int_D h\boldsymbol{u}(t,y)\boldsymbol{\psi}(t,y)dy. \tag{4.37}$$

This can be easily proven using test function $\boldsymbol{\psi} = \zeta(y,s)\varphi_\epsilon(s)$, where $\zeta \in H^1(0,T;X), \varphi_\epsilon(s) = \max\{0, \min\{1, \frac{t+\epsilon-s}{\epsilon}\}\}$ and passing $\epsilon \to 0$, cf. [41].

Analogously as in [15, Lemma 6.2] using the property (4.37) for the special test function $\boldsymbol{\psi} = [\boldsymbol{u}]_{\Delta t}$, cf. (5.7), we obtain the following property of the distributive time derivative

$$\int_0^t \Big\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{u}\Big\rangle_{\boldsymbol{V}} = \frac{1}{2}\int_0^t \int_D |\boldsymbol{u}|^2 \frac{\partial h}{\partial t} + \frac{1}{2}\int_D |\boldsymbol{u}|^2(t)h(t) \tag{4.38}$$

for the pairing between $W^{1,p}(D)$ and $(W^{1,p}(D))^*$.

Let us summarize the existence result of this section in the following theorem.

Theorem 4.1 (Existence of (κ, ε, h) - approximate weak solution)**.**
Let $\varepsilon,\ \kappa$ be fixed. Assume (3.1)–(3.4), (2.1), $q_{in}, q_{out} \in L^{p'}(0,T;L^2(0,1))$, $q_w \in L^{p'}(0,T;L^2(0,L))$.
Then there exists an approximated weak solution of problem (1.1)–(1.12) transformed to the fixed domain, in the sense of integral identity (2.13). Moreover,

$$\frac{\partial(h\boldsymbol{u})}{\partial t} \in \left\{ \begin{array}{l} L^{p'}(0,T;\boldsymbol{V}^*) \text{ for } 2 < p < \infty, \\ L^{p'}(0,T;\boldsymbol{V}^*) \oplus L^{4/3}((0,T)\times D), \\ \qquad\qquad \text{for } p = 2, \end{array} \right. \quad \frac{\partial(hq)}{\partial t} \in L^2(0,T;H^{-1}(D)),$$

such that

$$\int_0^T \big\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi}\big\rangle dt = -\int_0^T \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} dy\, dt$$

and the properties (4.37), (4.38) hold.

5 Problem with $\varepsilon = 0,\ \kappa = \infty$

We have proved the existence of weak solution, which is depending on the parameters ε, κ. Passing to the limit for $\varepsilon \to 0$, $\kappa \to \infty$ we obtain the weak solution of the original problem (1.1)–(1.12) for $\Omega(\eta^{(k)})$ for a fixed k. By this procedure we will prove the existence for one iteration with respect to the domain deformation $\eta^{(k)}$. We realize the limiting process by passing to the limit in both parameters at once, taking $\kappa = \varepsilon^{-1}$ and letting $\kappa \to \infty$.

We point out the dependence of weak solution on the parameters in the following way $\boldsymbol{u}_\kappa$, q_κ, σ_κ. Analogously as in Section 4.1 we obtain the first a priori estimate by testing (2.13) with $(\boldsymbol{u}_\kappa, q_\kappa, \sigma_\kappa)$ and using property (4.38).

$$
\begin{aligned}
&\max_{0\le t\le T} \int_D h(t)\left(|\boldsymbol{u}_\kappa|^2 + \varepsilon|q_\kappa|^2\right)(t)dy + \frac{E}{2}\int_0^L |\sigma_\kappa(t)|^2 dy_1 \\
&+ \int_0^T \int_D \tilde{\delta}|\nabla \boldsymbol{u}_\kappa|^p + \frac{2\alpha\varepsilon}{2+K^2}|\nabla q_\kappa|^2 dy + E\int_0^L c\left|\frac{\partial^2\sigma_\kappa}{\partial y_1^2}\right|^2 dy_1\, dt \\
&+ \int_0^L \frac{aE}{2}\left|\int_0^t \frac{\partial\sigma_\kappa(s)}{\partial y_1} ds\right|^2 + \frac{bE}{2}\left|\int_0^t \sigma_\kappa(s)s\right|^2 dy_1 \\
&+ \int_0^T\int_0^L 2\kappa\left|\sigma_\kappa - u_{2\kappa}\right|^2 dy_1\, dt \le \tilde{M}\int_0^T \mathcal{P}^{p'} + c_1\left\|\frac{\partial^2 R_0}{\partial y_1^2}\right\|^2_{L^2(0,L)} dt,
\end{aligned}
\tag{5.1}
$$

where $c_1 = c_1(p, E, a, c)$, $\tilde{M} = \tilde{M}(p, K, \alpha)$, see (4.7) and $\mathcal{P} := \|q_{in}\|_{L^2(0,1)} + \|q_{out}\|_{L^2(0,1)} + \|q_w\|_{L^2(0,L)}$, cf. (4.4). Note that the right hand side is independent on ε, κ.

5.1 Limiting process $\kappa = \varepsilon^{-1} \to \infty$

First of all we would like to point out, that the estimate (5.1) implies the weak convergence of

$$
\begin{aligned}
&(\boldsymbol{u}_\kappa, \sqrt{\varepsilon}q_\kappa, \sigma_\kappa) \rightharpoonup (\boldsymbol{u}, \tilde{q}, \sigma) \\
&\quad \text{in } \ L^p(0,T;\boldsymbol{V}) \times L^2(0,T;H^1(D)) \times L^2(0,T;H^2(0,L))
\end{aligned}
\tag{5.2}
$$

as $\kappa \to \infty$. Moreover, after inserting test functions $(\mathbf{0}, \phi, 0)$ into (2.13) for sufficiently smooth ϕ we obtain

$$
\begin{aligned}
&\int_0^T\int_D h\phi \operatorname{div} \boldsymbol{u}_\kappa \le \\
&\quad \sqrt{\varepsilon}C\|\sqrt{\varepsilon}q_\kappa\|_{L^2(0,T;H^1(D))}\left(\|\phi\|_{L^2(0,T;H^1(D))} + \|\partial_t\phi\|_{L^2((0,T)\times D)}\right).
\end{aligned}
\tag{5.3}
$$

Using the boundedness of $\sqrt{\varepsilon}q_\kappa$ in $L^2(0,T;H^1(D))$ and letting $\varepsilon = \kappa^{-1} \to 0$ we get

$$
\operatorname{div}_h \boldsymbol{u} = 0 \qquad a.e. \ \text{ on } \ (0,T)\times D.
$$

This fact allows us to confine later the space of test functions to the solenoidal space, i.e. $\mathrm{div}_h\boldsymbol{\psi} = 0$ a.e. on D.

As pointed out before, using the same techniques as in Section 4.1 we get estimates of time derivatives $\partial_t \boldsymbol{u}_\kappa$, $\partial_t\sigma_\kappa$, (4.13), (4.9), which depend on κ. Therefore in the limiting process for $\kappa \to \infty$ we cannot use the Lions-Aubin lemma as in Lemma 4.2 in order to obtain strong convergences in appropriate spaces for $(\boldsymbol{u}_\kappa, \sigma_\kappa) \to (\boldsymbol{u}, \sigma)$.

In fact, we have to use another argument to obtain the strong convergence. We follow the lines of [15, Section 8] and use the equicontinuity in time as in Alt, Luckhaus cf. [2, Lemma 1.9]. We show that

$$\int_0^{T-\tau}\int_D |(h\boldsymbol{u}_\kappa)(t+\tau) - (h\boldsymbol{u}_\kappa)(t)|^2 + \varepsilon|(hq_\kappa)(t+\tau) - (hq_\kappa)(t)|^2 dy dt \\ + \int_0^{T-\tau}\int_0^L |(h\sigma_\kappa)(t+\tau) - (h\sigma_\kappa)(t)|^2 dy_1 dt \le C(K,\alpha)\tau, \tag{5.4}$$

where C is a positive constant independent on $\tau, \kappa, \varepsilon$. To obtain (5.4) we test (2.13) with separable test functions $(\chi^\star \boldsymbol{w}, \chi^\star p, \chi^\star E v)$, where $\chi^\star(t)$ is a smooth approximation of the characteristic function of interval $(t, t+\tau)$ and $(\boldsymbol{w}(y), p(y), v(y_1)) \in V$, cf. (3.12). We put

$$\boldsymbol{w}(y) = \partial_t^\tau(h\boldsymbol{u}_\kappa), \quad p(y) = \partial_t^\tau(hq_\kappa), \quad \sigma(y_1) = \partial_t^\tau(h\sigma_\kappa),$$

where $\partial_t^\tau f := f(t+\tau) - f(t)$, use the property (4.37) and integrate with respect to t over $(0, T-\tau)$. We arrive at

$$\int_0^{T-\tau}\int_D |\partial_t^\tau(h\boldsymbol{u}_\kappa)|^2 + |\partial_t^\tau(hq_\kappa)|^2 dy + E\int_0^L h|\partial_t^\tau(\sigma_\kappa)|^2 dy_1\, dt \\ = -E\int_0^{T-\tau}\int_0^L \sigma_\kappa \partial_t^\tau(\sigma_\kappa)\partial_t^\tau h\, dy_1\, dt + \tag{5.5} \\ + \int_0^{T-\tau}\int_t^{t+\tau}\Big\{((\boldsymbol{u}_\kappa(s), \partial_t^\tau(h\boldsymbol{u}_\kappa))) - \int_D h(s)q_\kappa(s)\,\mathrm{div}\partial_t^\tau(h\boldsymbol{u}_\kappa)dy \\ + \int_D h(s)\mathrm{div}\boldsymbol{u}(s)\partial_t^\tau(hq_\kappa) + \dots dy + \int_0^L \dots dy_1 + \int_0^1 \dots dy_2\Big\} ds\, dt.$$

The property (5.3) implies, that the right hand side of (5.5) does not depend on ε. Moreover, it does not depend on κ, since the corresponding boundary term is bounded

$$\kappa\tau\int_0^{T-\tau}\int_0^L \left[u_{2,\kappa} - \sigma_\kappa\right]_\tau(t)\partial_t^\tau\left(h\big(u_{2\kappa} - \sigma_\kappa\big)\right) dy_1 dt \le C\tau \tag{5.6}$$

independently on κ. Here the notation for the so-called Steklov average is used

$$[\phi]_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau}\phi(s)ds. \tag{5.7}$$

Indeed, it holds $\|[\phi]_\tau\|_{L^2((0,T-\tau)\times D)} \leq \|\phi\|_{L^2((0,T)\times D)}$, which implies (5.6) with a use of (5.1), see also [15, Section 8].
In what follows we will use the following property

$$|\tau_{ij}(\hat{e}(\boldsymbol{u}))| \leq C_5(1+|\hat{e}(\boldsymbol{u})|)^{p-1}, \tag{5.8}$$

which is derived from (3.1), (3.4), cf. [28, Lemma 1.19].

Now we concentrate on the new viscous term $((\boldsymbol{u}_\kappa(s), \partial_t^\tau h\boldsymbol{u}_\kappa))$ on the right hand side of (5.5) and show, that it is bounded with $C\tau$. Indeed, we get (for the sake of simplicity we omit indices κ)

$$\begin{aligned}
&\tau \int_0^{T-\tau}\!\!\int_D \frac{1}{\tau}\int_t^{t+\tau} \tau_{ij}\big(\hat{e}(\boldsymbol{u}(s))\big)ds\big\{\hat{e}_{ij}(h\boldsymbol{u}(t+\tau)) - \hat{e}_{ij}(h\boldsymbol{u}(t))\big\}dy\,dt \qquad (5.9)\\
&\overset{(5.8)}{\leq} C_5\tau \int_0^{T-\tau}\!\!\int_D \big[1+|\hat{e}(\boldsymbol{u})|^{p-1}\big]_\tau(t)\big\{\hat{e}_{ij}(h\boldsymbol{u}(t+\tau)) - \hat{e}_{ij}(h\boldsymbol{u}(t))\big\}dy\,dt\\
&\leq c\tau\big\|[1+|\hat{e}(\boldsymbol{u})|^{p-1}]_\tau(t)\big\|_{L^{p'}(Q_{T-\tau})}\Big\||\hat{e}(\boldsymbol{u}(t+\tau))| + |\hat{e}(\boldsymbol{u}(t))|\Big\|_{L^p(Q_{T-\tau})},
\end{aligned}$$

where $p' = p/(p-1)$ and $Q_{T-\tau} = (0,T-\tau)\times D$. For the Steklov average it is not difficult to show

$$\|[\phi]_\tau\|_{L^r((0,T-\tau)\times D)} \leq \|\phi\|_{L^r((0,T)\times D)} \quad \forall r > 1. \tag{5.10}$$

Since $\||\boldsymbol{u}(t)|^{p-1}\|_{L^{p'}((0,T)\times D)} = \|\boldsymbol{u}(t)\|^{p-1}_{L^p((0,T)\times D)}$ we conclude from (5.9) and (5.10) that

$$\int_0^{T-\tau}\int_t^{t+\tau} ((\boldsymbol{u}_\kappa(s), \partial_t^\tau h\boldsymbol{u}_\kappa))\,ds\,dt \leq c\tau.$$

Estimates of other terms on the right hand side of (5.5) has been done in [41] and [15] and we omit them here. The proof of estimate (5.4) is now complete.

