On the weak solution of the fluid-structure interaction problem for shear-dependent fluids

Anna Hundertmark, Mária Lukáčová-Medviďová and Šárka Nečasová

Abstract. In this paper the coupled fluid-structure interaction problem for incompressible non-Newtonian shear-dependent fluid flow in two-dimensional time-dependent domain is studied. One part of the domain boundary consists of an elastic wall. Its temporal evolution is governed by the generalized string equation with action of the fluid forces by means of the Neumann type boundary condition. The aim of this work is to present the limiting process for the auxiliary (κ, ε, k) - problem. The weak solution of this auxiliary problem has been studied in our recent work [9].

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1. Problem Definition

The problem of a fluid interaction with a moving or deformable structure is important in many applications like biomechanics, hydroelasticity, aeroelasticity, sedimantation, modeling of blood flow, etc. We consider a two-dimensional fluid motion governed by the momentum and the continuity equation

$$\rho\partial_t \boldsymbol{v} + \rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} - \mathrm{div}\boldsymbol{\tau} + \nabla\pi = 0, \quad \mathrm{div}\,\boldsymbol{v} = 0, \tag{1.1}$$

with ρ denoting the constant density of fluid, $\boldsymbol{v} = (v_1, v_2)$ the velocity vector, π the pressure and $\boldsymbol{\tau}$ the shear stress tensor.

Let us first specify the shear-dependent fluids that will be considered in this paper. We assume that

$$\boldsymbol{\tau} = \boldsymbol{\tau}(e(\boldsymbol{v})) = 2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v}), \quad \text{where } e(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$$

is the symmetric deformation tensor. Moreover we assume that there exists a potential $\mathcal{U} \in C^2(\mathbb{R}^{2\times 2})$ of the stress tensor $\boldsymbol{\tau}$, such that for some $1 < p < \infty,\ C_1,\ C_2 > 0$ we have for all $\eta,\ \xi \in \mathbb{R}^{2\times 2}_{sym}$ and $i,\ j,\ k,\ l \in \{1, 2\}$, cf. [11]

$$\frac{\partial \mathcal{U}(\eta)}{\partial \eta_{ij}} = \boldsymbol{\tau}_{ij}(\eta), \qquad \mathcal{U}(\mathbf{0}) = \frac{\partial \mathcal{U}(\mathbf{0})}{\partial \eta_{ij}} = 0, \tag{1.2}$$

$$\frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{mn}\partial \eta_{rs}}\xi_{mn}\xi_{rs} \geq C_1\ (1 + |\eta|)^{p-2}|\xi|^2, \tag{1.3}$$

$$\left|\frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{ij}\partial \eta_{kl}}\right| \leq C_2(1 + |\eta|)^{p-2}. \tag{1.4}$$

One particular example satisfying the above properties is a stress tensor, which contains shear-dependent viscosity obeying the power-law model, cf. [8, 11, 12, 16]

$$\mu(|e(\boldsymbol{v})|) = \mu(1 + |e(\boldsymbol{v})|^2)^{\frac{p-2}{2}} \quad p > 1. \tag{1.5}$$

For $p < 2$ the viscosity is a decreasing function of the shear rate, i.e., shear-thinning. For $p > 2$ we have shear-thickening property and this model is an analogy of the so-called Ladyzhenskaja's fluid; for $p = 3$ it yields the Smagorinskij model of turbulence. In numerical simulations presented in our recent papers [10], [8] the shear-thinning model of Carreau has been used in order to model blood flow in compliant vessels. For the simplicity of presentation we will consider here only the case of shear-thickening fluids, i.e. $p \geq 2$. The generalization for shear-thinning fluids may be done in an analogous way as here, using an appropriate techniques for shear-thinning fluids, see results of Diening, Růžička and Wolf [5, 15].

The two-dimensional deformable computational domain

$$\Omega(\eta(t)) \equiv \{(x_1, x_2);\ 0 < x_1 < L,\ 0 < x_2 < R_0(x_1) + \eta(x_1, t)\},\ 0 < t < T$$

is given by a reference radius function $R_0(x_1)$ and the unknown free boundary function $\eta(x_1, t)$ describing the domain deformation. The fluid and the geometry of the computational domain are coupled through the following Dirichlet boundary condition on the deformable part of the boundary $\Gamma_w(t)$

$$\boldsymbol{v}(x_1, R_0(x_1) + \eta(x_1, t), t) = \left(0, \frac{\partial \eta(x_1, t)}{\partial t}\right), \tag{1.6}$$

where $\Gamma_w(t) = \{(x_1, x_2);\ x_2 = R_0(x_1) + \eta(x_1, t),\ x_1 \in (0, L)\}$. The normal component of the fluid stress tensor $\mathbf{T}_f\boldsymbol{n}$ and the outside pressure P_w provide the forcing terms for the boundary displacement η, that is modeled by the generalized string equation:

$$\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta + c\frac{\partial^5 \eta}{\partial t \partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2} = \frac{-g}{\tilde{E}\rho}\left[\tilde{\mathbf{T}}_f + \tilde{P_w}\mathbf{I}\right]\tilde{\boldsymbol{n}} \cdot \boldsymbol{e}_2 \text{ on } \Gamma_w^0. \tag{1.7}$$

Here $\Gamma_w^0 := \Gamma_w(t)|_{t=0}$, $[(\tilde{\mathbf{T}}_f + \tilde{P_w}\mathbf{I})\tilde{\boldsymbol{n}}](\tilde{x}) = [(\mathbf{T}_f + P_w\mathbf{I})\boldsymbol{n}](x)$, $x \in \Gamma_w(t)$, $\tilde{x} \in \Gamma_w^0$ and $\mathbf{T}_f = \boldsymbol{\tau} - \pi\mathbf{I}$. Moreover, $\boldsymbol{n}, \tilde{\boldsymbol{n}}$ denote the unit outward normals on $\Gamma_w(t), \Gamma_w^0$, respectively and $\boldsymbol{n}|\boldsymbol{n}| = (-\partial_{x_1}(R_0+\eta), 1)^T$. The coefficient $g = \frac{(R_0+\eta)\sqrt{1+(\partial_{x_1}(R_0+\eta))^2}}{R_0\sqrt{1+(\partial_{x_1}R_0)^2}}$ arrises from the transformation from the Eulerian frame of the fluid forces into the Lagrangian formulation of the string.

Equation (1.7) is equipped with the following boundary and initial conditions

$$\eta(0,t) = \eta_{x_1}(0,t) = \eta(L,t) = \eta_{x_1}(L,t) = \eta(x_1,0) = \frac{\partial \eta}{\partial t}(x_1,0) = 0. \quad (1.8)$$

Positive coefficients $\tilde{E},\ a,\ b,\ c$ appearing in (1.7) are given as follows [8],

$$\tilde{E} = \rho_w \hbar, \qquad a = \frac{|\sigma_z|}{\left(1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right)^2}, \quad b = \frac{\mathcal{E}}{(R_0+\eta)R_0}, \qquad c > 0,$$

where $\mathcal{E}$ is the Young modulus, $\hbar$ the wall thickness, ρ_w the density of the vessel wall tissue, the coefficient $c = \gamma/(\rho_w\hbar)$, γ positive constant. $|\sigma_z| = G\kappa$ is the longitudinal stress, $\kappa = 1$ is the Timoshenko's shear correction factor and G is the shear modulus, equal to $G = \mathcal{E}/2(1+\sigma)$ with $\sigma = 1/2$ for incompressible materials. Note that the coefficients a, b are non-constant, however, according to the assumption (2.1) below they are upper- and down-bounded. In what follows, we linearize the term $b = \frac{\mathcal{E}}{(R_0+\eta)R_0}$ by $\frac{\mathcal{E}}{\rho_w R_0^2}$ and for the sake of simplicity we work with constant coefficients $a,\ b,\ c$.

The equation (1.7) can be transformed as follows.

$$E\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \left[-\mathbf{T}_f\boldsymbol{n}|\boldsymbol{n}|\cdot\boldsymbol{e}_2 - P_w\right](x_1, R_0(x_1)+\eta(x_1,t),t), \quad (1.9)$$

$x_1 \in (0,L)$. Here $E = \tilde{E}\sqrt{1+(\partial_{x_1}R_0)^2}$. We assume that E is bounded.

We complete the system (1.1) with the following boundary and initial conditions: on the inflow part of the boundary, which we denote Γ_{in}, we set

$$v_2(0,x_2,t) = 0, \ \left(2\mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_1} - \pi + P_{in} - \frac{\rho}{2}|v_1|^2\right)(0,x_2,t) = 0 \quad (1.10)$$

for any $0 < x_2 < R_0(0)$, $0 < t < T$ and for a given function $P_{in} = P_{in}(x_2,t)$. On the opposite, outflow part of the boundary Γ_{out}, we set

$$v_2(L,x_2,t) = 0, \ \left(2\mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_1} - \pi + P_{out} - \frac{\rho}{2}|v_1|^2\right)(L,x_2,t) = 0 \quad (1.11)$$

for any $0 < x_2 < R_0(L)$, $0 < t < T$ and for a given function $P_{out} = P_{out}(x_2,t)$. Note that we require that the so-called kinematic pressure is prescribed on the inflow and outflow boundary. This implies that the fluxes of kinetic energy on inflow and outflow boundary will disappear in the weak formulation. Finally, on the remaining part of the boundary, Γ_c, we set the flow symmetry condition

$$v_2(x_1,0,t) = 0\ , \qquad \mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_2}(x_1,0,t) = 0 \tag{1.12}$$

for any $0 < x_1 < L$, $0 < t < T$. The initial conditions read

$$\boldsymbol{v}(x_1,x_2,0) = \mathbf{0} \qquad \text{for any } 0 < x_1 < L,\ 0 < x_2 < R_0(x_1). \tag{1.13}$$

The problem defined in (1.1)–(1.13) is a generalization of the problem for Newtonian fluid previously studied by Filo and Hundertmark in [6, 17]. Here the original generalized string model of Quarteroni [13, 14] with a regularization term η_{txx} has been used. The iterative process with respect to the domain deformation, cf. item 3 below and Section 4, has been be completed only for the (κ,ε) - approximation of the original problem and the convergence with respect to domain deformation was an open problem. In the present paper, similarly as in [4], we use a modified model for the structure equation having a viscoelastic term η_{txxxx}. For this model we show global existence in time of weak solution of unsteady, fully coupled fluid-structure problem. The existence result holds until a contact of the elastic boundary with a fixed boundary part. The question of existence of weak solution of fully coupled fluid-structure interaction problem with the original Quarteroni's generalized string model for generalized Newtonian fluids is still an open problem.

The main result of this paper is formulated in Theorem 1.2. For the existence proof a suitable approximation of the problem (1.1)–(1.13), see Section 2, is constructed.

1. $\underline{\varepsilon \text{ - approximation}}$ (2.7): the space of solenoidal functions on a moving domain is approximated by the artificial compressibility approach,
2. $\underline{\kappa \text{ - approximation}}$ (2.5), (2.6): the boundary conditions (1.6)–(1.7) has been splitted and the deformable boundary becomes semi-pervious for $\kappa < \infty$,
3. $\underline{h \text{ - approximation}}$: the domain deformation is assume to be given by a sufficiently smooth function $\delta(x_1,t)$; the weak formulation on a deformable domain $\Omega(\delta(t)) =: \Omega(h(t))$ is transformed to a reference domain $D = (0,L)\times(0,1)$ using the known radius $h := R_0 + \delta$, see (2.8).

Letting $\varepsilon \to 0$, $\kappa \to \infty$ and finally the fixed point procedure for the domain deformation complete the proof. In [9] the above fluid-structure interaction problem has been studied and the existence of weak solution for fixed parameters κ,ε and given deformation δ, such that $h = R_0 + \delta$, i.e., to the (κ,ε,h) - approximation of the problem (1.1)–(1.13) has been proven in details. In this work we only present the limiting processes for $\varepsilon \to 0$, $\kappa \to \infty$ and the fixed point procedure with respect to the domain deformation regarding the geometric nonlinearity of our problem.

1.1. Weak formulation

In this subsection our aim is to present the weak formulation of the problem (1.1)–(1.13). Assuming that η is enough regular (see below) and taking into account the results from [4] we can define the functional spaces that gives sense to the trace of velocity from $W^{1,p}(\Omega(\eta(t)))$ and thus to define the weak solution of the problem. We assume that $R_0 \in C_0^2(0,L)$.

Definition 1.1 (Weak formulation).
We say that $(\boldsymbol{v},\eta)$ is a weak solution of (1.1)–(1.13) on $[0,T)$ if the following conditions hold

- $\boldsymbol{v} \in L^p(0,T;W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0,T;L^2(\Omega(\eta(t))))$,
- $\eta \in W^{1,\infty}(0,T;L^2(0,L)) \cap H^1(0,T;H_0^2(0,L))$,
- $\operatorname{div}\boldsymbol{v} = 0$ a.e. on $\Omega(\eta(t))$,
- $\boldsymbol{v}\big|_{\Gamma_w(t)} = (0,\eta_t)$ for a.e. $x \in \Gamma_w(t),\ t \in (0,T)$, $v_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = 0$,

$$\begin{aligned}
&\int_0^T \int_{\Omega(\eta(t))} \Big\{ -\rho \boldsymbol{v}\cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i}\boldsymbol{\varphi}_j \Big\} dx\, dt \\
&+ \int_0^T \int_0^{R_0(L)} \Big(P_{out} - \frac{\rho}{2}|v_1|^2\Big) \varphi_1(L,x_2,t)\, dx_2\, dt \qquad (1.14)\\
&- \int_0^T \int_0^{R_0(0)} \Big(P_{in} - \frac{\rho}{2}|v_1|^2\Big) \varphi_1(0,x_2,t)\, dx_2\, dt \\
&+ \int_0^T \int_0^L P_w \varphi_2(x_1,R_0(x_1)+\eta(x_1,t),t) - a\frac{\partial^2 R_0}{\partial x_1^2}\xi\, dx_1\, dt \\
&+ \int_0^T \int_0^L -\frac{\partial \eta}{\partial t}\frac{\partial \xi}{\partial t} + c\,\frac{\partial^3 \eta}{\partial x_1^2 \partial t}\frac{\partial^2 \xi}{\partial x_1^2} + a\frac{\partial \eta}{\partial x_1}\frac{\partial \xi}{\partial x_1} + b\eta\,\xi\, dx_1\, dt \;=\; 0
\end{aligned}$$

for every test functions

$$\begin{aligned}
&\boldsymbol{\varphi}(x_1,x_2,t) \in H^1(0,T;W^{1,p}(\Omega(\eta(t)))) \text{ such that} \qquad (1.15)\\
&\operatorname{div}\boldsymbol{\varphi} = 0 \text{ a.e on } \Omega(\eta(t)),\\
&\varphi_2\big|_{\Gamma_w(t)} \in H^1(0,T;H_0^2(\Gamma_w(t))),\quad \varphi_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = \varphi_1\big|_{\Gamma_w(t)} = 0 \quad \text{and}\\
&\xi(x_1,t) = E\rho\, \varphi_2(x_1,R_0(x_1)+\eta(x_1,t),t).
\end{aligned}$$

Theorem 1.2 (Main result: existence of a weak solution).
Let $p \geq 2$. Assume that the boundary data fulfill $P_{in} \in L^{p'}(0,T;L^2(0,R_0(0)))$, $P_{out} \in L^{p'}(0,T;L^2(0,R_0(L)))$, $P_w \in L^{p'}(0,T;L^2(0,L))$, $\frac{1}{p}+\frac{1}{p'}=1$. Furthermore, assume that the properties (1.2)–(1.4) for the viscous stress tensor hold. Then for $T \leq T^*$, T^* depending on the data R_0, P_{in}, P_{out}, P_w, K, α, cf. (2.1), and $\alpha \leq \min\{R_{min}, \frac{1}{R_{min}+R_{max}}\}$, $R_{min} \leq R_0 \leq R_{max}$ there exists a weak solution $(\boldsymbol{v},\eta)$ of the problem (1.1)-(1.13) such that
i) $\boldsymbol{v} \in L^p(0,T;W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0,T;L^2(\Omega(\eta(t))))$,
$\eta \in W^{1,\infty}(0,T;L^2(0,L)) \cap H^1(0,T;H_0^2(0,L))$,
ii) $\boldsymbol{v}\big|_{\Gamma_w(t)} = (0,\eta_t)$ *for a.e.* $x \in \Gamma_w(t),\ t \in (0,T)$, $v_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = 0$,
iii) $\boldsymbol{v}$ *satisfies the condition* $\operatorname{div}\boldsymbol{v} = 0$ *a.e on* $\Omega(\eta(t))$ *and (1.14) holds.*

Remark 1.3. Let us point out that T^* denotes a time when the elastic boundary reaches the bottom boundary. If $T^* = \infty$ we have an existence of the global weak solution, otherwise the existence of the weak solution holds until the elastic boundary reaches the bottom boundary, see Section 4 for further details.

2. Auxiliary problem: (κ, ε, h) - approximation

In what follows we will formulate a suitable approximation of the original problem (1.1)–(1.13).

First of all we approximate the deformable boundary Γ_w by a given function $h = R_0 + \delta$, $\delta \in H^1(0,T; H_0^2(0,L)) \cap W^{1,\infty}(0,T; L^2(0,L))$, $R_0(x_1) \in C^2[0,L]$ satisfying for all $x_1 \in [0,L]$

$$0 < \alpha \le h(x_1,t) \le \alpha^{-1}, \quad \left|\frac{\partial h(x_1,t)}{\partial x_1}\right| + \int_0^T \left|\frac{\partial h(x_1,t)}{\partial t}\right|^2 dt \le K < \infty \tag{2.1}$$

$h(0,t) = R_0(0)$, $\;h(L,t) = R_0(L)$.

We look for a solution $(\boldsymbol{v}, \pi, \eta)$ of the following problem

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \operatorname{div} \boldsymbol{\tau} - \nabla \pi \quad \text{in} \quad \Omega(h(t)), \tag{2.2}$$

and for all $x_1 \in (0,L)$, see (1.9), $0 < t < T$

$$\begin{aligned} &-E\rho \left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta + c\frac{\partial^5 \eta}{\partial t \partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \\ &\left[\mu(|e(\boldsymbol{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial h}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - \pi + P_w\right](\bar{x},t), \end{aligned} \tag{2.3}$$

$$\boldsymbol{v}(\bar{x},t) = \left(0, \frac{\partial \eta}{\partial t}(x_1,t)\right), \tag{2.4}$$

$\bar{x} = (x_1, h(x_1,t))$.

Furthermore, in the analysis of problem (1.1)–(1.13) the boundary condition (1.6)-(1.7), cf. (2.3)-(2.4), is splitted in the following way, see [6]

$$\begin{aligned} &\left[\mu(|e(\boldsymbol{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial h}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - \pi + P_w\right](\bar{x},t) \\ &\qquad -\frac{\rho}{2}v_2\Big(v_2(\bar{x},t) - \frac{\partial h}{\partial t}(x_1,t)\Big) = \rho\kappa\left[\frac{\partial \eta}{\partial t}(x_1,t) - v_2(\bar{x},t)\right] \end{aligned} \tag{2.5}$$

and

$$-E\left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta + c\frac{\partial^5 \eta}{\partial t \partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \kappa\left[\frac{\partial \eta}{\partial t}(x_1,t) - v_2(\bar{x},t)\right]$$

with $\kappa \gg 1$. (2.6)

We will show later, that the approximation with κ is reasonable. One of the possible physical interpretations for introducing finite κ comes from the mathematical modeling of semi-pervious boundary, where this type of boundary condition occurs. In our case, the boundary Γ_w seems to be partly permeable for finite κ, but letting $\kappa \to \infty$ it becomes impervious. In fact, we prove the existence of solution if $\kappa \to \infty$ and thus we get the original boundary condition (2.3)-(2.4).

Furthermore, we overcome the difficulties with solenoidal spaces by means of the artificial compressibility. We approximate the continuity equation similarly as in [6] with

$$\varepsilon\left(\frac{\partial \pi_\varepsilon}{\partial t} - \Delta \pi_\varepsilon\right) + \operatorname{div} \boldsymbol{v}_\varepsilon = 0 \quad \text{in } \Omega(h(t)),\ t \in (0,T)$$

$$\frac{\partial \pi_\varepsilon}{\partial \boldsymbol{n}} = 0, \quad \text{on } \partial\Omega(h(t)),\ t \in (0,T), \qquad \pi_\varepsilon(0) = 0 \text{ in } \Omega(h(0)) \quad \varepsilon > 0. \tag{2.7}$$

By letting $\varepsilon \to 0$ we show that $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$, where $\boldsymbol{v}$ is the weak solution of (1.1). For fixed ε, due to the lack of solenoidal property for velocity, we have the additional term in momentum equation $\frac{\rho}{2} v_i \operatorname{div} \boldsymbol{v}$, which we include into the convective term, see (2.12).