The estimate (5.4) and the compactness argument from [2, Lemma 1.9] imply the following strong convergences for $\kappa \to \infty$

$$\boldsymbol{u}_\kappa \to \boldsymbol{u} \text{ in } L^1((0,T)\times D), \qquad \sigma_\kappa \to \sigma \text{ in } L^1((0,T)\times(0,L)).$$

Using the standard interpolations of spaces $L^r(Q_T)$ and $L^s(S_T)$, $Q_T = (0,T)\times D$, $S_T = (0,T)\times(0,L)$ and boundedness of $\boldsymbol{u}, \sigma$ in $L^4(Q_T)$, $L^6(S_T)$, respectively, we obtain

$$\boldsymbol{u}_\kappa \to \boldsymbol{u} \text{ in } L^r((0,T)\times D), \qquad \sigma_\kappa \to \sigma \text{ in } L^s((0,T)\times(0,L)),$$

where $1 \leq r < 4,\ 1 \leq s < 6$ for $\kappa \to \infty$.

Now let us consider test functions $\boldsymbol{\psi} \in L^p(0,T;X), \boldsymbol{\psi}(T) = 0$,

$$\begin{aligned}
X &= \{\boldsymbol{\psi} \in \boldsymbol{V}_{div}\,;\ \boldsymbol{\psi}_2\big|_{S_w} \in H_0^2(0,L)\}, &(5.11)\\
\boldsymbol{V}_{div} &= \{f \in \boldsymbol{V},\ \mathrm{div}_h f = 0 \ a.e. \text{ on } D\},\ \text{cf. (2.9)}
\end{aligned}$$

and $\xi = E\boldsymbol{\psi}_2\big|_{S_w}$ in (2.13). With this choice of test functions the boundary terms with κ are canceled.

Now, we can pass to the limit as $\kappa \to \infty$ in (2.13), where $\kappa = \varepsilon^{-1}$. We use the weak convergences of $\boldsymbol{u}_\kappa$ in $L^p(0,T;\boldsymbol{V}_{div})$, $\sqrt{\varepsilon}q_\kappa$ in $L^2(0,T;H^1(D))$, σ_κ in $L^2(0,T;H^2(0,L))$, see (5.2), the strong convergence of $h\boldsymbol{u}_\kappa$ for in $L^r((0,T)\times D)$, $0 \le r < 4$ and the Minty Trick for the viscous term.The limiting process in the viscous term is analogous to the limiting process for $n \to \infty$ in Section 4.2.

The convergence of the convective term for $\boldsymbol{\psi} \in H^1(0,T;X)$ can be obtained for all $p > 2$ in following way. For case $p = 2$ see [15, Section 8]. In order to obtain $\int_0^T b(\boldsymbol{u}_\kappa, \boldsymbol{u}_\kappa, \boldsymbol{\psi}) \to \int_0^T b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})$ one needs to show that $\int_0^T |B(\boldsymbol{u}_\kappa - \boldsymbol{u}, \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$, $\int_0^T |B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$. Indeed, using the Hölder inequality and imbedding $L^{\frac{2p}{p-2}}(D) \hookrightarrow W^{1,p}(D)$ we have

$$\begin{aligned}\int_0^T |B(\boldsymbol{u}_\kappa - \boldsymbol{u}, \boldsymbol{u}_\kappa, \boldsymbol{\psi})| &\le C(K,\alpha) \int_0^T \|\boldsymbol{u}_\kappa - \boldsymbol{u}\|_2 \|\boldsymbol{u}_\kappa\|_{1,p} \|\boldsymbol{\psi}\|_{\frac{2p}{p-2}} \\ &\le C(K,\alpha)\|\boldsymbol{\psi}\|_{H^1(0,T;W^{1,p}(D))} \|\boldsymbol{u}_\kappa - \boldsymbol{u}\|_{L^2((0,T)\times D)} \|\boldsymbol{u}_\kappa\|_{L^p(0,T;W^{1,p}(D))}.\end{aligned} \tag{5.12}$$

Thus $\int_0^T |B(\boldsymbol{u}_\kappa - \boldsymbol{u}, \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$. Further $\int_0^T |B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$ due to the weak convergence of $\boldsymbol{u}_\kappa$ in $L^p(0,T;\boldsymbol{V}_{div})$.

The convergence of the terms containing $\sqrt{\varepsilon}q_\kappa$ can be realized by the weak convergence in the corresponding spaces. The term $\int_0^T \int_D h q_\kappa \mathrm{div}_h \boldsymbol{\psi}$ is canceled due to the solenoidal test functions.

Finally, after the limiting process $\kappa \to \infty$ in (2.13) using above considerations for all $\boldsymbol{\psi} \in H_0^1(0,T;X)$, $\xi = E\boldsymbol{\psi}_2\big|_{S_w}$ we arrive at

$$\begin{aligned}\int_0^T \int_D \left\{ h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} \right\} dy = \\ \int_0^T \Bigg\{ \; ((\boldsymbol{u},\boldsymbol{\psi}))_h \; + \; b_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}) \\ + \int_0^1 h(L) q_{out}(y_2,t)\psi_1\,(L,y_2,t) - h(0) q_{in}(y_2,t)\psi_1\,(0,y_2,t)\,dy_2 \\ + \int_0^L \left(q_w + \frac{1}{2}\frac{\partial h}{\partial t} u_2 \right) \psi_2\,(y_1,1,t)\,dy_1 \\ + \int_0^L -\sigma \frac{\partial \xi}{\partial t} + c\frac{\partial^2 \sigma}{\partial y_1^2}\frac{\partial^2 \xi}{\partial y_1^2} + a\frac{\partial}{\partial y_1}\int_0^t \sigma(y_1,s)ds \frac{\partial \xi}{\partial y_1} \\ - a\frac{\partial^2 R_0}{\partial y_1^2}\xi + b\int_0^t \sigma(y_1,s)ds\,\xi(y_1,t)\,dy_1 \Bigg\}\,dt.\end{aligned} \tag{5.13}$$

In order to investigate the meaning of the left hand side of the above equality we define the ALE-type time derivative $\bar{\partial}_t$

$$\bar{\partial}_t(h\boldsymbol{u}) := \frac{\partial (h\boldsymbol{u})}{\partial t} - \frac{\partial h}{\partial t}\frac{1}{h}\frac{\partial (y_2 h \boldsymbol{u})}{\partial y_2}. \tag{5.14}$$

Note that $\bar{\partial}_t(h\boldsymbol{u}) = h\partial_t^y \boldsymbol{u}$, where $\partial_t^y := \left(\frac{\partial}{\partial t} - \frac{\partial h}{\partial t}\frac{y_2}{h}\frac{\partial}{\partial y_2}\right)$ denotes in fact the time derivative transformed to the rectangle domain D, i.e., in coordinates (y_1, y_2).

The right hand side of (5.13) is bounded for every $\boldsymbol{\psi} \in \mathcal{M}$,

$$\begin{aligned} \mathcal{M} \;=\; & \{\omega \in L^p(0,T;X) \text{ for } p > 2; \\ & \omega \in L^p(0,T;X) \cap L^4((0,T)\times D) \text{ for } p = 2\}. \end{aligned} \tag{5.15}$$

Thus it can be identified with some functional $\chi \in \mathcal{M}^*$. Then using integration by parts with respect to y_2 on the left hand side, backward transformation from D to the moving domain $\Omega(h(t))$ and the separation of variables it can be shown that $\chi = \bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*)$, see Appendix A for more details. Thus we can replace

$$\int_0^T \int_D \left\{ h\boldsymbol{u}\cdot\frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2\boldsymbol{u})}{\partial y_2}\cdot\boldsymbol{\psi} \right\} dy\,dt = -\int_0^T \left\langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi} \right\rangle_X .$$

Finally, we transform (5.13) from the rectangle D to the moving domain $\Omega(h(t))$ and obtain the existence of a weak solution to our original problem (1.1)–(1.12) with the Dirichlet boundary condition $\partial_t \eta = v_2|_{\Gamma_w(h(t))}$ for a prescribed domain deformation h.

Theorem 5.1 (Existence of weak solution for $\varepsilon = 0$, $\kappa = \infty$).
Assume that $h \in H^1(0,T;H_0^2(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L))$ satisfies (2.1). Let the boundary data fulfill $q_{in}, q_{out} \in L^{p'}(0,T;L^2(0,1))$, $q_w \in L^{p'}(0,T;L^2(0,L))$. Furthermore, assume that the properties (3.1)–(3.4) for the viscous stress tensor hold. Then there exists a weak solution $(\boldsymbol{v}, \eta)$ of the problem (1.1)–(1.12), such that

i) $(\boldsymbol{u}, \eta) \in [L^p(0,T;\boldsymbol{V}) \times H^1(0,T;H_0^2(0,L))] \cap [L^\infty(0,T;L^2(D)) \times W^{1,\infty}(0,T;L^2(0,L))]$*, where $\boldsymbol{u}$ is defined in (2.8),*

ii) the time derivative $\bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*)$ for $p > 2$ and $\bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*) \oplus L^{4/3}((0,T)\times D)$ for $p = 2$,

$$\int_0^T \int_D \left\{ h\boldsymbol{u}\cdot\frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2\boldsymbol{u})}{\partial y_2}\cdot\boldsymbol{\psi} \right\} dy\,dt = -\int_0^T \left\langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi} \right\rangle dt,$$

where $\bar{\partial}_t(h\boldsymbol{u}) = \frac{\partial(h\boldsymbol{u})}{\partial t} - \frac{1}{h}\frac{\partial h}{\partial t}\frac{\partial(y_2 h\boldsymbol{u})}{\partial y_2} = h\partial_t^y\boldsymbol{u}$,
for every test function $\boldsymbol{\psi} \in \mathcal{M} \cap H_0^1(0,T;X)$,

iii) $\boldsymbol{v}$ satisfies the condition div $\boldsymbol{v} = 0$ a.e on $\Omega(h(t))$,
$v_2(x_1, h(x_1,t), t) = \partial_t\eta(x_1,t)$ for a.e. $x_1 \in (0,L)$, $t \in (0,T)$

and the following integral identity holds

$$
\begin{aligned}
&\int_0^T \int_{\Omega(h(t))} \Big\{ -\rho \boldsymbol{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\boldsymbol{v})|) e(\boldsymbol{v}) e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i} \boldsymbol{\varphi}_j \Big\} dx\, dt \\
&+ \int_0^T \int_0^{R_0(L)} \left(P_{out} - \frac{\rho}{2} |v_1|^2 \right) \varphi_1(L, x_2, t)\, dx_2\, dt \\
&- \int_0^T \int_0^{R_0(0)} \left(P_{in} - \frac{\rho}{2} |v_1|^2 \right) \varphi_1(0, x_2, t)\, dx_2\, dt \\
&+ \int_0^T \int_0^L \left(P_w - \frac{\rho}{2} v_2 \left(v_2 - \frac{\partial h}{\partial t} \right) \right) \varphi_2(x_1, h(x_1,t), t)\, dx_1\, dt \\
&+ \int_0^T \int_0^L - \frac{\partial \eta}{\partial t} \frac{\partial \xi}{\partial t} + c \frac{\partial^3 \eta}{\partial x_1^2 \partial t} \frac{\partial^2 \xi}{\partial x_1^2} + a \frac{\partial \eta}{\partial x_1} \frac{\partial \xi}{\partial x_1} dx_1\, dt \\
&+ \int_0^T \int_0^L -a \frac{\partial^2 R_0}{\partial x_1^2} \xi + b \eta\, \xi\, dx_1\, dt \;=\; 0
\end{aligned}
$$

for every test functions

$$
\begin{aligned}
&\boldsymbol{\varphi}(x_1, x_2, t) = \boldsymbol{\psi} \left(x_1, \frac{x_2}{h(x_1,t)}, t \right) \quad \textit{such that} \\
&\boldsymbol{\psi} \in H_0^1(0,T; \boldsymbol{V}), \;\; \boldsymbol{\psi}_2\big|_{S_w} \in H_0^1(0,T; H_0^2(0,L)), \\
&div\, \boldsymbol{\varphi} = 0 \quad \textit{a.e. on } \Omega(h(t)), \\
&\textit{and} \quad \xi(x_1, t) = E\rho\, \varphi_2(x_1, h(x_1,t), t).
\end{aligned}
$$