Our approximated problem is defined on a moving domain depending on function $h = R_0 + \delta$. Now we will reformulate it to a fixed rectangular domain. Set

$$\begin{aligned}
\boldsymbol{u}(y_1, y_2, t) &\stackrel{\text{def}}{=} \boldsymbol{v}(y_1, h(y_1,t)y_2, t)\\
q(y_1, y_2, t) &\stackrel{\text{def}}{=} \rho^{-1}\pi(y_1, h(y_1,t)y_2, t)\\
\sigma(y_1, t) &\stackrel{\text{def}}{=} \frac{\partial \eta}{\partial t}(y_1, t)
\end{aligned} \tag{2.8}$$

for $y \in D = \{(y_1, y_2);\ 0 < y_1 < L,\ 0 < y_2 < 1\}$, $0 < t < T$.

We define the following space

$$\begin{aligned}
\boldsymbol{V} &\equiv \left\{\boldsymbol{w} \in W^{1,p}(D):\ w_1 = 0 \ \text{ on } S_w,\ w_2 = 0 \ \text{ on } S_{in} \cup S_{out} \cup S_c\right\},\\
S_w &= \{(y_1, 1) : 0 < y_1 < L\}, \qquad S_{in} = \{(0, y_2) : 0 < y_2 < 1\},\\
S_{out} &= \{(L, y_2) : 0 < y_2 < 1\}, \qquad S_c = \{(y_1, 0) : 0 < y_1 < L\}.
\end{aligned} \tag{2.9}$$

Let us introduce the following notations

$$\operatorname{div}_h \boldsymbol{u} \stackrel{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2},$$

$$\begin{aligned}
a_1(q,\phi) = \int_D \Bigg\{ &\left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_1}\\
&+ \left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_2}\Bigg\} dy,
\end{aligned} \tag{2.10}$$

viscous term

$$((\boldsymbol{u}, \boldsymbol{\psi})) = \int_D h \boldsymbol{\tau}_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{\psi}) dy, \tag{2.11}$$

$$\boldsymbol{\tau}_{ij}(\hat{e}(\boldsymbol{u})) = 2\rho^{-1}\mu(|\hat{e}(\boldsymbol{u})|)\hat{e}_{ij}(\boldsymbol{u}), \qquad \hat{e}_{ij}(\boldsymbol{u}) = \frac{1}{2}(\hat{\partial}_i(u_j) + \hat{\partial}_j(u_i)),$$

$$\hat{\partial}_1 = \left(\frac{\partial}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_2}\right), \qquad \hat{\partial}_2 = \frac{1}{h}\frac{\partial}{\partial y_2},$$

convective term

$$
\begin{aligned}
b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) = \int_D & \left(hu_1 \left(\frac{\partial \boldsymbol{z}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{z}}{\partial y_2} \right) \cdot \boldsymbol{\psi} + \frac{h}{2}\boldsymbol{z}\cdot\boldsymbol{\psi}\, \mathrm{div}_h \boldsymbol{u}\, dy \\
& - \frac{1}{2}\int_0^1 R_0 u_1 z_1 \psi_1 \,(L, y_2)\, dy_2 + \frac{1}{2}\int_0^1 R_0 u_1 z_1 \psi_1\,(0,y_2)\, dy_2 \\
& - \frac{1}{2}\int_0^L u_2 z_2 \psi_2\,(y_1, 1)\, dy_1. \qquad (2.12)
\end{aligned}
$$

Remark 2.1. Note, that the transformed stress tensor $\boldsymbol{\tau}_{ij} = 2\rho^{-1}\mu(|\hat{e}(\boldsymbol{u})|)\hat{e}_{ij}(\boldsymbol{u})$ from (2.11) with $\mu(|\hat{e}(\boldsymbol{u})|)$ defined in (1.5) also satisfies (1.2)–(1.4).

Remark 2.2. Since the terms defined in (2.10), (2.11) and (2.12) are dependent on the domain deformation h, it will be sometimes useful to denote this explicitely, e.g., $b(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) = b_h(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi})$, $\hat{e}(\boldsymbol{u}) = \hat{e}_h(\boldsymbol{u})$.

Definition 2.3 (Weak solution of (κ,ε,h) - approximate problem).
Let $\boldsymbol{u} \in L^p(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D))$, $q \in L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ and $\sigma \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$. A triple $\boldsymbol{w} = (\boldsymbol{u},q,\sigma)$ is called a weak solution of the regularized problem (1.1)–(1.13) if the following equation holds

$$
\begin{aligned}
& -\int_0^T \left\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \; = \\
& \int_0^T \int_D \left(-\frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}) - h\, q\, \mathrm{div}_h \boldsymbol{\psi} \right) dy \; + ((\boldsymbol{u},\boldsymbol{\psi}))\, dt \\
& + \int_0^T \int_0^1 h(L,t) q_{out} \psi_1 \;(L, y_2, t) - h(0,t) q_{in} \psi_1 \;(0, y_2, t)\; dy_2 dt \\
& + \int_0^T \int_0^L \left(q_w + \frac{1}{2} u_2 \frac{\partial h}{\partial t} + \kappa\,(u_2 - \sigma) \right) \psi_2 \;(y_1, 1, t)\, dy_1 dt \\
& + \varepsilon \int_0^T \left\langle \frac{\partial (hq)}{\partial t}, \phi \right\rangle dt \qquad (2.13) \\
& + \int_0^T \int_D \left(-\varepsilon \frac{\partial h}{\partial t}\frac{\partial (y_2 q)}{\partial y_2}\phi + \varepsilon a_1(q,\phi) + \; h\, \mathrm{div}_h \boldsymbol{u}\; \phi \right) dy\, dt \\
& + \frac{\varepsilon}{2} \int_0^T \int_0^L \frac{\partial h}{\partial t}(y_1,t) q \phi(y_1,1,t)\, dy_1 dt \; + \\
& + \int_0^T \int_0^L \left(\frac{\partial \sigma}{\partial t}\xi + c\frac{\partial^2 \sigma}{\partial y_1^2}\frac{\partial^2 \xi}{\partial y_1^2} + a\frac{\partial}{\partial y_1}\int_0^t \sigma(y_1,s) ds \frac{\partial \xi}{\partial y_1} \right. \\
& \qquad \left. + b\int_0^t \sigma(y_1,s) ds\; \xi - a\frac{\partial^2 R_0}{\partial y_1^2}\xi + \frac{\kappa}{E}\,(\sigma - u_2)\,\xi \right) (y_1,t)\, dy_1 dt
\end{aligned}
$$

for every $(\boldsymbol{\psi},\phi,\xi) \in H_0^1(0,T;\boldsymbol{V}) \times L^2(0,T;H^1(D)) \times L^2(0,T;H_0^2(0,L))$.

Here we remind $E = \tilde{E}\sqrt{1 + (\partial_{y_1} R_0)^2}$. For simplicity and without lost of generality we assume in what follows that E, a, b, c are constant, cf. (2.1).

2.1. Existence of $(\kappa, \varepsilon,\, h)$ - approximate weak solution

For the proof of weak solution to the auxiliary problem defined in Definition 2.3 following properties of viscous and convective forms are useful[1]; for their proofs see [9, Section 3.1].

Lemma 2.4 (Coercivity of the viscous form).
The viscous form defined in (2.11) *satisfies for any* $2 \le p < \infty$ *the following estimates: there exists* $\tilde{\delta} = \tilde{\delta}(K, \alpha) > 0$ *such that*

$$
\begin{aligned}
&1)\ ((\boldsymbol{u}, \boldsymbol{u})) \ge \tilde{\delta}\|\boldsymbol{u}\|_{1,p}^{p} + \tilde{\delta}\|\boldsymbol{u}\|_{1,2}^{2},\\
&2)\ ((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)),\\
&\qquad\qquad\qquad \ge \tilde{\delta} \int_D |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^2 + |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^p,\\
&3)\ ((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)) \ge 0.
\end{aligned}
$$

Lemma 2.5 (Boundedness of the viscous form).
Let $\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{V}$, *then for* $2 \le p < \infty$ *it holds*

$$
((\boldsymbol{u}, \boldsymbol{v})) \le C\|\boldsymbol{u}\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p} + C_0\|\boldsymbol{u}\|_{1,p}\|\boldsymbol{v}\|_{1,p}, \quad C_0 > 0. \tag{2.14}
$$

Lemma 2.6 (Nonlinear convective term $b(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\psi})$).
For the trilinear form $b(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\psi})$, *defined in* (2.12) *the following properties hold*

$$
\begin{aligned}
b(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\psi}) &= \frac{1}{2}B(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\psi}) - \frac{1}{2}B(\boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{z}), \qquad (2.15)\\
\text{where}\quad B(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\psi}) &\equiv \int_D \left(h u_1 \left(\frac{\partial \boldsymbol{z}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{z}}{\partial y_2} \right) \cdot \boldsymbol{\psi}\, dy.
\end{aligned}
$$

Moreover for $p \ge 2$ *we have*

$$
|B(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\psi})| \le c\|\boldsymbol{u}\|_{1,p}\|\boldsymbol{z}\|_{1,p}\|\boldsymbol{\psi}\|_{1,p}.
$$

In our recent work [9] the following result has been proved.

Theorem 2.7 (Existence of (κ, ε, h) - approximate weak solution).
Let $\varepsilon,\ \kappa$, *be fixed. Assume* (1.2)–(1.4), *a given function* h, *such that* (2.1) *holds,* $q_{in}, q_{out} \in L^{p'}(0,T; L^2(0,1)), q_w \in L^{p'}(0,T; L^2(0,L))$. *Then there exists a weak solution of the* (κ, ε, h) - *approximated problem transformed to the fixed domain, in the sense of integral identity* (2.13). *Moreover,*

$$
\frac{\partial (h\boldsymbol{u})}{\partial t} \in \left\{ \begin{array}{l} L^{p'}(0,T; \boldsymbol{V}^*) \ \text{for}\ 2 < p < \infty,\\ L^{p'}(0,T; \boldsymbol{V}^*) \oplus L^{4/3}((0,T) \times D),\\ \hfill \text{for}\ p = 2, \end{array} \right. \quad \frac{\partial (hq)}{\partial t} \in L^2(0,T; H^{-1}(D)),
$$

such that

$$
\int_0^T \Big\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \Big\rangle dt = -\int_0^T \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} dy\, dt.
$$

Proof. See [9, Section 3 and 4]. □

---

[1] We use here notations $\|\cdot\|_p := \|\cdot\|_{L^p(D)}, \|\cdot\|_{1,p} := \|\cdot\|_{W^{1,p}(D)}$.