Note that the structure equation is fulfilled in a slightly modified sense,

$$
\begin{aligned}
E\rho \left[\frac{\partial^2 \eta}{\partial t^2} - a \frac{\partial^2 \eta}{\partial x_1^2} + b\eta + c \frac{\partial^5 \eta}{\partial t \partial x_1^4} - a \frac{\partial^2 R_0}{\partial x_1^2} \right] (x_1, t) = \\
\left[-(\mathbf{T}_f + P_w \mathbf{I}) \boldsymbol{n} |\boldsymbol{n}| \cdot \boldsymbol{e}_2 + \frac{\rho}{2} \partial_t \eta (\partial_t \eta - \partial_t h) \right] (x_1, h(x_1,t), t)
\end{aligned}
$$

a.e. on $(0,T) \times (0,L)$, compare (1.6).

6 Fixed point iterations

Until now we have proved the existence of weak solution of the original problem in a domain given by a known deformation function, i.e., $h(x_1,t) = R_0(x_1) + \delta(x_1,t)$, $\delta \in H^1(0,T; H_0^2(0,L)) \cap W^{1,\infty}(0,T; L^2(0,L))$, $R_0(x_1) \in C^2[0,L]$. The aim of this section is to show the existence of the weak solution of (1.13), which implies, that the domain deforms according to the function $\eta(x_1,t)$, i.e., $h = R_0 + \eta$. To this end we apply the Schauder fixed point theorem and we obtain the final result: existence of weak solution for a fully coupled fluid structure interaction problem (1.1)–(1.12).

Let us denote the space $Y = H^1(0,T; L^2(0,L))$. For each test function $\boldsymbol{\psi} \in L^p(0,T; X)$, $\boldsymbol{\psi}(T) = 0$, recalling (5.11), $\xi = E\boldsymbol{\psi}_2\big|_{S_w}$ and for any $h =$

$R_0+\delta \in Y$, such that (2.1) holds we construct solution $(\boldsymbol{u},\, \eta)$ of the following problem defined on the reference domain D, $\sigma = \partial_t \eta$

$$
\begin{aligned}
& -\int_0^T \left\langle \bar{\partial}_t (h\boldsymbol{u}), \boldsymbol{\psi} \right\rangle \, dt \qquad (6.16)\\
&= \int_0^T \Bigg\{ \; ((\boldsymbol{u}, \boldsymbol{\psi}))_h \; + \; b_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}) \\
&\quad + \int_0^1 h(L) q_{out}(y_2, t) \psi_1\,(L, y_2, t) - h(0) q_{in}(y_2, t) \psi_1\,(0, y_2, t)\, dy_2 \\
&\quad + \int_0^L \Big(q_w + \frac{1}{2}\frac{\partial h}{\partial t}\sigma \Big) \psi_2\,(y_1, 1, t)\, dy_1 \\
&\quad + \langle \partial_t \sigma, \xi \rangle + \int_0^L c\, \frac{\partial^2 \sigma}{\partial y_1^2}\frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1} \int_0^t \sigma(y_1, s) ds \frac{\partial \xi}{\partial y_1} \\
&\qquad\qquad - a \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \int_0^t \sigma(y_1, s) ds\, \xi(y_1, t)\, dy_1 \Bigg\} \, dt \, .
\end{aligned}
$$

Further, let the ball $B_{\alpha,K}$ be defined by

$$
\begin{aligned}
B_{\alpha,K} = \Big\{ & \delta \in Y;\; \|\delta\|_Y \le C(\alpha, K),\; 0 < \alpha \le R_0(y_1) + \delta(y_1, t) \le \alpha^{-1}, \\
& \Big| \frac{\partial \delta(y_1, t)}{\partial y_1} \Big| \le K, \;\; \delta(y_1, 0) = 0, \; \forall y_1 \in [0, L], \; \forall t \in [0, T], \\
& \int_0^T \Big| \frac{\partial \delta(y_1, t)}{\partial t} \Big|^2 dt \le K, \; \forall y_1 \in [0, L] \Big\},
\end{aligned}
$$

where $C(\alpha, K)$ is a suitable constant large enough with respect to K, α and the data.

By choosing $\delta \in B_{\alpha,K}$ the following energy estimate holds for all $2 \le p < \infty$ uniformly in δ,

$$
\begin{aligned}
& \|\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D))} + \|\boldsymbol{u}\|^p_{L^p(0,T;W^{1,p}(D))} \qquad (6.17)\\
&\quad + \|\eta_t\|^2_{L^\infty(0,T;L^2(0,L))} + \|\eta_t\|^2_{L^2(0,T;H^2(0,L))} + \|\eta\|^2_{L^\infty(0,T;H^1(0,L))} \\
&\quad \le c(T, p, K, \alpha) \left(\|\mathcal{P}\|^{p'}_{L^{p'}(0,T)} + \|R_0\|^2_{C^2[0,L]} \right) .
\end{aligned}
$$

This estimate is obtained by multiplying (6.16) by $\boldsymbol{\psi} = \boldsymbol{u}$ and $\xi = E u_2|_{S_w} = E\eta_t$, cf. (5.1).

Now, let us define the following mapping by (6.16),

$$
\begin{aligned}
& \mathcal{F} : B_{\alpha,K} \to Y; \\
& \mathcal{F}(\delta) = \eta, \quad (\delta = h - R_0).
\end{aligned}
$$

Our aim is to apply the Schauder fixed point theorem and prove that the mapping $\mathcal{F}$ has at least one fixed point. This implies the existence of the weak solution to our problem (6.16).

First we have to check that $\mathcal{F}(B_{\alpha,K}) \subset B_{\alpha,K}$. Note that our a priori estimate (6.17) yields $\|\eta_{y_1}\|_{C([0,T]\times[0,L])} \le K$, $\|\eta_t\|_{L^2(0,T;C[0,L])} \le K$ and $\|\eta\|_Y \le C(\alpha, K)$ for given data P_{in}, P_{out}, P_w, R_0, given K, α; $\alpha < R_{min} := \min_{y_1\in[0,L]} R_0(y_1)$ and for sufficiently small time $\tilde{T}$. Moreover, since $H^1(0,T; H^2(0,L)) \hookrightarrow C(0,T;C^1[0,L])$ and $\eta(y_1, 0) = 0$, there exist a maximal time T_{max}, such that

i) $\|\eta\|_\infty := \|\eta\|_{C([0,T_{max}]\times[0,L])} \le R_{min} - \alpha$.
This yields that $\min_{t\in(0,T_{max})} \min_{y_1\in(0,L)} (R_0 + \eta) \ge R_{min} - \|\eta\|_\infty \ge \alpha$. Thus we can avoid a contact of the regularized deforming wall with the solid bottom.

ii) Further, we require that the domain deformation is bounded from above, $\|R_0 + \eta\|_\infty \le \alpha^{-1}$.

Having *i)*, the condition *ii)* is satisfied if $R_{min} - \alpha \le \alpha^{-1} - R_{max}$. Thus, for instance if $\alpha^{-1} \ge R_{min} + R_{max}$.
Consequently, $\mathcal{F}(B_{\alpha,K}) \subset B_{\alpha,K}$ as far as $t \le T^* := \min\{T_{max}, \tilde{T}\}$ for given data P_{in}, P_{out}, P_w, R_0, K and α such that $\alpha \le \min\{R_{min}, \frac{1}{R_{min}+R_{max}}\}$.

Secondly, we need to verify that $\mathcal{F}(\delta) = \eta$ is relatively compact in Y. Let us consider a sequence $\{\delta^{(k)}\}_{k=1}^\infty$ in $B_{\alpha,K}$. Let us denote by $\boldsymbol{u}^{(k)}$ and $\eta^{(k)} \equiv \mathcal{F}(\delta^{(k)})$ the weak solution of (6.16) for $h = h^{(k)} := R_0 + \delta^{(k)}$. Due to the apriori estimate (6.17) we have the weak convergences of $\eta^{(k)}$, $\boldsymbol{u}^{(k)}$ in corresponding spaces. In order to obtain strong convergences of $\eta^{(k)}$ in Y (and of $\boldsymbol{u}^{(k)}$ in $L^2(0,T;L^2(D))$) we use the result on the equicontinuity in time. For the formulation in the Eulerian coordinates this yields

$$\int_0^{T-\tau}\int_{B_M} |\chi_{t+\tau}^{(k)} \bar{\boldsymbol{v}}^{(k)}(t+\tau) - \chi_t^{(k)} \bar{\boldsymbol{v}}^{(k)}(t)|^2 + \int_0^{T-\tau}\int_0^L |\eta_t^{(k)}(t+\tau) - \eta_t^{(k)}(t)|^2 \le C(K,\alpha)(\tau^{1/p} + \tau^{1/2}). \tag{6.18}$$

Here $B_M \in \mathbb{R}^2$ is the fixed rectangle domain $(0, L) \times (0, M)$, $M > \alpha^{-1}$, cf. (9.1), such that $\Omega(h^{(k)}(t)) \subset B_M$ for all k, $\chi_t^{(k)}$ is the characteristic function of $\Omega(h^{(k)}(t))$ on B_M and $\bar{\boldsymbol{v}}^{(k)}$ is an extension of the weak solution $\boldsymbol{v}^{(k)}$ to the B_M defined in (9.2). Note, that the constant $C(K, \alpha)$ does not depend on k. This estimate can be obtained using a suitable extension of the weak solution to a fixed domain B_M and specific divergence free test functions. The proof of (6.18) is realized in analogous way as in [9, Lemma 9], the details can be found in Appendix B, Lemma 9.1.

Consequently, the Riesz-Fréchet-Kolmogorov compactness argument [4, Theorem IV.26] based on (6.18) implies the relative compactness of $\partial_t\eta^{(k)}$, $\bar{\boldsymbol{v}}^{(k)}$

in $L^2(0,T;L^2(0,L))$, $L^2(0,T;L^2(B_M))$, respectively. Additionally, the standard interpolations give us the compactness of $\bar{\boldsymbol{v}}^{(k)}$ in $L^r((0,T)\times B_M)$, $1\leq r<4$ and $\partial_t\eta^{(k)}$ in $L^s((0,T)\times(0,L))$, $1\leq s<6$.

Finally, we need to check that the mapping $\mathcal{F}$ is continuous with respect to the strong topology in Y. We have to prove that for any convergent subsequence $\delta^{(k)}\in B_{\alpha,K}$, $\delta^{(k)}\to\delta$ in Y

$$\mathcal{F}(\delta^{(k)})=\eta^{(k)}\to\mathcal{F}(\delta)=\eta.$$

As already shown above $\eta^{(k)}$ converges strongly to some η in Y, i.e., we have $\eta^{(k)}\to\eta$ in $H^1(0,T;L^2(0,L))$ as $k\to\infty$. Due to the boundedness of η from the apriori estimate (6.17) and the imbeddings in one dimension we have even stronger result - the uniform convergence of $\partial_{y_1}\eta^{(k)}$ in $C([0,T]\times[0,L])$. Indeed,

$$\begin{aligned} L^\infty(0,T;H^2(0,L))\cap W^{1,\infty}(0,T;L^2(0,L)) \\ \hookrightarrow C^{0,1-\beta}(0,T;H^{2\beta}(0,L)) \end{aligned}\tag{6.19}$$

for $0<\beta<1$. From the continuous imbedding of $H^{2\beta}(0,L)$ into $H^{2\beta-\epsilon}(0,L)$ and the Arzelá-Ascoli Lemma we conclude that a subsequence of $\eta^{(k)}$ converges strongly in $C([0,T];H^s(0,L))$, $0<s<2$. Since for $s>3/2$ we also have continuous imbedding $H^s(0,L)\hookrightarrow C^1[0,L]$, we can conclude, that $\eta^{(k)}\to\eta$ strongly in $C(0,T;C^1[0,L])$.

Before we start the limiting process in (6.16), let us summarize available convergences.

$$\begin{aligned}
\boldsymbol{u}^{(k)} &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^p(0,T;W^{1,p}(D)),\\
\bar{\boldsymbol{v}}^{(k)} &\to \bar{\boldsymbol{v}} && \text{strongly in } L^r((0,T)\times B_M),\ 1\leq r<4,\\
\boldsymbol{u}^{(k)} &\to \boldsymbol{u} && \text{strongly in } L^r((0,T)\times D),\ 1\leq r<4,\\
\eta^{(k)} &\rightharpoonup \eta && \text{weakly in } H^1(0,T;H^2(0,L)),\\
\eta^{(k)} &\rightharpoonup^* \eta && \text{weakly* in } L^\infty(0,T;L^2(0,L))\\
\eta^{(k)} &\to \eta && \text{uniformly in } C(0,T;C^1[0,L]),\\
\partial_t\eta^{(k)} &\to \partial_t\eta && \text{strongly in } L^s((0,T)\times(0,L)),\ 1\leq s<6
\end{aligned}\tag{6.20}$$

We have to verify, that the limit η from (6.20) is the weak solution associated with δ and thus $\mathcal{F}(\delta)=\eta$.

Limiting process

Let us consider (6.16), where now $\boldsymbol{u}^{(k)}$, $h^{(k)}$ stay instead of $\boldsymbol{u}$, h and $\sigma^{(k)}=\partial_t\eta^{(k)}$ instead of σ and let $k\to\infty$. First of all we have to realize, that due to the solenoidal property, which depends on $h^{(k)}$, the test functions are implicitly dependent on k. This fact present a difficulty when we pass with $k \to \infty$. Nevertheless we can construct sufficiently smooth test functions $\tilde{\boldsymbol{\psi}}(y,t) = \tilde{\boldsymbol{\varphi}}(x,t)$, which are independent on k and divergence free in $\Omega(h)$, $h = R_0 + \delta$ (i.e. $\mathrm{div}_h \tilde{\boldsymbol{\psi}} = 0$). They are also well defined on infinitely many approximate domains $\Omega(h^{(k)})$ and dense in the space of admissible test functions $L^p(0,T;X)$, cf. (5.11). Such a test functions $\tilde{\boldsymbol{\varphi}}$ can be constructed on $(0,T) \times B_M$ as algebraic sum, see [9, Remark 3]

$$\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_1,$$

where $\boldsymbol{\varphi}_0$ is a smooth function with compact support in $\Omega(h)$, $\mathrm{div}\boldsymbol{\varphi}_0 = 0$ on $\Omega(h)$ and $\boldsymbol{\varphi}_0$ is extended by 0 to $(0,T) \times B_M$. Further, having $\xi \in H^1(0,T;H_0^2(0,L))$ we define $\boldsymbol{\varphi}_1 \overset{\text{def}}{=} (0, \xi(x_1)/E)$ on $B_M \backslash B_\alpha$, $B_\alpha = (0,L) \times (0,\alpha) \in \mathbb{R}^2$, the constant E comes from (1.14). Note that $\mathrm{div}\boldsymbol{\varphi}_1 = 0$ on $B_M \backslash B_\alpha$. Moreover, $\boldsymbol{\varphi}_1$ such that $\int_{\partial B_\alpha} \boldsymbol{\varphi}_1 \cdot n = \int_0^\alpha \boldsymbol{\varphi}_1^1(L,x_2,t) - \boldsymbol{\varphi}_1^1(0,x_2,t)dx_2 + \int_0^L \frac{\xi}{E}(x_1,t)dx_1 = 0$ can be extended into B_α by a divergence-free extension, whereas remaining boundary conditions on Γ_{in}, Γ_{out}, Γ_c are preserved, see e.g., [18, p.144]. Note, that due to the uniform convergence of $\eta^{(k)}$ the function $\boldsymbol{\varphi}_0$ is defined on each $\Omega(h^{(N)})$ for sufficiently large N. Moreover $\boldsymbol{\varphi}_1$ is defined on $\Omega(h^{(k)})$ for each k. For more details on this construction we refer a reader to [8, Section 7, pp. 35-36], compare [9].

Having $\tilde{\boldsymbol{\psi}}(y,t) = \tilde{\boldsymbol{\psi}}(x_1, \frac{x_2}{h(x_1,t)}, t) = \tilde{\boldsymbol{\varphi}}(x,t)$, $x \in \Omega(h)$, $y \in D$, let us construct the set of admissible test functions $\boldsymbol{\psi}^{(k)}$ by transformation of $\tilde{\boldsymbol{\varphi}}$ from $\Omega(h^{(k)})$ into D,

$$\boldsymbol{\psi}^{(k)}(y_1,y_2,t) := \tilde{\boldsymbol{\psi}}(x_1, \frac{x_2}{h^{(k)}(x_1,t)}, t) = \tilde{\boldsymbol{\varphi}}(x_1,x_2,t), \qquad x \in \Omega(h^{(k)}),\ y \in D. \tag{6.21}$$

The test functions (6.21) have the following property

$$\boldsymbol{\psi}^{(k)} : D \to \mathbb{R}^2; \quad \mathrm{div}_{h^{(k)}} \boldsymbol{\psi}^{(k)} = 0, \quad E\boldsymbol{\psi}_2^{(k)}(y_1,1,t) = \xi(y_1,t), \quad \text{and}$$
$$\left.\begin{array}{l} \boldsymbol{\psi}^{(k)} \to \tilde{\boldsymbol{\psi}}, \\ \hat{e}_{h^{(k)}}(\boldsymbol{\psi}^{(k)}) \to \hat{e}(\tilde{\boldsymbol{\psi}}) \end{array}\right\} \quad \text{uniformly on } (0,T) \times D.$$

This property follows from the special construction of $\tilde{\boldsymbol{\varphi}}$, the property (3.8) and the uniform convergence of $\delta^{(k)}$ and $\partial_{y_1}\delta^{(k)}$ that follows from (6.19).

Thus it is enough to consider test functions $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}}$, which are independent on k and smooth enough. The limiting process in the test functions follows afterwards using the uniform convergence $\boldsymbol{\psi}^{(k)}$ and $\hat{e}(\boldsymbol{\psi}^{(k)})$.

In the following lines we will present the limiting process for $k \to \infty$ in chosen non-linear terms. Let us first consider the convective term and show

$$\int_0^T \Big(b_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)}, \boldsymbol{\psi}) - b_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}) \Big)\, dt \to 0.$$

Recalling (3.11), the following terms appear in the above expression

$$\int_0^T B_h(\boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{\psi}) + B_{h^{(k)}}(\boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)}, \boldsymbol{\psi}) + B_{(h^{(k)}-h)}(\boldsymbol{u}, \boldsymbol{u}^{(k)}, \boldsymbol{\psi}) dt.$$

To show the convergence of above integrals, we restrict ourselves only to the terms containing $\partial y_1 h^{(k)}$, convergence of terms with $h^{(k)}$ is analogous. Let us consider

$$\int_0^T \int_D \left(\frac{\partial \boldsymbol{u}^{(k)}}{\partial y_2} - \frac{\partial \boldsymbol{u}}{\partial y_2} \right) \cdot \boldsymbol{\psi} u_1 \frac{\partial h}{\partial y_1} + \frac{\partial \boldsymbol{u}^{(k)}}{\partial y_2} \cdot \boldsymbol{\psi} \left(u_1^{(k)} - u_1 \right) \frac{\partial h^{(k)}}{\partial y_1}$$
$$\frac{\partial \boldsymbol{u}^{(k)}}{\partial y_2} \cdot \boldsymbol{\psi} \left(\frac{\partial h^{(k)}}{\partial y_1} - \frac{\partial h}{\partial y_1} \right) u_1^{(k)} dy\, dt.$$

The convergence of the first term is obvious due to the weak convergence of $\boldsymbol{u}^{(k)}$ in $L^p(0,T;W^{1,p}(D))$. The strong convergence of $\boldsymbol{u}^{(k)}$ in $L^{p'}((0,T) \times (D))$ and the uniform convergence of $\partial_{y_1} h^{(k)}$ imply the convergence in the remaining two terms.
In what follows we denote $\hat{e}^{(k)} := (\hat{e})_{h^{(k)}}$, $\hat{e} := (\hat{e})_h$, cf. (2.11, 3.8). The limiting process in the viscous term will be realized as follows.

$$\begin{aligned}
&\int_0^T ((\boldsymbol{u}^{(k)}, \boldsymbol{\psi}))_{h^{(k)}} - ((\boldsymbol{u}, \boldsymbol{\psi}))_h dt \qquad (6.22)\\
&= \int_0^T \int_D h \left[\tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)}))\hat{e}_{ij}^{(k)}(\boldsymbol{\psi}) - \tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{\psi}) \right] \\
&\qquad + \left[h^{(k)} - h \right] \tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)}))\hat{e}_{ij}^{(k)}(\boldsymbol{\psi})\, dy\, dt \\
&= \underbrace{\int_0^T \int_D h\, \tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)})) \left[\hat{e}_{ij}^{(k)}(\boldsymbol{\psi}) - \hat{e}_{ij}(\boldsymbol{\psi}) \right] dy\, dt}_{\text{(I)}} \\
&+ \underbrace{\int_0^T \int_D h \left[\tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)})) - \tau_{ij}(\hat{e}(\boldsymbol{u})) \right] \hat{e}_{ij}(\boldsymbol{\psi})\, dy\, dt}_{\text{(II)}} \\
&+ \underbrace{\int_0^T \int_D \left[h^{(k)} - h \right] \tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)}))\hat{e}_{ij}^{(k)}(\boldsymbol{\psi})\, dy\, dt}_{\text{(III)}}.
\end{aligned}$$

It is easy to see that the term (III) goes to zero. Using the definition of $\hat{e} \equiv (\hat{e})_h = \nabla \boldsymbol{u} F(h, y_1) + (\nabla \boldsymbol{u} F(h, y_1))^T$, cf. (3.8), due to the uniform convergence of $h^{(k)}$ in $C(0,T;C^1[0,L])$ the convergence in all components of F is obvious and we obtain that (I) $\to 0$.