3. Problem with $\varepsilon = 0,\ \kappa = \infty$

The weak solution from Theorem 2.7 depends on the parameters ε, κ and k. Keeping k fixed but passing to the limit with $\varepsilon \to 0$, $\kappa \to \infty$ we will obtain the weak solution of the original problem (1.1)–(1.13) defined on $\Omega(\eta^{(k-1)})$. By this procedure we will prove the existence for one iteration with respect to the domain deformation $\eta^{(k)}$. We realize the limiting process by passing to the limit in both parameters at once, taking $\kappa = \varepsilon^{-1}$ and letting $\kappa \to \infty$.

In what follows we point out the dependence of weak solution on parameters ε, κ: $\boldsymbol{u}_\kappa$, q_κ, σ_κ. Since k is fixed in this section, we omit the notation of the dependence on k.

In analogy to the estimate [9, Section 4.1, estimate (4.7)] we obtain the following a priori estimate by testing (2.13) with $(\boldsymbol{u}_\kappa, q_\kappa, \sigma_\kappa)$, using the lemmas from Section 2.1 and after analogous manupilations as in [9]. Note, that the right hand side is independent on ε, κ.

$$
\begin{aligned}
&\max_{0\le t\le T}\int_D h(t)\left(|\boldsymbol{u}_\kappa|^2+\varepsilon|q_\kappa|^2\right)(t)dy+\frac{E}{2}\int_0^L|\sigma_\kappa(t)|^2dy_1 \qquad (3.1)\\
&+\int_0^T\int_D\tilde{\delta}|\nabla\boldsymbol{u}_\kappa|^p+\frac{2\alpha\varepsilon}{2+K^2}|\nabla q_\kappa|^2dy+E\int_0^L c\left|\frac{\partial^2\sigma_\kappa}{\partial y_1^2}\right|^2dy_1\,dt\\
&+\int_0^L\frac{aE}{2}\left|\int_0^t\frac{\partial\sigma_\kappa(s)}{\partial y_1}ds\right|^2+\frac{bE}{2}\left|\int_0^t\sigma_\kappa(s)s\right|^2dy_1\\
&+\int_0^T\!\!\int_0^L 2\kappa\left|\sigma_\kappa-u_{2\kappa}\right|^2dy_1\,dt\le\tilde{M}\int_0^T\mathcal{P}^{p'}+c_1\left\|\frac{\partial^2R_0}{\partial y_1^2}\right\|^2_{L^2(0,L)}dt\,.
\end{aligned}
$$

Here $c_1 = c_1(p, E, a, c)$, $\tilde{M} = \tilde{M}(p, K, \alpha)$ and $\mathcal{P} := \|q_{in}\|_{L^2(0,1)} + \|q_{out}\|_{L^2(0,1)} + \|q_w\|_{L^2(0,L)}$.

3.1. Limiting process $\kappa = \varepsilon^{-1} \to \infty$

First of all let us investigate the solenoidal property of the weak solution in the limiting case, i.e., for $\kappa = \infty$. The estimate (3.1) implies the weak convergence of

$$
\begin{aligned}
&(\boldsymbol{u}_\kappa, \sqrt{\varepsilon}q_\kappa, \sigma_\kappa) \rightharpoonup (\boldsymbol{u}, \tilde{q}, \sigma) \qquad (3.2)\\
&\quad\text{in}\ \ L^p(0,T;\boldsymbol{V})\times L^2(0,T;H^1(D))\times L^2(0,T;H^2(0,L))
\end{aligned}
$$

as $\kappa \to \infty$. Moreover, after inserting test functions $(\mathbf{0}, \phi, 0)$ into (2.13) for sufficiently smooth ϕ we obtain

$$
\begin{aligned}
&\int_0^T\int_D h\phi\operatorname{div}\boldsymbol{u}_\kappa \le \qquad (3.3)\\
&\sqrt{\varepsilon}C\|\sqrt{\varepsilon}q_\kappa\|_{L^2(0,T;H^1(D))}\left(\|\phi\|_{L^2(0,T;H^1(D))}+\|\partial_t\phi\|_{L^2((0,T)\times D)}\right),
\end{aligned}
$$

Using the boundedness of $\sqrt{\varepsilon}q_\kappa$ in $L^2(0,T;H^1(D))$ and letting $\varepsilon = \kappa^{-1} \to 0$ in (3.3) we get

$$
\operatorname{div}_h\boldsymbol{u} = 0 \qquad a.e. \quad \text{on}\ \ (0,T)\times D.
$$

This fact allows us to confine later the space of test functions to the solenoidal space, i.e. $\mathrm{div}_h \boldsymbol{\psi} = 0$ a.e. on D.

In the limiting process for $\kappa \to \infty$ we cannot use the Lions-Aubin lemma in order to obtain strong convergences in appropriate spaces for $(\boldsymbol{u}_\kappa, \sigma_\kappa) \to (\boldsymbol{u}, \sigma)$, since the estimates of the time derivatives $\partial_t \boldsymbol{u}_\kappa,\ \partial_t \sigma_\kappa$, depend on κ, see [9, Section 4.1]. In fact, we have to use another argument to obtain the strong convergence. We follow the lines of [6, Section 8] and use the equicontinuity in time as in Alt, Luckhaus cf. [1, Lemma 1.9]. It can be shown that

$$\begin{aligned}&\int_0^{T-\tau}\int_D |(h\boldsymbol{u}_\kappa)(t+\tau) - (h\boldsymbol{u}_\kappa)(t)|^2 + \varepsilon|(hq_\kappa)(t+\tau) - (hq_\kappa)(t)|^2 dydt \\ &+ \int_0^{T-\tau}\int_0^L |(h\sigma_\kappa)(t+\tau) - (h\sigma_\kappa)(t)|^2 dy_1 dt \le C(K,\alpha)\tau, \end{aligned} \tag{3.4}$$

where C is a positive constant independent on $\tau, \kappa, \varepsilon$. The proof of (3.4) can be found in [9, Section 5.1] and we omit its presentation here.

The estimate (3.4) and the compactness argument from [1, Lemma 1.9] imply the following strong convergences for $\kappa \to \infty$

$$\boldsymbol{u}_\kappa \to \boldsymbol{u} \ \text{ in } L^1((0,T)\times D), \qquad \sigma_\kappa \to \sigma \ \text{ in } L^1((0,T)\times(0,L)).$$

Using the standard interpolations of spaces $L^r(Q_T)$ and $L^s(S_T)$, $Q_T = (0,T)\times D$, $S_T = (0,T)\times(0,L)$ and boundedness of $\boldsymbol{u}, \sigma$ in $L^4(Q_T)$, $L^6(S_T)$, respectively, we obtain

$$\boldsymbol{u}_\kappa \to \boldsymbol{u} \ \text{ in } L^r((0,T)\times D), \qquad \sigma_\kappa \to \sigma \ \text{ in } L^s((0,T)\times(0,L)),$$

where $1 \le r < 4,\ 1 \le s < 6$ for $\kappa \to \infty$.

Now let us consider test functions $\boldsymbol{\psi} \in L^p(0,T;X)$, $\boldsymbol{\psi}(T) = 0$, $\xi = E\boldsymbol{\psi}_2\big|_{S_w}$, where

$$\begin{aligned} X &= \{\boldsymbol{\psi} \in \boldsymbol{V}_{div}\,;\ \boldsymbol{\psi}_2\big|_{S_w} \in H_0^2(0,L)\}, \\ \boldsymbol{V}_{div} &:= \{f \in \boldsymbol{V},\ \mathrm{div}_h f = 0 \ a.e. \text{ on } D\},\ \text{cf. } (2.9) \end{aligned} \tag{3.5}$$

in (2.13). With this choice of test functions the boundary terms with κ are canceled.

Now, we can pass to the limit in $\kappa \to \infty$ in (2.13), where $\kappa = \varepsilon^{-1}$. We use the weak convergences of $\boldsymbol{u}_\kappa$ in $L^p(0,T;\boldsymbol{V}_{div})$, $\sqrt{\varepsilon}q_\kappa$ in $L^2(0,T;H^1(D))$, σ_κ in $L^2(0,T;H^2(0,L))$, see (3.2) and strong convergence of $h\boldsymbol{u}_\kappa$ in $L^r((0,T)\times D)$, $0 \le r < 4$. The convergence of the viscous term follows from the monotonicity of the viscous operator and the Minty-Browder theorem, see also [9, Section 4.1]. Analogous arguments in order to obtain convergence in the viscous term when $k \to \infty$ will be presented in Section 4.

The convergence of the convective term for $\boldsymbol{\psi} \in H^1(0,T;X)$ can be obtained for all $p > 2$ in following way. For case $p = 2$ see [6, Section 8]. In order to obtain $\int_0^T b(\boldsymbol{u}_\kappa, \boldsymbol{u}_\kappa, \boldsymbol{\psi}) \to \int_0^T b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})$ one needs to show that

$\int_0^T |B(\boldsymbol{u}_\kappa - \boldsymbol{u}, \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$, $\int_0^T |B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$. Indeed, using the Hölder inequality and imbedding $L^{\frac{2p}{p-2}}(D) \hookrightarrow W^{1,p}(D)$ we have

$$\int_0^T |B(\boldsymbol{u}_\kappa - \boldsymbol{u}, \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \le C(K,\alpha) \int_0^T \|\boldsymbol{u}_\kappa - \boldsymbol{u}\|_2 \|\boldsymbol{u}_\kappa\|_{1,p} \|\boldsymbol{\psi}\|_{\frac{2p}{p-2}} \tag{3.6}$$
$$\le C(K,\alpha)\|\boldsymbol{\psi}\|_{H^1(0,T;W^{1,p}(D))} \|\boldsymbol{u}_\kappa - \boldsymbol{u}\|_{L^2((0,T)\times D)} \|\boldsymbol{u}_\kappa\|_{L^p(0,T;W^{1,p}(D))}.$$

Thus $\int_0^T |B(\boldsymbol{u}_\kappa - \boldsymbol{u}, \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$. Further $\int_0^T |B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\kappa, \boldsymbol{\psi})| \to 0$ due to the weak convergence of $\boldsymbol{u}_\kappa$ in $L^p(0,T;\boldsymbol{V}_{div})$.