In order to show the convergence in the second term (II), we will use the Minty Trick argument. Let us denote for better readability $\xi^k := \hat{e}^{(k)}(\boldsymbol{u}^{(k)})$,

$\xi := \hat{e}(\boldsymbol{u})$ and $\phi := \hat{e}(\boldsymbol{\psi})$. Now we have the operator $\mathcal{A}$, $\mathcal{A} : L^p((0,T)\times D) \to L^{p'}((0,T)\times D)$,

$$\left\langle \mathcal{A}(\xi^k), \phi \right\rangle := \int_0^T \int_D h\,\tau_{ij}(\hat{e}_{h^{(k)}}(\boldsymbol{u}^{(k)}))(\hat{e}_{ij})_h(\boldsymbol{\psi})\,dy\,dt.$$

From Lemma 3.4 we know that the operator $\mathcal{A}$ is monotonous, i.e. $\left\langle \mathcal{A}(\xi^k) - \mathcal{A}(\xi), \xi^k - \xi \right\rangle \geq 0$. Thus, we have

$$\begin{aligned} &\liminf_{k\to\infty} \left\langle \mathcal{A}(\xi^k) - \mathcal{A}(\xi), \xi^k - \xi \right\rangle = \\ &\quad \liminf_{k\to\infty} \left\{ -\left\langle \mathcal{A}(\xi), \xi^k - \xi \right\rangle - \left\langle \mathcal{A}(\xi^k), \xi \right\rangle + \left\langle \mathcal{A}(\xi^k), \xi^k \right\rangle \right\} \geq 0. \end{aligned} \tag{6.23}$$

Further, from Lemma 3.5, assumptions (2.1) on $h^{(k)}$ and the fact that $\boldsymbol{u}, \boldsymbol{\psi} \in L^p(0,T;W^{1,p}(D))$ we have for any k, cf. (4.12),

$$\left| \left\langle \mathcal{A}(\xi^k), \phi \right\rangle \right| \leq C(K, \alpha).$$

Therefore $\mathcal{A}(\xi^k)$ is bounded in $L^{p'}((0,T)\times D)$ and thus $\mathcal{A}(\xi^k) \rightharpoonup f$. Moreover, from the weak convergence of $\nabla \boldsymbol{u}^{(k)}$ and the uniform convergence of $h^{(k)}$ in $C(0,T;C^1[0,L])$ we obtain that $\liminf_{k\to\infty} \left\langle \mathcal{A}(\xi), \xi^k - \xi \right\rangle = 0$ for $\xi^k = \hat{e}^{(k)}(\boldsymbol{u}^{(k)})$. Thus (6.23) implies that $\liminf_{k\to\infty} \langle \mathcal{A}(\xi^k), \xi^k \rangle \geq \langle f, \xi \rangle$. Moreover, analogously as in Section 4.2 we obtain by limiting in the weak formulation that $\lim_{k\to\infty} \langle \mathcal{A}(\xi^k), \xi^k \rangle = \langle f, \xi \rangle$. Thus, the Minty Trick argument concludes that $f = \mathcal{A}(\xi)$, i.e.

$$\mathcal{A}(\xi^k) \rightharpoonup \mathcal{A}(\xi) \quad \text{and thus} \quad \left\langle \mathcal{A}(\xi^k), \phi \right\rangle \to \langle \mathcal{A}(\xi), \phi \rangle$$

for any $\phi \in L^p((0,T)\times D)$ as $k \to \infty$.

This concludes the limiting process in (6.16). We found out that $\mathcal{F}(\delta^{(k)}) \to \mathcal{F}(\delta)$ as $k \to \infty$ and that $\mathcal{F}(\delta) = \eta$, i.e. η is the weak solution of (6.16) associated with the limit δ, $(h = R_0 + \delta)$.

Finally, the Schauder fixed point theorem implies that there exists at least one fixed point of the mapping $\mathcal{F}$ defined by the weak formulation (6.16), $\mathcal{F}(\eta) = \eta$. Thus, we obtain the existence of at least one weak solution of the original unsteady fluid-structure interaction problem (1.1) – (1.12). The proof of the Theorem 1.1 is now completed. ■

We summarize the result of this section. For all $p \geq 2$ there exists at least one weak solution to the original *fluid-structure interaction* problem (1.1) – (1.12) such that

i) $\boldsymbol{v} \in L^p(0,T;W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0,T;L^2(\Omega(\eta(t))))$,
$\eta \in W^{1,\infty}(0,T;L^2(0,L)) \cap H^1(0,T;H_0^2(0,L))$,

ii) $\operatorname{div} \boldsymbol{v} = 0$ a.e. on $\Omega(\eta(t))$,

iii) $\boldsymbol{v}\big|_{\Gamma_w(t)} = (0, \eta_t)$ for a.e. $x \in \Gamma_w(t),\ t \in (0,T)$, $v_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = 0$,

and the following integral identity holds

$$\begin{aligned}
&\int_0^T \int_{\Omega(\eta(t))} \Big\{ -\rho \boldsymbol{v}\cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i}\boldsymbol{\varphi}_j \Big\}\, dx\, dt \\
&+ \int_0^T \int_0^{R_0(L)} \Big(P_{out} - \frac{\rho}{2}|v_1|^2\Big)\varphi_1(L, x_2, t)\, dx_2\, dt \\
&- \int_0^T \int_0^{R_0(0)} \Big(P_{in} - \frac{\rho}{2}|v_1|^2\Big)\varphi_1(0, x_2, t)\, dx_2\, dt \\
&+ \int_0^T \int_0^L P_w \varphi_2(x_1, R_0(x_1) + \eta(x_1,t), t)\, dx_1\, dt \\
&+ \int_0^T \int_0^L -\frac{\partial \eta}{\partial t}\frac{\partial \xi}{\partial t} + c\,\frac{\partial^3 \eta}{\partial x_1^2 \partial t}\frac{\partial^2 \xi}{\partial x_1^2} + a\frac{\partial \eta}{\partial x_1}\frac{\partial \xi}{\partial x_1}\ dx_1\, dt \\
&+ \int_0^T \int_0^L -a\frac{\partial^2 R_0}{\partial x_1^2}\xi + b\eta\,\xi\ dx_1\, dt \quad = \quad 0
\end{aligned} \tag{6.24}$$

for every test functions $\boldsymbol{\varphi}$ with the property (1.14).

Remarks

1) It should be pointed out that we have obtained the existence of weak solution until some time T^* in Section 6. We remind that this time is obtained in order to achieve the fixed point of the mapping $\mathcal{F}$ and to avoid the contact of the elastic boundary $\Gamma_w(t)$ with the fixed boundary for given data P_{in}, P_{out}, P_w, R_0 and α, K. Similarly as in [9, Grandmont et al.], we can prolongate the solution in time and even obtain the global existence until the contact with the solid bottom.

Indeed, we can construct a non-decreasing sequence of times $\{T^* = T_1^*, \ldots, T_{m-1}^*, T_m^*, \ldots\}$, such that for given α, K, $\alpha \le \min\{R_{min}, \frac{1}{R_{min}+R_{max}}\}$, starting from initial data in time T_{m-1}^*, we have the existence of weak solution for some time $T_{m-1}^* + T := T_m^*$. We distinguish between two situations. Either $\sup T_m^* = \infty$, which means, that the contact with the solid bottom never happens and we obtain global existence. Otherwise $\sup T_m^* := T^{**} < \infty$ for given α. In this case we can decrease α. If the time interval of the existence cannot be prolongated for chosen α, we have to decrease α again. This is repeated until α reaches 0. The later represents the contact with the solid boundary at some time $T^{**} + \bar{T}$, where $\bar{T} \ge 0$.

2) Our result on the existence of weak solution for the coupled fluid-structure interaction problem for shear-thickening power-law fluids is shown

for the generalized string equation (1.4) with a regularizing term of type $\Delta^2\eta_t$. The same existence result can be obtained for other regularizing terms in the structure equation. Instead of

$$-a\Delta\eta + c\Delta^2\eta_t \quad \text{we can consider} \quad a\Delta^2\eta - c\Delta\eta_t.$$

The regularity of the domain deformation coming from the term $\Delta^2\eta$ is essential to obtain condition $|\eta_{x_1}| \leq K$, cf. (2.1). This is a necessary condition for generalized Korn's inequality for $p \neq 2$, see (3.8), (3.9).

In [9] the unsteady fluid-structure interaction between Navier-Stokes fluid in three dimensions and elastic plate has been analyzed. Note, that such a condition for η_{x_1} is not required for the Korn's 'equality' for the moving domain $\Omega(\eta)$ in [9]. Thus, for a two dimensional Newtonian fluid and one-dimensional structure a less regular string model may be used.

3) We would like to point out, that for the Navier-Stokes equations ($p = 2$) the integral equicontinuity for η_t and $\boldsymbol{u}$ can be obtained by a different method than that presented in Section 6 and proved in Appendix B. More precisely, we can follow the method of transformation of the solution from the domain in one time instance to the second one, previously used by Padula et al. in [20]. By this procedure the solenoidal property of test functions is preserved. For the Navier-Stokes equations we would have enough regularity to show then equicontinuity in time. For the non-Newtonian case ($p > 2$) however the regularity of η is not sufficient for such an approach. In this case we needed the construction of suitable test functions using an appropriate extension of the solution to the fixed domain, as it has been done in Lemma 9.1. Analogous construction was previously presented in [9], see also the reference [16] therein.

7 Conclusion

In the present paper we have proven the existence of weak solution to the fully nonlinear fluid-structure interaction problem for the shear-thickening fluid coupled with viscoelastic string.

The nonlinear stress tensor satisfies the polynomial growth conditions (3.1)–(3.4). For shear-thickening fluids ($p \geq 2$) this allows us to use the energy method and monotonicity arguments based on the Minty-Browder theorem to study the existence of the (κ, ε, h) - approximate solutions defined in Section 2. The existence of weak solutions on a time-depedent domain $\Omega(h(t))$ deforming according to a given, sufficiently smooth function $h(t)$ has been shown by limiting $\kappa \to \infty$ and $\varepsilon \to 0$. For the limiting processes additional compactness argument due to the integral equicontinuity in time has been used.

The final step regarding to the geometric nonlinearity of the fluid-structure interaction problem, i.e. the existence of a weak solution on the moving domain $\Omega(\eta(t))$ is proved in the last section by the fixed point procedure applying the Schauder fixed point theorem. Consequently we have obtained the existence of the weak solution to the shear-thickening non-Newtonian fluid coupled with an elastic string membrane, the main result is formulated in Theorem 1.1.

Our result generalizes the previous result [15] of one of the authors, where only the Newtonian fluid have been studied. In [15] the existence of a unique weak solution on the deforming domain $\Omega(\eta(t))$ with unknown interface η has been completed using Banach's fixed point approach only for the (κ, ε) - approximation of the coupled system. Furthermore, our result also generalizes the recent result of Čanić, Muha [8] and of Chambolle et al. [9] for the case of non-Newtonian shear thickening fluids.

In future we would like to study a generalization to three-dimensional geometries and more complex structural models as well as the generalization for shear thinning fluids.

Acknowledgment
The present research has been financed by the DFG project ZA 613/1-1, the Nečas Centrum for Mathematical Modelling LC06052 (financed by MSMT) and the Grant of the Czech Republic, No. P201/11/1304. It also has been partially supported by the 6th EU-Framework Programme under the Contract No. DEASE: MEST-CT-2005-021122 and the DST-DAAD project based personnel exchange program with Indian Institute of Science, Bangalore. We would like to thank Ján Filo (Comenius University, Bratislava) for fruitful discussions on the topic.

Appendix A (On distributive time derivative)

Our aim is to show that $\int_0^T \int_D \Big\{ h\boldsymbol{u}\cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2}\cdot \boldsymbol{\psi}\Big\}\, dy\, dt$ $= -\int_0^T \big\langle \bar{\partial}_t (h\boldsymbol{u}), \boldsymbol{\psi}\big\rangle_X$. Let us first recall the weak formulation of the κ - approximate problem, cf. (2.13),

$$
\begin{aligned}
& -\int_0^T \left\langle \frac{\partial (h\boldsymbol{u}_\kappa)}{\partial t}, \boldsymbol{\psi}\right\rangle + \int_D \frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u}_\kappa)}{\partial y_2}\cdot \boldsymbol{\psi}\; dt \; = \\
& \int_0^T \int_D \big\{ b(\boldsymbol{u}_\kappa, \boldsymbol{u}_\kappa, \boldsymbol{\psi}) - h\, q_\kappa \,\mathrm{div}_h \boldsymbol{\psi}\big\} dy \; + ((\boldsymbol{u}_\kappa, \boldsymbol{\psi}))\, dt \\
& + \int_0^T \int_0^1 h(L,t) q_{out} \psi_1 \; (L, y_2, t) - h(0,t) q_{in} \psi_1 \; (0, y_2, t) \; dy_2 dt \\
& + \int_0^T \int_0^L \left\{ q_w + \frac{1}{2} u_{2\kappa} \frac{\partial h}{\partial t} + \kappa\,(u_{2\kappa} - \sigma_\kappa)\right\} \psi_2 \; (y_1, 1, t)\, dy_1 dt \\
& + \varepsilon \int_0^T \left\langle \frac{\partial (h q_\kappa)}{\partial t}, \phi \right\rangle dt \qquad\qquad (8.1) \\
& + \int_0^T \int_D \left\{ -\varepsilon \frac{\partial h}{\partial t}\frac{\partial (y_2 q_\kappa)}{\partial y_2}\phi + \varepsilon a_1(q_\kappa, \phi) + \; h\, \mathrm{div}_h \boldsymbol{u}_\kappa \; \phi \right\} dy\, dt \\
& + \frac{\varepsilon}{2} \int_0^T \int_0^L \frac{\partial h}{\partial t}(y_1, t) q_\kappa \phi(y_1, 1, t)\, dy_1 dt \; + \\
& + \int_0^T \int_0^L \left\{ \frac{\partial \sigma_\kappa}{\partial t}\xi + c\,\frac{\partial^2 \sigma_\kappa}{\partial y_1^2}\frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1}\int_0^t \sigma_\kappa(y_1, s) ds \frac{\partial \xi}{\partial y_1} \right. \\
& \qquad \left. + b \int_0^t \sigma_\kappa(y_1, s) ds\; \xi - a \frac{\partial^2 R_0}{\partial y_1^2}\xi + \frac{\kappa}{E}\,(\sigma_\kappa - u_{2\kappa})\,\xi \right\} (y_1, t)\, dy_1 dt.
\end{aligned}
$$

The right hand side of (8.1) is bounded for each test function $\boldsymbol{\psi} \in \mathcal{M}$ defined in (5.15) independently on $\kappa = \varepsilon^{-1}$. Thus, taking into account (4.36) we obtain

$$
\partial_t (h\boldsymbol{u}_\kappa) - \partial_t h[\partial_{y_2}(y_2 \boldsymbol{u}_\kappa)] := \bar{\partial}_t (h\boldsymbol{u}_\kappa) \;\rightharpoonup\; \chi \in \mathcal{M}^* \;\; \text{as } \kappa \to \infty.
$$

In what follows we investigate the representation of the functional χ. For simplicity we restrict here on the case $2 < p < \infty$. The case $p = 2$ is analogous, but we need to consider $\boldsymbol{\psi} \in L^p(0,T;X) \cap L^4((0,T)\times D)$.
After the limiting process in κ we obtain for all $\boldsymbol{\psi} \in H_0^1(0,T;X)$

$$
-\int_0^T \int_D h\boldsymbol{u}\cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2}\cdot \boldsymbol{\psi} dy = \int_0^T \big\langle \chi, \boldsymbol{\psi}\big\rangle_X, \qquad (8.2)
$$

space X is defined in (5.11). Using integration by parts with respect to y_2 on the left hand side of (8.2) we obtain

$$
\int_0^T \int_D -h\boldsymbol{u}\cdot \left(\frac{\partial}{\partial t} - \frac{\partial h}{\partial t}\frac{y_2}{h}\frac{\partial}{\partial y_2}\right) \boldsymbol{\psi} dy - \int_0^L \frac{\partial h}{\partial t} u_2 \boldsymbol{\psi}_2 (y_1, 1, t) dy_1\, dt.
$$

Here we denote $\left(\frac{\partial}{\partial t} - \frac{\partial h}{\partial t}\frac{y_2}{h}\frac{\partial}{\partial y_2}\right) := \partial_t^y$. Note, that due to the transformation to the domain $\Omega(h(t))$ we have $\partial_t \boldsymbol{\varphi}(x,t) = \partial_t^y \boldsymbol{\psi}(y,t),\ x \in \Omega(h(t)),\ y \in D$. Moreover, the boundary term in the above expression is bounded for any $\boldsymbol{\psi} \in L^p(0,T;X)$ and can be added to the right hand side of (8.2), which we denote by a functional χ^b, i.e. $\int_0^T \left\langle \chi^b, \boldsymbol{\psi} \right\rangle_X = \int_0^T \left\langle \chi, \boldsymbol{\psi} \right\rangle_X + \int_0^L \frac{\partial h}{\partial t} u_2 \boldsymbol{\psi}_2(y_1,1,t)dy_1\, dt$. After the transformation of equation (8.2) to the moving domain we obtain the following equality in the Eulerian coordinates $x \in \Omega(h(t))$:

$$
\begin{aligned}
-\int_0^T \int_{\Omega(h(t))} \rho \boldsymbol{v} \partial_t \boldsymbol{\varphi}\, dx\, dt &= \int_0^T \left\langle h^{-1}\chi_x^b, \boldsymbol{\varphi} \right\rangle_{X_\Omega} dt, \qquad \text{where} \\
X_\Omega \quad = \quad & \{\boldsymbol{\varphi} \in W^{1,p}(\Omega);\ \operatorname{div} \varphi = 0 \text{ a.e. on } \Omega,\ \varphi_2|_{\Gamma_w} \in H_0^2(0,L), \\
& \varphi_1|_{\Gamma_w} = 0,\ \varphi_2|_{\Gamma_{in} \cup \Gamma_{out} \cup \Gamma_c} = 0\}, \quad \Omega = \Omega(h(t)).
\end{aligned}
$$

Since the divergence operator in the Eulerian coordinates does not depend on time, we can consider separable test functions $\boldsymbol{\varphi}$. Using $\boldsymbol{\varphi}(x_1,x_2,t) = \boldsymbol{w}(x_1,x_2)\xi(t),\ \boldsymbol{w} \in X_\Omega,\ \xi(t) \in C_0^1(0,T)$, we obtain from above

$$
-\int_0^T \left\langle \rho \boldsymbol{v}, \boldsymbol{w} \right\rangle_{X_\Omega} \xi'(t) dt = \int_0^T \left\langle h^{-1}\chi_x^b, \boldsymbol{w} \right\rangle_{X_\Omega} \xi(t) dt,
$$

which yields that $h^{-1}\chi_x^b = \partial_t(\rho \boldsymbol{v})$ in distributive sense. Therefore it holds $\int_0^T \left\langle h^{-1}\chi_x^b, \boldsymbol{\varphi} \right\rangle_{X_\Omega} dt = -\int_0^T \int_{\Omega(h(t))} \rho \boldsymbol{v} \partial_t \boldsymbol{\varphi} dx\, dt$. By transformation to the fixed domain D and using the definition of χ^b we get

$$
\int_0^T \left\langle \chi, \boldsymbol{\psi} \right\rangle_X dt = -\int_0^T \int_D h\boldsymbol{u} \partial_t^y \boldsymbol{\psi}\, dy\, dt - \int_0^T \int_0^L \frac{\partial h}{\partial t} u_2 \boldsymbol{\psi}_2(y_1,1,t)\, dy_1\, dt.
$$

Using the definition of ∂_t^y and the integration by parts with respect to y_2 we find out, that the boundary terms on the right hand side are canceled and

$$
\int_0^T \left\langle \chi, \boldsymbol{\psi} \right\rangle_X dt = -\int_0^T \int_D \boldsymbol{u} \frac{\partial (h\boldsymbol{\psi})}{\partial t} + y_2 \frac{\partial h}{\partial t} \frac{\partial \boldsymbol{u}}{\partial y_2} \boldsymbol{\psi}\, dy\, dt.
$$

This means that $\chi + y_2 \frac{\partial h}{\partial t} \frac{\partial \boldsymbol{u}}{\partial y_2}$ is the distributive time derivative $h\partial_t \boldsymbol{u}$. Thus the limit χ equals

$$
\chi = h\frac{\partial \boldsymbol{u}}{\partial t} - y_2 \frac{\partial h}{\partial t} \frac{\partial \boldsymbol{u}}{\partial y_2} \equiv \bar{\partial}_t(h\boldsymbol{u}) = h\partial_t^y \boldsymbol{u}. \tag{8.3}
$$

Finally we have obtained $\chi = \bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*)$ and we can replace in (5.13)

$$
\int_0^T \int_D \left\{ h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} \right\} dy\, dt \quad \text{by} \qquad -\int_0^T \left\langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi} \right\rangle_X dt.
$$

Since we have proven that χ is the distributive ALE-type time derivative of $\boldsymbol{u}$ in the sense of (8.3), due to $u_2(y_1,1,t)=\eta_t(y_1,t)$ we have also shown, that σ has the distributive time derivative

$$\partial_t\sigma = \partial_t^y u_2|_{S_w}.$$

Thus, we have in (5.13) $-\int_0^T\int_0^L \sigma\frac{\partial\xi}{\partial t} = \int_0^T\langle\partial_t\sigma,\xi\rangle$.

It remains to show the property of weak time derivative analogous to (4.38). Indeed, for fixed κ using the definition of the derivative $\bar{\partial}_t$ (5.14), the property (4.38) and the partial integration with respect to the y_2 we obtain

$$\int_0^T \left\langle\bar{\partial}_t(h\boldsymbol{u}_\kappa),\boldsymbol{u}_\kappa\right\rangle_X dt = \frac{1}{2}\int_D |\boldsymbol{u}_\kappa|^2(t)h(t) - \frac{1}{2}\int_0^T\int_0^L \frac{\partial h}{\partial t}|u_{\kappa 2}|^2\,dy_1\,dt.$$

Now letting $\kappa\to\infty$, using the strong and the weak convergences from Section 5 and the weak limit (8.3) we obtain the desired property

$$\int_0^T \left\langle\bar{\partial}_t(h\boldsymbol{u}),\boldsymbol{u}\right\rangle_X dt = \frac{1}{2}\int_D |\boldsymbol{u}|^2(t)h(t) - \frac{1}{2}\int_0^T\int_0^L \frac{\partial h}{\partial t}|u_2|^2 dy_1 dt. \quad (8.4)$$

Appendix B (Equicontinuity in time)

The aim of this section is to show the integral equicontinuity in time. Lemma 9.1 provides the equicontinuity result that holds independently on k. To this end we need to find suitable divergence free test functions in order to control difference of velocity at different time instances. In order to obtain such test functions we follow a construction presented in [9], see also the reference [16] therein.

We introduce, in analogy to [9, Lemma 3], the following extensions of the domain and the weak solution.
We define an extension of the moving domain $\Omega(h^{(k)}(t))$ to a box domain

$$B_M \equiv (0,L)\times(0,M)\in\mathbb{R}^2 \quad (9.1)$$

for some $M>\alpha^{-1}$ specified later. Moreover we define an extension into B_M of solution $\boldsymbol{u}^{(k)}(y,t)=\boldsymbol{v}^{(k)}(x,t)$ of (6.16),

$$\bar{\boldsymbol{v}}^{(k)} = \begin{cases} \boldsymbol{v}^{(k)} & \text{in } \Omega(h^{(k)}(t)) \\ (0,\eta_t^{(k)}) & \text{in } B_M\backslash\Omega(h^{(k)}(t)). \end{cases} \quad (9.2)$$

Further, for $\gamma>1$ and any function $\boldsymbol{f}(x_1,x_2)$ we define $\boldsymbol{f}_\gamma$ as follows

$$\boldsymbol{f}_\gamma(x_1,x_2) = (\gamma f_1(x_1,\gamma x_2), f_2(x_1,\gamma x_2)).$$

Note that if $\boldsymbol{f}$ is divergence free, then $\boldsymbol{f}_\gamma$ is divergence free, too.

Lemma 9.1. *For the weak solution* $(\boldsymbol{v}^{(k)}, \eta_t^{(k)}) = (\boldsymbol{u}^{(k)}, \sigma^{(k)})$ *of the problem (6.16) it holds*

$$\int_0^{T-\tau}\int_{B_M} \chi_t^{(k)} |\bar{\boldsymbol{v}}^{(k)}(t+\tau) - \bar{\boldsymbol{v}}^{(k)}(t)|^2 + \int_0^{T-\tau}\int_0^L |\eta_t^{(k)}(t+\tau) - \eta_t^{(k)}(t)|^2 \leq C(\tau^{1/p} + \tau^{1/2}). \quad (9.3)$$

Here $\chi_t^{(k)}$ *denotes the characteristic function of* $\Omega(h^{(k)}(t))$. *The constant* $C = C(K,\alpha)$ *does not depend on* k.

Proof. We recall that $h^{(k)} = R_0 + \delta^{(k)}$, but for the sake of simplicity we omit the superscript (k) in this proof and we denote $h := R_0 + \delta^{(k)}$, $\bar{\boldsymbol{v}} := \bar{\boldsymbol{v}}^{(k)}$, $\eta := \eta^{(k)}$.
To prove the statement of this lemma, we will use following two properties.