The convergence of the terms containing $\sqrt{\varepsilon} q_\kappa$ can be realized by the weak convergence in the corresponding spaces. The term $\int_0^T \int_D h q_\kappa \operatorname{div}_h \boldsymbol{\psi}$ is canceled due to the solenoidal test functions.

Finally, after the limiting process $\kappa \to \infty$ in (2.13) using above considerations for all $\boldsymbol{\psi} \in H_0^1(0,T;X)$ and $\xi = E\boldsymbol{\psi}_2\big|_{S_w}$ we arrive at

$$\int_0^T \int_D \left\{ h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} \right\} dy = \tag{3.7}$$
$$\int_0^T \Bigg\{ ((\boldsymbol{u}, \boldsymbol{\psi}))_h + b_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})$$
$$+ \int_0^1 h(L) q_{out}(y_2,t) \psi_1(L, y_2, t) - h(0) q_{in}(y_2,t) \psi_1(0,y_2,t)\, dy_2$$
$$+ \int_0^L \left(q_w + \frac{1}{2}\frac{\partial h}{\partial t} u_2 \right) \psi_2(y_1, 1, t)\, dy_1$$
$$+ \int_0^L -\sigma \frac{\partial \xi}{\partial t} + c \frac{\partial^2 \sigma}{\partial y_1^2} \frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1} \int_0^t \sigma(y_1, s) ds \frac{\partial \xi}{\partial y_1}$$
$$- a \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \int_0^t \sigma(y_1,s) ds\, \xi(y_1,t)\, dy_1 \Bigg\} dt.$$

In order to investigate the meaning of the left hand side of the above equality we define the ALE-type time derivative $\bar{\partial}_t$

$$\bar{\partial}_t(h\boldsymbol{u}) := \frac{\partial (h\boldsymbol{u})}{\partial t} - \frac{\partial h}{\partial t} \frac{1}{h} \frac{\partial (y_2 h \boldsymbol{u})}{\partial y_2}. \tag{3.8}$$

Note that $\bar{\partial}_t(h\boldsymbol{u}) = h \partial_t^y \boldsymbol{u}$, where $\partial_t^y := \left(\frac{\partial}{\partial t} - \frac{\partial h}{\partial t} \frac{y_2}{h} \frac{\partial}{\partial y_2} \right)$ denotes in fact the time derivative transformed to the rectangle domain D, i.e., in coordinates (y_1, y_2).

The right hand side of (3.7) is bounded for every $\boldsymbol{\psi} \in \mathcal{M}$,

$$\begin{aligned} \mathcal{M} \quad = \quad & \{\omega \in L^p(0,T;X) \text{ for } p > 2; \\ & \omega \in L^p(0,T;X) \cap L^4((0,T)\times D) \text{ for } p = 2\}. \end{aligned} \tag{3.9}$$

Thus it can be identified with some functional $\chi \in \mathcal{M}^*$. Then using integration by parts with respect to y_2 on the left hand side, backward transformation from D to the moving domain $\Omega(h(t))$ and the separation of variables it can be shown that $\chi = \bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*)$, see [9, Appendix A] for more details. Thus we can replace

$$\int_0^T \int_D \left\{ h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} \right\} dy\, dt = -\int_0^T \left\langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi} \right\rangle_X .$$

Finally, we transform (3.7) from the rectangle D to the moving domain $\Omega(h(t))$ and obtain the existence of a weak solution to our original problem (1.1)–(1.13) with the Dirichlet boundary condition $\partial_t \eta = v_2|_{\Gamma_w(h(t))}$ for a prescribed domain deformation h.

Theorem 3.1 (Existence of weak solution for $\varepsilon = 0$, $\kappa = \infty$). *Assume that $h \in H^1(0,T;H_0^2(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L))$ satisfies (2.1). Let the boundary data fulfill $q_{in}, q_{out} \in L^{p'}(0,T;L^2(0,1))$, $q_w \in L^{p'}(0,T;L^2(0,L))$. Furthermore, assume that the properties (1.2)–(1.4) for the viscous stress tensor hold. Then there exists a weak solution $(\boldsymbol{v}, \eta)$ of the problem (1.1)–(1.13), such that*

i) $(\boldsymbol{u}, \eta) \in [L^p(0,T;\boldsymbol{V}) \times H^1(0,T;H_0^2(0,L))] \cap [L^\infty(0,T;L^2(D)) \times W^{1,\infty}(0,T;L^2(0,L))]$, *where $\boldsymbol{u}$ is defined in (2.8),*

ii) *the time derivative $\bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*)$ for $p > 2$ and $\bar{\partial}_t(h\boldsymbol{u}) \in L^{p'}(0,T;X^*) \oplus L^{4/3}((0,T) \times D)$ for $p = 2$,*

$$\int_0^T \int_D \left\{ h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial (y_2 \boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} \right\} dy\, dt = -\int_0^T \Big\langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi} \Big\rangle dt,$$

where $\bar{\partial}_t(h\boldsymbol{u}) = \frac{\partial (h\boldsymbol{u})}{\partial t} - \frac{1}{h}\frac{\partial h}{\partial t}\frac{\partial (y_2 h\boldsymbol{u})}{\partial y_2} = h\partial_t^y \boldsymbol{u}$,
for every test function $\boldsymbol{\psi} \in \mathcal{M} \cap H_0^1(0,T;X)$,

iii) *$\boldsymbol{v}$ satisfies the condition div $\boldsymbol{v} = 0$ a.e on $\Omega(h(t))$,*
$v_2(x_1, h(x_1,t), t) = \partial_t \eta(x_1, t)$ for a.e. $x_1 \in (0,L)$, $t \in (0,T)$

and the following integral identity holds

$$\begin{aligned}
&\int_0^T \int_{\Omega(h(t))} \Big\{ -\rho \boldsymbol{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\boldsymbol{v})|) e(\boldsymbol{v}) e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i} \boldsymbol{\varphi}_j \Big\} dx\, dt \\
&+ \int_0^T \int_0^{R_0(L)} \left(P_{out} - \frac{\rho}{2}|v_1|^2 \right) \varphi_1(L, x_2, t)\, dx_2\, dt \\
&- \int_0^T \int_0^{R_0(0)} \left(P_{in} - \frac{\rho}{2}|v_1|^2 \right) \varphi_1(0, x_2, t)\, dx_2\, dt \\
&+ \int_0^T \int_0^L \left(P_w - \frac{\rho}{2} v_2 \left(v_2 - \frac{\partial h}{\partial t} \right) \right) \varphi_2(x_1, h(x_1,t), t)\, dx_1\, dt \\
&+ \int_0^T \int_0^L - \frac{\partial \eta}{\partial t}\frac{\partial \xi}{\partial t} + c\, \frac{\partial^3 \eta}{\partial x_1^2 \partial t} \frac{\partial^2 \xi}{\partial x_1^2} + a \frac{\partial \eta}{\partial x_1}\frac{\partial \xi}{\partial x_1} dx_1\, dt \\
&+ \int_0^T \int_0^L -a \frac{\partial^2 R_0}{\partial x_1^2} \xi + b\eta\, \xi\, dx_1\, dt \; = \; 0
\end{aligned}$$

for every test functions

$$
\begin{aligned}
&\boldsymbol{\varphi}(x_1,x_2,t) = \boldsymbol{\psi}\left(x_1, \frac{x_2}{h(x_1,t)}, t\right) \quad \textit{such that}\\
&\boldsymbol{\psi} \in H_0^1(0,T;\boldsymbol{V}), \ \ \boldsymbol{\psi}_2\big|_{S_w} \in H_0^1(0,T;H_0^2(0,L)),\\
&div\, \boldsymbol{\varphi} = 0 \quad a.e.\ on\ \Omega(h(t)),\\
&and \quad \xi(x_1,t) = E\rho\, \varphi_2(x_1,h(x_1,t),t).
\end{aligned}
$$

Note that the structure equation is fulfilled in a slightly modified sense,

$$
\begin{aligned}
&E\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) =\\
&\qquad\qquad \left[-(\mathbf{T}_f + P_w\mathbf{I})\boldsymbol{n}|\boldsymbol{n}|\cdot\boldsymbol{e}_2 + \frac{\rho}{2}\partial_t\eta(\partial_t\eta - \partial_t h)\right](x_1,h(x_1,t),t)
\end{aligned}
$$

a.e. on $(0,T)\times(0,L)$, compare (1.9).

4. Fixed point iterations

Until now we have proved the existence of weak solution of the original problem in a domain given by a known deformation function δ, i.e. $h = R_0 + \delta$, $\delta \in H^1(0,T;H_0^2(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L))$, $R_0(x_1) \in C^2[0,L]$. In this section we show the existence of the weak solution of (1.14), which implies, that the domain deforms according to the function $\eta(x_1,t)$, i.e. $h = R_0 + \eta$. This will be realized with the use of the Schauder fixed point theorem. First, applying the compactness argument based on the equicontinuity in time we obtain that bounded sequence sequence $(\boldsymbol{v}^{(k)}, \eta^{(k)})$ defined on $\Omega(\delta^{(k)})$ for some sequence $\delta^{(k)} \to \delta$ converges to the limit $(\boldsymbol{v},\eta)$ defined on $\Omega(\delta)$. Consequently, the Schauder fixed point argument implies, that the weak solution η is associated with the time dependent domain $\Omega(\eta)$. Finally we obtain the main result: existence of weak solution for a fully coupled fluid structure interaction problem (1.1)–(1.13).