1. The distributive time derivative $\bar{\partial}$ we have:
   For each $\boldsymbol{\psi} \in H^1(0,T;X)$, cf. (5.11), $\boldsymbol{\psi}(T) = 0$ it holds, cf. (4.37).

   $$\begin{aligned} &-\int_0^{\tilde{\tau}} \langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi}\rangle dt \qquad (9.4)\\ &= \int_0^{\tilde{\tau}}\int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\boldsymbol{\psi} dy dt - \int_D h\boldsymbol{u}(\tilde{\tau}, y)\boldsymbol{\psi}(\tilde{\tau}, y) dy. \end{aligned}$$

   For classical time derivative, this property is clear. For our distributive derivative $\bar{\partial}$ it can be proven analogously as in (4.37).

2. By inserting any time independent test function $\boldsymbol{\psi} = \boldsymbol{\psi}(y)$ into (9.4) and subtracting (9.4) for $\tilde{\tau} = t+\tau$, and $\tilde{\tau} = t$ we obtain

   $$\begin{aligned} &-\int_t^{t+\tau} \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}(x)\rangle_{X_\Omega} ds \qquad (9.5)\\ &= \int_t^{t+\tau}\int_D \frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\boldsymbol{\psi}(y) dy ds - \int_D [h\boldsymbol{u}(t+\tau) - h\boldsymbol{u}(t)]\boldsymbol{\psi}(y) dy. \end{aligned}$$

   Here the integral on the left hand side has been transformed into $\Omega(h(t))$, $\boldsymbol{\psi} = \boldsymbol{\psi}(y) = \boldsymbol{\varphi}(x)$, $y \in D$, $x \in \Omega(h(t))$, $X_\Omega = X_{\Omega(h(t))}$ was defined in Appendix A.

Now, let us integrate (9.5) over $\int_0^{T-\tau} dt$. The first term on the right hand side (integrated over $\int_0^{T-\tau}$) can be bounded with $C\tau$ independently on k for test functions (9.10) specified later . The second term on the right hand of (9.5) can be rewritten due to the transformation to the $\Omega(h)$

$$\int_0^{T-\tau}\int_{\Omega(h(t+\tau))} \boldsymbol{v}(x_{t+\tau}, t+\tau)\boldsymbol{\varphi}(x_{t+\tau}) dx - \int_{\Omega(h(t))} \boldsymbol{v}(x_t, t)\boldsymbol{\varphi}(x_t)\, dx\, dt. \quad (9.6)$$

Note, that the space coordinate $x_t \equiv x(t) \in \Omega(h(t))$ depends on time, hence the test functions $\boldsymbol{\varphi}$ implicitly depend on time, which is pointed out above.

Using the previously defined extensions of the solution $\bar{\boldsymbol{v}}$ and some further manipulations we can rewrite (9.6) as follows

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\Omega(h(t))} \bar{\boldsymbol{v}}(x_{t+\tau}, t+\tau)\boldsymbol{\varphi}(x_{t+\tau}) - \boldsymbol{v}(x_t, t)\boldsymbol{\varphi}(x_t)dx \\
&+ \int_{B_M} (\chi_{t+\tau} - \chi_t)\bar{\boldsymbol{v}}(x_{t+\tau}, t+\tau)\boldsymbol{\varphi}(x_{t+\tau})\, dx\, dt = \qquad (9.7)\\
&\int_0^{T-\tau}\int_{\Omega(h(t))} \underbrace{[\bar{\boldsymbol{v}}\,(x_{t+\tau}, t+\tau) - \boldsymbol{v}(x_t, t)]\boldsymbol{\varphi}(x_t)}_{\text{(I)}} + \underbrace{[\boldsymbol{\varphi}(x_{t+\tau}) - \boldsymbol{\varphi}(x_t)]\bar{\boldsymbol{v}}(x_{t+\tau}, t+\tau)}_{\text{(II)}} \\
&+ \int_{B_M} \underbrace{(\chi_{t+\tau} - \chi_t)\bar{\boldsymbol{v}}(x_{t+\tau}, t+\tau)\boldsymbol{\varphi}(x_{t+\tau})}_{\text{(III)}}\, dx\, dt.
\end{aligned}$$

Here χ_t, $\chi_{t+\tau}$ are the characteristic functions of $\Omega(h(t))$, $\Omega(h(t+\tau))$, respectively.
In what follows we estimate the term (II) for any test function $\boldsymbol{\varphi} \in L^p(0,T;X_\Omega)$. Further, we take specific test functions and concentrate on the terms (I), (III).

Since $\delta \in L^\infty(0,T;H^2(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L))$, from the imbedings in one dimension (6.19) it follows that $\delta \in C^{0,1/2}([0,T];H^1(0,L))$. Thus

$$\|\delta(t+\tau) - \delta(t)\|_{L^\infty((0,T)\times(0,L))} \le C\sqrt{\tau}. \qquad (9.8)$$

Using (9.8) we can estimate the term (II):

$$\begin{aligned}
\text{(II)} &\le \int_0^{T-\tau}\left(\int_{\Omega(h(t))} |\boldsymbol{\varphi}(x_{t+\tau}) - \boldsymbol{\varphi}(x_t)|^2 dx\right)^{1/2} \|\bar{\boldsymbol{v}}\|_{L^2(\Omega(h(t)))}\|dt \qquad (9.9)\\
&= \int_0^{T-\tau}\left(\int_{\Omega(h(t))}\left|\int_{x_2(t)}^{x_2(t+\tau)} \partial_s\boldsymbol{\varphi}(x_1,s)ds\right|^2 dx\right)^{1/2} \|\bar{\boldsymbol{v}}\|_{L^2(\Omega(h(t)))}\|dt\\
&\le \int_0^{T-\tau}\left(\int_{B_M} |\nabla\boldsymbol{\varphi}|^2 dx |x_2(t+\tau) - x_2(t)|^2\right)^{1/2} \|\bar{\boldsymbol{v}}\|_{L^2(\Omega(h(t)))}\|dt\\
&\le \|\boldsymbol{\varphi}\|_{L^2(0,T;H^1(B_M))}\|\delta(t+\tau) - \delta(t)\|_{L^\infty((0,T)\times(0,L))}\|\bar{\boldsymbol{v}}\|_{L^2((0,T)\times B_M)}\\
&\le C\sqrt{\tau}.
\end{aligned}$$

Now we specify proper test functions, that will be used in what follows. For $x_t = x(t) \in \Omega(h(t))$, $\gamma > 1$ and fixed t, τ we set

$$\begin{aligned}
\boldsymbol{\varphi}(x_t) &= \bar{\boldsymbol{v}}_\gamma(x_{t+\tau}, t+\tau) - \bar{\boldsymbol{v}}_\gamma(x_t, t), \qquad (9.10)\\
\xi(x_1) &= E(\partial_t\eta(x_1, t+\tau) - \partial_t\eta(x_1, t)).
\end{aligned}$$

Note that since $\boldsymbol{v}$ is divergence-free, the test function $\boldsymbol{\varphi}$ is also divergence-free[1]. Moreover, taking into account (9.8), for $\gamma \geq 1 + \frac{C\sqrt{\tau}}{\alpha}$ and $x_2 \in \Gamma_w(t)$ the coordinate γx_2 exceeds the moving domain $\Omega(h)$, since we have $\gamma(R_0 + \delta(s)) \geq R_0 + \delta(s) + \|\delta(t+\tau) - \delta(t)\|_\infty$, $s = t, t+\tau$. According to the construction, such a test function fulfill the boundary condition

$$E\boldsymbol{\varphi}(x_1, R_0(x_1) + \delta(x_1, t)) = E(0, \partial_t\eta(x_1, t+\tau) - \partial_t\eta(x_1, t)) \equiv (0, \xi(x_1)).$$

Let us estimate now the term (III). Since $\partial_t\eta$ is bounded in $L^\infty(0,T; L^2(0,L))$ independently on k, we have

$$\int_{B_M} |\chi_{t+\tau} - \chi_t|^2 = \int_0^L |\delta(t+\tau) - \delta(t)|^2 = \int_0^L \left| \int_t^{t+\tau} \partial_t\delta(s)ds \right|^2 \leq C\tau. \tag{9.11}$$

Thus, the term (III) can be bounded for $\boldsymbol{\varphi}$ from (9.10) as follows.

$$(\mathrm{III}) \leq \int_0^{T-\tau} \|\chi_{t+\tau} - \chi_t\|_{L^2(B_M)} \|\bar{\boldsymbol{v}}\|_{L^4(B_M)} \|\boldsymbol{\varphi}\|_{L^4(B_M)} dt \leq C\sqrt{\tau}. \tag{9.12}$$

For the test functions from (9.10) the term (I) equals

$$\begin{aligned}(\mathrm{I}) &= \int_0^{T-\tau} \int_{\Omega(h(t))} [\bar{\boldsymbol{v}}(t+\tau) - \bar{\boldsymbol{v}}(t)][\bar{\boldsymbol{v}}_\gamma(t+\tau) - \bar{\boldsymbol{v}}_\gamma(t)]dxdt \\ &= \int_0^{T-\tau} \int_{\Omega(h(t))} \underbrace{[\bar{\boldsymbol{v}}(t+\tau) - \bar{\boldsymbol{v}}(t)]^2}_{(\mathrm{Ia})} + \\ &\quad \underbrace{[\bar{\boldsymbol{v}}(t+\tau) - \bar{\boldsymbol{v}}(t)].\Big([\bar{\boldsymbol{v}}_\gamma(t+\tau) - \bar{\boldsymbol{v}}(t+\tau)] - [\bar{\boldsymbol{v}}_\gamma(t) - \bar{\boldsymbol{v}}(t)]\Big)}_{(\mathrm{Ib})} dxdt\end{aligned} \tag{9.13}$$

For the simplicity we used shorter notations here, e.g., $\bar{\boldsymbol{v}}(t+\tau) := \bar{\boldsymbol{v}}(x_{t+\tau}, t+\tau)$. The term (Ia) appears on the left hand side of the assertion of this lemma; the term (Ib) need to be estimated from above. We illustrate the estimate of some chosen terms of (Ib) as follows. Estimates of other terms are analogous.

In the sequel we take $\gamma = 1 + \frac{C\sqrt{\tau}}{\alpha}$ and $M \geq 2\alpha^{-1}$. For these parameters we have according to Lemma 9.2,

$$\begin{aligned}&\int_0^{T-\tau} \int_{\Omega(h(t))} \bar{\boldsymbol{v}}(t+\tau)[\bar{\boldsymbol{v}}_\gamma(t) - \bar{\boldsymbol{v}}(t)]dxdt \leq \\ &C_\alpha\sqrt{\tau} \int_0^{T-\tau} \|\bar{\boldsymbol{v}}(t+\tau)\|_{L^2(B_M)} \|\bar{\boldsymbol{v}}(t)\|_{H^1(B_M)} dt \leq C_\alpha\sqrt{\tau}.\end{aligned}$$

[1] Since $\boldsymbol{\varphi}(x_{t+\tau}) = \bar{\boldsymbol{v}}_\gamma(x_{t+2\tau}, t+2\tau) - \bar{\boldsymbol{v}}_\gamma(x_{t+\tau}, t+\tau)$, we have to integrate over $\int_0^{T-2\tau} dt$ in the estimate of the term (II), or we define $\boldsymbol{\varphi}(x_{t+\tau}) = 0$ if $t + \tau > T$.

To complete the proof, the remaining terms coming from the fluid equations, i.e., the convective term, the viscous term, boundary terms and the equation for η have to be estimated. We illustrate here only the calculations for the nonlinear viscous term and omit tedious but standard calculations for other terms, previously performed also in [15].