Let us denote the space $Y = H^1(0,T;L^2(0,L))$. For each test function $\boldsymbol{\psi} \in L^p(0,T;X)$, $\boldsymbol{\psi}(T) = 0$, $\xi = E\boldsymbol{\psi}_2\big|_{S_w}$, recalling (3.5), and for any $h = R_0 + \delta \in Y$, such that (2.1) holds, we construct solutions $\boldsymbol{u}$, η of the following problem defined on the reference domain D, ($\sigma = \partial_t\eta$),

$$
\begin{aligned}
&-\int_0^T \left\langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi}\right\rangle dt \qquad\qquad (4.1)\\
&= \int_0^T \Bigg\{ ((\boldsymbol{u},\boldsymbol{\psi}))_h + b_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})\\
&\quad + \int_0^1 h(L)q_{out}(y_2,t)\psi_1(L,y_2,t) - h(0)q_{in}(y_2,t)\psi_1(0,y_2,t)\,dy_2\\
&\quad + \int_0^L \left(q_w + \frac{1}{2}\frac{\partial h}{\partial t}\sigma\right)\psi_2(y_1,1,t)\,dy_1
\end{aligned}
$$

$$+ \langle \partial_t \sigma, \xi \rangle + \int_0^L c \frac{\partial^2 \sigma}{\partial y_1^2} \frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1} \int_0^t \sigma(y_1, s) ds \frac{\partial \xi}{\partial y_1}$$
$$- a \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \int_0^t \sigma(y_1, s) ds \, \xi(y_1, t) \, dy_1 \Big\} dt \, .$$

Further, let the ball $B_{\alpha,K}$ be defined by

$$B_{\alpha,K} = \Big\{ \delta \in Y; \ \|\delta\|_Y \le C(\alpha, K), \ 0 < \alpha \le R_0(y_1) + \delta(y_1, t) \le \alpha^{-1},$$
$$\left| \frac{\partial \delta(y_1, t)}{\partial y_1} \right| \le K, \quad \delta(y_1, 0) = 0, \ \forall y_1 \in [0, L], \ \forall t \in [0, T],$$
$$\int_0^T \left| \frac{\partial \delta(y_1, t)}{\partial t} \right|^2 dt \le K, \ \forall y_1 \in [0, L] \Big\},$$

where $C(\alpha, K)$ is a suitable constant large enough with respect to K, α and the data.

By choosing $\delta \in B_{\alpha,K}$ the following energy estimate holds for all $2 \le p < \infty$ uniformly in δ,

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D))} + \|\boldsymbol{u}\|^p_{L^p(0,T;W^{1,p}(D))} \\ &\quad + \|\eta_t\|^2_{L^\infty(0,T;L^2(0,L))} + \|\eta_t\|^2_{L^2(0,T;H^2(0,L))} + \|\eta\|^2_{L^\infty(0,T;H^1(0,L))} \\ &\le c(T, p, K, \alpha) \left(\|\mathcal{P}\|^{p'}_{L^{p'}(0,T)} + \|R_0\|^2_{C^2[0,L]} \right) . \end{aligned} \tag{4.2}$$

This estimate is obtained by multiplying (4.1) by $\boldsymbol{\psi} = \boldsymbol{u}$ and $\xi = E u_2|_{S_w} = E\eta_t$, cf. (3.1).

Now, let us define the following mapping by (4.1),

$$\begin{aligned} &\mathcal{F} : B_{\alpha,K} \to Y; \\ &\mathcal{F}(\delta) = \eta, \quad (\delta = h - R_0). \end{aligned}$$

Our aim is to apply the Schauder fixed point theorem and prove that the mapping $\mathcal{F}$ has at least one fixed point. This implies the existence of the weak solution to our problem (4.1).

First we check that $\mathcal{F}(B_{\alpha,K}) \subset B_{\alpha,K}$. Note that our a priori estimate (4.2) yields $\|\eta_{y_1}\|_{C([0,T]\times[0,L])} \le K$, $\|\eta_t\|_{L^2(0,T;C[0,L])} \le K$ and $\|\eta\|_Y \le C(\alpha, K)$ for given data P_{in}, P_{out}, P_w, R_0, given K, α; $\alpha < R_{min} := \min_{y_1 \in [0,L]} R_0(y_1)$ and for sufficiently small time $\tilde{T}$. Moreover, since $H^1(0, T; H^2(0, L)) \hookrightarrow C(0, T; C^1[0, L])$ and $\eta(y_1, 0) = 0$, there exist a maximal time T_{max}, such that

i) $\|\eta\|_\infty := \|\eta\|_{C([0,T_{max}]\times[0,L])} \le R_{min} - \alpha$.
This yields that $\min_{t \in (0,T_{max})} \min_{y_1 \in (0,L)} (R_0 + \eta) \ge R_{min} - \|\eta\|_\infty \ge \alpha$. Thus we can avoid a contact of the regularized deforming wall with the solid bottom.

ii) Further, we require that the domain deformation is bounded from above, $\|R_0 + \eta\|_\infty \le \alpha^{-1}$.

Having i), the condition ii) is satisfied if $R_{min} - \alpha \leq \alpha^{-1} - R_{max}$. Thus, for instance if $\alpha^{-1} \geq R_{min} + R_{max}$.
Consequently, $\mathcal{F}(B_{\alpha,K}) \subset B_{\alpha,K}$ as far as $t \leq T^* := \min\{T_{max}, \tilde{T}\}$ for given data P_{in}, P_{out}, P_w, R_0, K and α such that $\alpha \leq \min\{R_{min}, \frac{1}{R_{min}+R_{max}}\}$.

Secondly, we verify the relative compactness of the mapping $\mathcal{F}$ in Y. Let us consider a sequence $\{\delta^{(k)}\}_{k=1}^{\infty}$ in $B_{\alpha,K}$. Denote by $\boldsymbol{u}^{(k)}$ and $\eta^{(k)} \equiv \mathcal{F}(\delta^{(k)})$ the weak solution of (4.1) for $h = h^{(k)} := R_0 + \delta^{(k)}$. Note, that due to the apriori estimate (4.2) we have weak convergences of $\eta^{(k)}$, $\boldsymbol{u}^{(k)}$ in the corresponding spaces. In Section 4.1, cf. Lemma 4.1, we show the equicontinuity in time, which implies the strong convergences of $\eta^{(k)}$ in Y as well as the strong convergence $\boldsymbol{u}^{(k)}$ in $L^2((0,T) \times D)$.

Finally, in Section 4.2 we check the continuity of the mapping $\mathcal{F}$ with respect to the strong topology in Y.

4.1. Relative compactness of the fixed point mapping $\mathcal{F}$

In this section we verify the relative compactness of the mapping $\mathcal{F}$ using the integral equicontinuity in time and the Riesz-Fréchet-Kolmogorov compactness argument. We prove Lemma 4.1, which provides the integral equicontinuity of $\eta^{(k)}$ and additionaly of $\boldsymbol{u}^{(k)}$, that holds independently on k. To this end we need to find suitable divergence free test functions in order to control difference of velocity at different time instances. In order to obtain such test functions we follow a construction presented in [4], see also the reference [16] therein.

We introduce, in analogy to [4, Lemma 3], the following extensions of the domain and the weak solution. Let B_M be a box domain

$$B_M \equiv (0, L) \times (0, M) \in \mathbb{R}^2 \tag{4.3}$$

for some $M > \alpha^{-1}$ specified later. Let us consider a sequence $\{\delta^{(k)}\}_{k=1}^{\infty}$ in $B_{\alpha,K}$ and $h^{(k)} := R_0 + \delta^{(k)}$. We define an extension of solution $\boldsymbol{u}^{(k)}(y,t) = \boldsymbol{v}^{(k)}(x,t)$ of (4.1) where $h = h^{(k)}$ into B_M,

$$\bar{\boldsymbol{v}}^{(k)} = \begin{cases} \boldsymbol{v}^{(k)} & \text{in } \Omega(h^{(k)}(t)) \\ (0, \eta_t^{(k)}) & \text{in } B_M \backslash \Omega(h^{(k)}(t)). \end{cases} \tag{4.4}$$

Further, for $\gamma > 1$ and any function $\boldsymbol{f}(x_1, x_2)$ we define $\boldsymbol{f}_\gamma$ as follows

$$\boldsymbol{f}_\gamma(x_1, x_2) = (\gamma f_1(x_1, \gamma x_2), f_2(x_1, \gamma x_2)).$$

Note that if $\boldsymbol{f}$ is divergence free, then $\boldsymbol{f}_\gamma$ is divergence free, too.

In what follows we will use the following property, which is valid for any $\boldsymbol{f} \in H^1(B_M)$, for $\gamma = 1 + \frac{C\sqrt{\tau}}{\alpha}$ and $M \geq 2\alpha^{-1}$, see [9, Lemma 9.2]

$$|\boldsymbol{f}_\gamma - \boldsymbol{f}| \leq |\boldsymbol{f}(x_1, \gamma x_2, t) - \boldsymbol{f}(x_1, x_2, t)| + (\gamma - 1)|\boldsymbol{f}(x_1, \gamma x_2, t)|. \tag{4.5}$$

Lemma 4.1. *For the weak solution* $(\boldsymbol{v}^{(k)}, \eta_t^{(k)}) = (\boldsymbol{u}^{(k)}, \sigma^{(k)})$ *of the problem* (4.1)*, where* $h = h^{(k)}$*, it holds*

$$\int_0^{T-\tau}\int_{B_M} \chi_t^{(k)} |\bar{\boldsymbol{v}}^{(k)}(t+\tau) - \bar{\boldsymbol{v}}^{(k)}(t)|^2 + \int_0^{T-\tau}\int_0^L |\eta_t^{(k)}(t+\tau) - \eta_t^{(k)}(t)|^2 \le C(\tau^{1/p} + \tau^{1/2}). \quad (4.6)$$

Here $\chi_t^{(k)}$ *denotes the characteristic function of* $\Omega(h^{(k)}(t))$*. The constant* $C = C(K, \alpha)$ *does not depend on* k*.*

Proof. We recall that $h^{(k)} = R_0 + \delta^{(k)}$, but for the sake of simplicity we omit the superscript (k) in this proof and we denote $h := R_0 + \delta^{(k)}$, $\bar{\boldsymbol{v}} := \bar{\boldsymbol{v}}^{(k)}$, $\eta := \eta^{(k)}$.

To prove the statement of this lemma, we will use following two properties.