After subtracting the weak formulation (6.16) for $\int_0^{t+\tau} ds - \int_0^t ds$, inserting test functions constructed above (independent on s) into (6.16) and integrating over $\int_0^{T-\tau} dt$ we obtain from the viscous term

$$\int_0^{T-\tau} \int_t^{t+\tau} \int_{\Omega(h(s))} \tau_{ij}(e[\boldsymbol{v}(s)]).e[\bar{\boldsymbol{v}}_\gamma(t+\tau) - \bar{\boldsymbol{v}}_\gamma(t)]dx\, ds\, dt.$$

For the simplicity, we set $\omega := \bar{\boldsymbol{v}}_\gamma(t+\tau)$ or $\omega := \bar{\boldsymbol{v}}_\gamma(t)$. The above expression can be bounded with use of (5.8) as follows,

$$\begin{aligned}
&\leq \int_0^{T-\tau} \int_t^{t+\tau} \int_{\Omega(h(s))} C_5(1+|e[\boldsymbol{v}(s)]|)^{p-1} e[\omega] dx\, ds\, dt \\
&\leq C(K,\alpha) \int_0^{T-\tau} \int_t^{t+\tau} \|1+\nabla \boldsymbol{v}(s)\|^{p-1}_{L^p(\Omega(h(s)))} \|\nabla\omega\|_{L^p(\Omega(h(s)))} ds\, dt \\
&\leq C(K,\alpha) \int_0^{T-\tau} \Big(\int_t^{t+\tau} \|1+\nabla \boldsymbol{v}(s)\|^{p}_{L^p(\Omega(h(s)))} ds \Big)^{\frac{p-1}{p}} \|\nabla\omega\|_{L^p(B_M)} \tau^{\frac{1}{p}} dt \\
&\leq C(K,\alpha)\tau^{\frac{1}{p}} \left(\int_0^{T} \|1+\nabla \boldsymbol{v}(s)\|^{p}_{L^p(\Omega(h(s)))} ds \right)^{\frac{p-1}{p}} \int_0^{T-\tau} \|\nabla\omega\|_{L^p(B_M)} dt \\
&\leq C(K,\alpha)\tau^{\frac{1}{p}} \|1+\nabla \boldsymbol{v}\|^{p-1}_{L^p(0,T;L^p(\Omega(h))} \|\nabla\omega\|_{L^1(0,T;L^p(B_M))} \leq C(K,\alpha)\tau^{\frac{1}{p}}.
\end{aligned}$$

We conclude, that the estimates of remaining terms on the right hand side are analogous as already show in the first part of the paper or in [15] and we leave them to the valued reader. The proof of the lemma is now completed. ■

Due to the (9.11) it is also easy to obtain from (9.3) that

$$\int_0^{T-\tau}\int_{B_M} |\chi^{(k)}_{t+\tau}\bar{\boldsymbol{v}}^{(k)}(t+\tau) - \chi^{(k)}_t \bar{\boldsymbol{v}}^{(k)}(t)|^2 \leq C(\tau^{1/p} + \tau^{1/2}). \tag{9.14}$$

This result implies that $\chi^{(k)}_t\bar{\boldsymbol{v}}^{(k)}(t)$, and consequently $\bar{\boldsymbol{v}}^{(k)}(t)$ is relatively compact in $L^2((0,T)\times B_M)$.

Lemma 9.2. *If $\gamma = 1 + \frac{C\sqrt{\tau}}{\alpha}$ and $M \geq 2\alpha^{-1}$, then for any $\boldsymbol{f} \in H^1(B_M)$ we have*

$$\int_{\Omega(h(t))} |\boldsymbol{f}_\gamma - \boldsymbol{f}|^2 dx \leq C_\alpha \tau \|\boldsymbol{f}\|^2_{H^1(B_M)}.$$

Proof. From the definition of $\boldsymbol{f}_\gamma$ it is obvious that

$$|\boldsymbol{f}_\gamma - \boldsymbol{f}| \leq |\boldsymbol{f}(x_1, \gamma x_2, t) - \boldsymbol{f}(x_1, x_2, t)| + (\gamma - 1)|\boldsymbol{f}(x_1, \gamma x_2, t)|. \tag{9.15}$$

Now consider $\boldsymbol{f}^\star$ - a smooth approximation of $\boldsymbol{f}$. We can write

$$\int_{\Omega(h(t))} |\boldsymbol{f}^\star(x_1, \gamma x_2, t) - \boldsymbol{f}^\star(x_1, x_2, t)|^2 dx = \int_{\Omega(h(t))} \left| \int_{x_2}^{\gamma x_2} \frac{\partial \boldsymbol{f}^\star}{\partial s}(x_1, s, t) ds \right|^2 dx.$$

Note that $x_2 \in \Omega(h(t)) \leq \alpha^{-1}$. Thus, for sufficiently small τ, $(\alpha > 0)$ and $\gamma = 1 + \frac{C\sqrt{\tau}}{\alpha}$ the above integral bound $\gamma x_2 < 2\alpha^{-1} = M$. Hence we can estimate

$$\int_{\Omega(h(t))} \left| \int_{x_2}^{\gamma x_2} \frac{\partial \boldsymbol{f}^\star}{\partial s}(x_1, s, t) ds \right|^2 dx \leq [(\gamma - 1)\alpha^{-1}]^2 \int_{B_M} |\nabla \boldsymbol{f}^\star|^2 dx$$

and by using the standard Sobolev approximation argument we finally get

$$\int_{\Omega(h(t))} |\boldsymbol{f}(x_1, \gamma x_2, t) - \boldsymbol{f}(x_1, x_2, t)|^2 dx \leq C_\alpha \tau \int_{B_M} |\nabla \boldsymbol{f}|^2 dx.$$

The above result and (9.15) imply the assertion of the lemma. ∎

References

[1] Adams R.A., *Sobolev Spaces*, American Mathematical Society, Graduate Studies in Mathematics, 1998.

[2] Alt H.W., Luckhaus S., Quasilinear elliptic-parabolic differential equations, *Math Z.* 1983; **183**: 311–341.

[3] Brezis H., *Functional Analysis, Sobolev spaces and Partial Differential Equations*, Springer Verlag, 2011.

[4] Brezis H., *Analyse Fonctionelle- Théorie et applications*, Masson, Paris, 1983.

[5] Beirão Da Veiga H., On the existence of strong solutions to a coupled fluid-structure evolution problem, *J. Math. Fluid Mech.* 2004; **6**(1): 21–52.

[6] Bucur D., Feireisl E., Nečasová Š., Influence of wall roughness on the slip behavior of viscous fluids, *Proc. R. Soc. Edinb.: Sect. A, Math.* 2008; **138**(5): 957–973.

[7] Bucur D., Feireisl E., Nečasová Š., On the asymptotic limit of flows past a ribbed boundary, *J. Math. Fluid Mech.* 2008; **10**(4): 554–568.

[8] Čanić S., Muha B., Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls, *Archives for Rational Mechanics and Analysis,* 2013; **207**(4): 919–968, DOI: 10.1007/s00205-012-0585-5

[9] Chambolle A., Desjardin B., Esteban M.J., Grandmont C., Existence of weak solutions for unsteady fluid-plate interaction problem, *J. Math. Fluid. Mech.* 2005; **4**(3): 368–404.

[10] Coutand D., Shkoller S., The interaction between quasilinear elastodynamics and the Navier-Stokes equations, *Arch. Rational Mech. Anal.* 2006; **179**: 303–352.

[11] Diening, L., Růžička M., Wolf, J., Existence of weak solutions for unsteady motions of generalized Newtonian fluids, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 2010; **(9)1**: 1–46.

[12] Diening L., Málek J., Steinhauer M., On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications, *ESAIM Control Optim. Calc. Var.* 2008; **14**: 211–232.

[13] Evans L.C., *Partial Differential Equations*, American Mathematical Society, Graduate Studies in Mathematics, 1998.

[14] Feistauer M., *Mathematical Methods in Fluid Dynamic*, John Wiley & Sons, Inc., New York, 1993.

[15] Filo J., Zaušková A., 2D Navier-Stokes equations in a time dependent domain with Neumann type boundary conditions, *J. Math. Fluid Mech.* 2010; **12**(1): 1–46.

[16] Frehse J., Málek J., Steinhauer M., On analysis of steady flow of fluid with shear dependent viscosity based on the Lipschitz truncation method, *SIAM J. Math. Anal.* 2003; **35**(5): 1064–1083.

[17] Frehse J., Málek J., Steinhauer M., An existence result for fluids with shear dependent viscosity, steady flows, *Nonlinear Anal.* 1997; **30**: 3041–3049.

[18] Galdi G.P., *An Introduction to the Theory of Navier-Stokes Equations I*, Springer-Verlag, New York, 1994

[19] Guidoboni G., Guidorzi M., Padula M., Continuous dependence on initial data in fluid-structure motion, *J. Math. Fluid Mech.* 2010; published online: http://dx.doi.org/10.1007/s00021-010-0031-0.

[20] Guidorzi M., Padula M., Plotnikov P.I., Hopf solutions to a fluid-elastic interaction model, *Math. Models Meth. Appl. Sci.* 2008; **18**(2): 215–269.

[21] Hlaváček I., Nečas J., On inequalities of Korn's type, I. Boundary-value problems for elliptic systems of partial differential equations, *Arch. Ration. Mech. An.* 1970; **36**(4): 305–311.

[22] Henry D., *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, 1981.

[23] Hundertmark-Zaušková A., Lukáčová-Medviďová M., Numerical study of shear-dependent non-Newtonian fluids in compliant vessels, *Comput. Math. Appl.* 2010; **60**: 572–590.

[24] Kaplický P., Regularity of flow of a non-Newtonian fluid in two dimensions subject to Dirichlet boundary conditions, *Journal for Analysis and its Applications* 2005; **24**(3): 467–486.

[25] Ladyzenskaya O.A., *The Mathematical Theory of Viscous Incompressible Flow,* Gordon and Beach, New York, 1969.

[26] Lions J.L., *Quelques Méthodes de Résolution des Problémes aux Limites Non-Linéares,* Dunod, Paris, 1969, (in French).

[27] Málek J., Nečas J., Růžička M., On the weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domans: the case $p \geq 2$, *Adv. Differ. Equat.* 2001, **6**(3): 257–302.

[28] Málek J., Nečas J., Rokyta M., Růžička M., *Weak and Measure-Valued Solutions to Evolutionary PDEs*; Chapman and Hall, London, 1996.

[29] Málek J., Rajagopal K. R., Mathematical Issues Concerning the Navier-Stokes Equations and Some of its Generalizations, In *Handbook of Differential Equations*, edited by C.M. Dafermos, E. Feireisl, North-Holland Publishing Company, 2005.

[30] Nečas J., *Sur les Normes Équivalentes dans* $W_p^{(k)}(\Omega)$ *et sur la Coercitivité des Formes Formellement Positives,* Les Presses de l'Université de Montréal, Janvier, 1962, 102–108.

[31] Neff, P., On Korn's first inequality with non-constant coefficients, *Proc. R. Soc. Edinb.: Section A Mathematics* 2002; **132**(1): 221–243.

[32] Neustupa J., Existence of weak solution to the Navier-Stokes equation in a general time-varying domain by Rothe method, *Math. Meth. Appl. Sci.* 2009; **32**(6): 653–683.

[33] Pompe W., Korn's first inequality with variable coefficients and its generalization, *Comment Math. Univ. Carolinae* 2003; **44**(1): 57–70.

[34] Quarteroni A., Mathematical and numerical simulation of the cardiovascular System. In *Proceedings of the ICM,* 3, Beijing, 2002; 839–850.

[35] Quarteroni A., Formaggia L., Computational Models in the Human Body, In *Handbook of Numerical Analysis, Volume XII*, Editor P.G. Ciarlet, Guest Editor N. Ayache, Elsevier North Holland, 2004.

[36] Surulescu C., On the stationary interaction of a Navier-Stokes fluid with an elastic tube wall, *Appl. Anal.* 2007; **86**(2): 149–165.

[37] Takahashi T., Tucsnak M., Global strong solution for the two-dimensional motion of an infinite cylinder in a viscous fluid, *J. Math. Fluid. Mech.* 2004; **6**(1): 52–77.

[38] Temam R., *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland Publishing Company, 1979.

[39] Yeleswarapu K.K., *Evaluation of Continuum Models for Characterizing the Constitutive Behavior of Blood*, PhD. Thesis, University of Pittsburgh, Pittsburgh, 1996.

[40] Wolf J., Existence of weak solution to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity, *J. Math. Fluid. Mech.*, 2007; **9**(1): 104–138.

[41] Zaušková A., *2D Navier–Stokes Equations in a Time Dependent Domain*. PhD. Thesis, Comenius University, Bratislava, 2007.

Mária Lukáčová, Anna Hundertmark
Institute for Mathematics
Johannes Gutenberg University,
Staudingerweg 9, Mainz
Germany

Šárka Nečasová
Academy of Sciences of Czech Republic
Žitná 25, Praha
Czech Republic