1. For each $\boldsymbol{\psi} \in H^1(0,T;X)$, cf. (3.5), $\boldsymbol{\psi}(T) = 0$ it holds
$$-\int_0^{\tilde{\tau}} \langle \bar{\partial}_t(h\boldsymbol{u}), \boldsymbol{\psi} \rangle dt \quad (4.7)$$
$$= \int_0^{\tilde{\tau}}\int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2}\boldsymbol{\psi} dy dt - \int_D h\boldsymbol{u}(\tilde{\tau}, y)\boldsymbol{\psi}(\tilde{\tau}, y) dy.$$
For classical time derivative, this property is clear. For our distributive derivative $\bar{\partial}$ it can be proven using test function $\boldsymbol{\psi} = \zeta(y,t)\varphi_\epsilon(t)$, where $\zeta \in H^1(0,T;X), \varphi_\epsilon(t) = \max\{0, \min\{1, \frac{\tilde{\tau}+\epsilon-t}{\epsilon}\}\}$ and passing $\epsilon \to 0$, cf.[17].
2. By inserting any time independent test function $\boldsymbol{\psi} = \boldsymbol{\psi}(y)$ into (4.7) and subtracting (4.7) for $\tilde{\tau} = t + \tau$, and $\tilde{\tau} = t$ we obtain
$$-\int_t^{t+\tau} \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}(x,t) \rangle_{X_\Omega} ds \quad (4.8)$$
$$= \int_t^{t+\tau}\int_D \frac{\partial h}{\partial t}\frac{\partial (y_2\boldsymbol{u})}{\partial y_2}\boldsymbol{\psi}(y) dy ds - \int_D [h\boldsymbol{u}(t+\tau) - h\boldsymbol{u}(t)]\boldsymbol{\psi}(y) dy.$$
Here the integral on the left hand side has been transformed into $\Omega(h(t))$, $\boldsymbol{\psi} = \boldsymbol{\psi}(y) = \boldsymbol{\varphi}(x)$, $y \in D$, $x \in \Omega(h(t))$ and the space X_Ω is defined as

$$X_\Omega = \{\boldsymbol{\varphi} \in W^{1,p}(\Omega);\ \operatorname{div}\varphi = 0 \text{ a.e. on } \Omega,$$
$$\varphi_2|_{\Gamma_w} \in H_0^2(0,L), \varphi_1|_{\Gamma_w} = \varphi_2|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = 0\}, \quad \Omega = \Omega(h(t)).$$

Now, let us integrate (4.8) over $\int_0^{T-\tau} dt$. The first term on the right hand side (integrated over $\int_0^{T-\tau}$) can be bounded with $C\tau$ independently on k for test functions (4.13) specified later . The second term on the right hand of (4.8) can be rewritten due to the transformation to the $\Omega(h(t))$

$$\int_0^{T-\tau}\int_{\Omega(h(t+\tau))} \boldsymbol{v}(x_{t+\tau}, t+\tau)\boldsymbol{\varphi}(x_{t+\tau}) dx - \int_{\Omega(h(t))} \boldsymbol{v}(x_t, t)\boldsymbol{\varphi}(x_t)\, dx\, dt. \quad (4.9)$$

Note, that the space coordinate $x_t \equiv x(t) \in \Omega(h(t))$ depends on time, hence the test functions $\boldsymbol{\varphi}$ implicitly depend on time, which is pointed out above.

Using the previously defined extensions of the solution $\bar{\boldsymbol{v}}$ and some further manipulations we can rewrite (4.9) as follows

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\Omega(h(t))}\bar{\boldsymbol{v}}(x_{t+\tau},t+\tau)\boldsymbol{\varphi}(x_{t+\tau})-\boldsymbol{v}(x_t,t)\boldsymbol{\varphi}(x_t)dx\\
&+\int_{B_M}(\chi_{t+\tau}-\chi_t)\bar{\boldsymbol{v}}(x_{t+\tau},t+\tau)\boldsymbol{\varphi}(x_{t+\tau})\,dx\,dt= \qquad (4.10)\\
&\int_0^{T-\tau}\int_{\Omega(h(t))}\underbrace{[\bar{\boldsymbol{v}}\,(x_{t+\tau},t+\tau)-\boldsymbol{v}(x_t,t)]\boldsymbol{\varphi}(x_t)}_{\text{(I)}}+\underbrace{[\boldsymbol{\varphi}(x_{t+\tau})-\boldsymbol{\varphi}(x_t)]\bar{\boldsymbol{v}}(x_{t+\tau},t+\tau)}_{\text{(II)}}\\
&+\int_{B_M}\underbrace{(\chi_{t+\tau}-\chi_t)\bar{\boldsymbol{v}}(x_{t+\tau},t+\tau)\boldsymbol{\varphi}(x_{t+\tau})}_{\text{(III)}}\,dx\,dt.
\end{aligned}$$

Here χ_t, $\chi_{t+\tau}$ are the characteristic functions of $\Omega(h(t))$, $\Omega(h(t+\tau))$, respectively. In what follows we estimate the term (II) for any test function $\boldsymbol{\varphi}\in L^p(0,T;X_\Omega)$. Further, we concentrate on the terms (I), (III) using specific test functions.

From the imbedings in one dimension we have $\delta\in C^{0,1/2}([0,T];H^1(0,L))$, cf. (4.19), thus

$$\|\delta(t+\tau)-\delta(t)\|_{L^\infty((0,T)\times(0,L))}\le C\sqrt{\tau}. \qquad (4.11)$$

Using (4.11) we can estimate the term (II):

$$\begin{aligned}
\text{(II)} &\le \int_0^{T-\tau}\left(\int_{\Omega(h(t))}|\boldsymbol{\varphi}(x_{t+\tau})-\boldsymbol{\varphi}(x_t)|^2dx\right)^{1/2}\|\bar{\boldsymbol{v}}\|_{L^2(\Omega(h(t))}\|dt \qquad (4.12)\\
&= \int_0^{T-\tau}\left(\int_{\Omega(h(t))}\Big|\int_{x_2(t)}^{x_2(t+\tau)}\partial_s\boldsymbol{\varphi}(x_1,s)ds\Big|^2dx\right)^{1/2}\|\bar{\boldsymbol{v}}\|_{L^2(\Omega(h(t))}\|dt\\
&\le \int_0^{T-\tau}\left(\int_{B_M}|\nabla\boldsymbol{\varphi}|^2dx|x_2(t+\tau)-x_2(t)|^2\right)^{1/2}\|\bar{\boldsymbol{v}}\|_{L^2(\Omega(h(t))}\|dt\\
&\le \|\boldsymbol{\varphi}\|_{L^2(0,T;H^1(B_M))}\|\delta(t+\tau)-\delta(t)\|_{L^\infty((0,T)\times(0,L))}\|\bar{\boldsymbol{v}}\|_{L^2((0,T)\times B_M)}\\
&\le C\sqrt{\tau}.
\end{aligned}$$

Now we specify proper test functions, that will be used in what follows. For $x_t=x(t)\in\Omega(h(t))$, $\gamma>1$ and fixed t, τ we set

$$\begin{aligned}
\boldsymbol{\varphi}(x_t) &= \bar{\boldsymbol{v}}_\gamma(x_{t+\tau},t+\tau)-\bar{\boldsymbol{v}}_\gamma(x_t,t), \qquad (4.13)\\
\xi(x_1) &= E(\partial_t\eta(x_1,t+\tau)-\partial_t\eta(x_1,t)).
\end{aligned}$$

Note that since $\boldsymbol{v}$ is divergence-free, the test function $\boldsymbol{\varphi}$ is also divergence-free [2]. Moreover, taking into account (4.11), for $\gamma\ge 1+\frac{C\sqrt{\tau}}{\alpha}$ and $x_2\in\Gamma_w(t)$ the coordinate γx_2 exceeds the moving domain $\Omega(h(s))$, since we have

[2]Since $\boldsymbol{\varphi}(x_{t+\tau})=\bar{\boldsymbol{v}}_\gamma(x_{t+2\tau},t+2\tau)-\bar{\boldsymbol{v}}_\gamma(x_{t+\tau},t+\tau)$, we have to integrate over $\int_0^{T-2\tau}dt$ in the estimate of the term (II), or we define $\boldsymbol{\varphi}(x_{t+\tau})=0$ if $t+\tau>T$.

$\gamma(R_0+\delta(s)) \geq R_0+\delta(s)+\|\delta(t+\tau)-\delta(t)\|_\infty$, $s=t,\ t+\tau$. According to the construction, such a test function fulfill the boundary condition

$$E\boldsymbol{\varphi}(x_1, R_0(x_1)+\delta(x_1,t)) = E(0, \partial_t\eta(x_1,t+\tau)-\partial_t\eta(x_1,t)) \equiv (0,\xi(x_1)).$$

Let us estimate now the term (III). Since $\partial_t\eta$ is bounded in $L^\infty(0,T; L^2(0,L))$ independently on k, we have

$$\int_{B_M} |\chi_{t+\tau}-\chi_t|^2 = \int_0^L |\delta(t+\tau)-\delta(t)|^2 = \int_0^L \left|\int_t^{t+\tau} \partial_t\delta(s)ds\right|^2 \leq C\tau. \tag{4.14}$$

Thus, the term (III) can be bounded for $\boldsymbol{\varphi}$ from (4.13) as follows.

$$\text{(III)} \leq \int_0^{T-\tau} \|\chi_{t+\tau}-\chi_t\|_{L^2(B_M)}\|\bar{\boldsymbol{v}}\|_{L^4(B_M)}\|\boldsymbol{\varphi}\|_{L^4(B_M)}dt \leq C\sqrt{\tau}. \tag{4.15}$$

For the test functions from (4.13) the term (I) equals

$$\begin{aligned}
\text{(I)} &= \int_0^{T-\tau}\int_{\Omega(h(t))} [\bar{\boldsymbol{v}}(t+\tau)-\bar{\boldsymbol{v}}(t)][\bar{\boldsymbol{v}}_\gamma(t+\tau)-\bar{\boldsymbol{v}}_\gamma(t)]dxdt \\
&= \int_0^{T-\tau}\int_{\Omega(h(t))} \underbrace{[\bar{\boldsymbol{v}}(t+\tau)-\bar{\boldsymbol{v}}(t)]^2}_{\text{(Ia)}} + \\
&\quad \underbrace{[\bar{\boldsymbol{v}}(t+\tau)-\bar{\boldsymbol{v}}(t)].\Big([\bar{\boldsymbol{v}}_\gamma(t+\tau)-\bar{\boldsymbol{v}}(t+\tau)]-[\bar{\boldsymbol{v}}_\gamma(t)-\bar{\boldsymbol{v}}(t)]\Big)}_{\text{(Ib)}}\,dxdt
\end{aligned} \tag{4.16}$$

For the simplicity we used shorter notations here, e.g., $\bar{\boldsymbol{v}}(t+\tau) := \bar{\boldsymbol{v}}(x_{t+\tau}, t+\tau)$. The term (Ia) appears on the left hand side of the assertion of this lemma; the term (Ib) need to be estimated from above. We illustrate the estimate of some chosen terms of (Ib) as follows. Estimates of other terms are analogous.

In the sequel we take $\gamma = 1+\frac{C\sqrt{\tau}}{\alpha}$ and $M \geq 2\alpha^{-1}$. For these parameters we have according to Lemma 4.5,

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\Omega(h(t))} \bar{\boldsymbol{v}}(t+\tau)[\bar{\boldsymbol{v}}_\gamma(t)-\bar{\boldsymbol{v}}(t)]dxdt \leq \\
&C_\alpha\sqrt{\tau}\int_0^{T-\tau} \|\bar{\boldsymbol{v}}(t+\tau)\|_{L^2(B_M)}\|\bar{\boldsymbol{v}}(t)\|_{H^1(B_M)}dt \leq C_\alpha\sqrt{\tau}.
\end{aligned}$$

To complete the proof, the remaining terms coming from the fluid equations, i.e., the convective term, the viscous term, boundary terms and the equation for η have to be estimated. We illustrate here only the calculations for the nonlinear viscous term and omit tedious but standard calculations for other terms, previously performed also in [6].

After subtracting the weak formulation (4.1) for $\int_0^{t+\tau} ds - \int_0^t ds$, inserting test functions constructed above (independent on s) into (4.1) and integrating over $\int_0^{T-\tau} dt$ we obtain from the viscous term

$$\int_0^{T-\tau}\int_t^{t+\tau}\int_{\Omega(h(s))} \tau_{ij}(e[\boldsymbol{v}(s)]).e[\bar{\boldsymbol{v}}_\gamma(t+\tau)-\bar{\boldsymbol{v}}_\gamma(t)]dx\,ds\,dt. \tag{4.17}$$

For the simplicity, we set $\omega := \bar{\boldsymbol{v}}_\gamma(t+\tau)$ or $\omega := \bar{\boldsymbol{v}}_\gamma(t)$ in (4.17). Using the fact, that $|\tau_{ij}(e(\boldsymbol{v}))| \le C_5(1+|e(\boldsymbol{v})|)^{p-1}$, which can be derived from (1.2), (1.4), cf. [11, Lemma 1.19], (4.17) can be bounded as follows,

$$
\begin{aligned}
&\le \int_0^{T-\tau}\int_t^{t+\tau}\int_{\Omega(h(s))} C_5(1+|e[\boldsymbol{v}(s)]|)^{p-1} e[\omega]\,dx\,ds\,dt \\
&\le C(K,\alpha)\int_0^{T-\tau}\int_t^{t+\tau} \|1+\nabla\boldsymbol{v}(s)\|^{p-1}_{L^p(\Omega(h(s)))}\|\nabla\omega\|_{L^p(\Omega(h(s)))}ds\,dt \\
&\le C(K,\alpha)\int_0^{T-\tau}\left(\int_t^{t+\tau}\|1+\nabla\boldsymbol{v}(s)\|^p_{L^p(\Omega(h(s)))}ds\right)^{\frac{p-1}{p}}\|\nabla\omega\|_{L^p(B_M)}\tau^{\frac{1}{p}}dt \\
&\le C(K,\alpha)\tau^{\frac{1}{p}}\left(\int_0^{T}\|1+\nabla\boldsymbol{v}(s)\|^p_{L^p(\Omega(h(s)))}ds\right)^{\frac{p-1}{p}}\int_0^{T-\tau}\|\nabla\omega\|_{L^p(B_M)}dt \\
&\le C(K,\alpha)\tau^{\frac{1}{p}}\|1+\nabla\boldsymbol{v}\|^{p-1}_{L^p(0,T;L^p(\Omega(h))}\|\nabla\omega\|_{L^1(0,T;L^p(B_M))} \le C(K,\alpha)\tau^{\frac{1}{p}}.
\end{aligned}
$$

The estimates of remaining terms on the right hand side can be obtained using the so-called Steklov averages analogously as in e.g., [9, Section 5] or [6, Section 8] and we leave them to the valued reader. The proof of the lemma is now completed. □

Due to the (4.14) it is also easy to obtain from (4.6) that

$$
\int_0^{T-\tau}\int_{B_M}|\chi^{(k)}_{t+\tau}\bar{\boldsymbol{v}}^{(k)}(t+\tau)-\chi^{(k)}_t\bar{\boldsymbol{v}}^{(k)}(t)|^2+\int_0^{T-\tau}\int_0^L|\eta^{(k)}_t(t+\tau)-\eta^{(k)}_t(t)|^2 \le C(\tau^{1/p}+\tau^{1/2}). \tag{4.18}
$$

This result implies that $\chi^{(k)}_t\bar{\boldsymbol{v}}^{(k)}(t)$, and consequently $\bar{\boldsymbol{v}}^{(k)}(t)$ is relatively compact in $L^2((0,T)\times B_M)$.

Consequently, the Riesz-Fréchet-Kolmogorov compactness argument [2, Theorem IV.26] based on (4.18) implies the relative compactness of $\partial_t\eta^{(k)}, \bar{\boldsymbol{v}}^{(k)}$ in $L^2(0,T;L^2(0,L))$, $L^2(0,T;L^2(B_M))$, respectively. Additionally, the standard interpolations give us the compactness of $\bar{\boldsymbol{v}}^{(k)}$ in $L^r((0,T)\times B_M)$, $1\le r<4$ and $\partial_t\eta^{(k)}$ in $L^s((0,T)\times(0,L))$, $1\le s<6$.

4.2. Continuity of the mapping $\mathcal{F}$

As already shown above $\eta^{(k)}$ converges strongly to some η in Y as $k\to\infty$. In this section we show by limiting process for $k\to\infty$ in (4.1) that for any convergent subsequence $\delta^{(k)}\in B_{\alpha,K}$, $\delta^{(k)}\to\delta$ in Y we have

$$
\mathcal{F}(\delta^{(k)}) = \eta^{(k)} \to \mathcal{F}(\delta) \text{ and that } \eta \equiv \mathcal{F}(\delta).
$$

First, we know that $\eta^{(k)}\to\eta$ in $H^1(0,T;L^2(0,L))$. Due to the boundedness of η from apriori estimate (4.2) and the imbeddings in one dimension we have even stronger result - the uniform convergence of $\partial_{y_1}\eta^{(k)}$ in $C([0,T]\times[0,L])$.

Indeed,

$$L^\infty(0,T;H^2(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L)) \hookrightarrow C^{0,1-\beta}(0,T;H^{2\beta}(0,L)) \tag{4.19}$$

for $0<\beta<1$. From the continuous imbedding of $H^{2\beta}(0,L)$ into $H^{2\beta-\epsilon}(0,L)$ and the Arzelá-Ascoli Lemma we conclude that a subsequence of $\eta^{(k)}$ converges strongly in $C([0,T];H^s(0,L))$, $0<s<2$. Since for $s>3/2$ we also have continuous imbedding $H^s(0,L)\hookrightarrow C^1[0,L]$, we can conclude, that $\eta^{(k)}\to\eta$ strongly in $C(0,T;C^1[0,L])$.

Let us summarize available convergences

$$\begin{aligned}
\boldsymbol{u}^{(k)} &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^p(0,T;W^{1,p}(D)),\\
\bar{\boldsymbol{v}}^{(k)} &\to \bar{\boldsymbol{v}} && \text{strongly in } L^r((0,T)\times B_M),\ 1\le r<4,\\
\boldsymbol{u}^{(k)} &\to \boldsymbol{u} && \text{strongly in } L^r((0,T)\times D),\ 1\le r<4,\\
\eta^{(k)} &\rightharpoonup \eta && \text{weakly in } H^1(0,T;H^2(0,L)),\\
\eta^{(k)} &\rightharpoonup^* \eta && \text{weakly* in } L^\infty(0,T;L^2(0,L))\\
\eta^{(k)} &\to \eta && \text{uniformly in } C(0,T;C^1[0,L]),\\
\partial_t\eta^{(k)} &\to \partial_t\eta && \text{strongly in } L^s((0,T)\times(0,L)),\ 1\le s<6.\\
\bar\partial_t(h\boldsymbol{u})^{(k)} &\rightharpoonup \chi && \text{weakly in } \begin{cases} L^{p'}(0,T;\boldsymbol{V}^*) \text{ for } 2<p<\infty\\ L^{p'}(0,T;\boldsymbol{V}^*)\oplus L^{4/3}((0,T)\times D) \text{ for } p=2.\end{cases}
\end{aligned} \tag{4.20}$$

The last statement of (4.20) follows from (4.2) and from the boundednes of the functional $\bar\partial_t(h\boldsymbol{u})^{(k)}$. Let us present the estimation of nonlinear terms on the right hand side. Indeed, from Lemma 2.5 it follows $\int_0^T((\boldsymbol{u}^k,\boldsymbol{\psi}))\le C(K,\alpha)\|\boldsymbol{\psi}\|_{L^p(0,T;W^{1,p}(D)}$. The non-linear convective term can be estimated using Lemma 2.6, the Hölder inequality and the imbedding of $W^{1,p}(D)$ into $L^{\frac{2p}{p-2}}(D)$ for $p>2$ as follows

$$\begin{aligned}\int_0^T B_{h^{(k)}}(\boldsymbol{u}^{(k)},\boldsymbol{u}^{(k)},\boldsymbol{\psi}) &\le C(K,\alpha)\int_0^T \|\boldsymbol{u}^{(k)}\|_{1,p}\|\boldsymbol{u}^{(k)}\|_2\|\boldsymbol{\psi}\|_{\frac{2p}{p-2}}\\
&\le C(K,\alpha)\|\boldsymbol{u}^{(k)}\|_{L^\infty(0,T;L^2(D))}\|\boldsymbol{u}^{(k)}\|_{L^p(0,T;W^{1,p}(D))}\|\boldsymbol{\psi}\|_{L^{p'}(0,T;W^{1,p}(D))},\end{aligned}$$

which is bounded due to (4.2) for all $\boldsymbol{\psi}\in L^p(0,T;\boldsymbol{V})$. Analogously the term $\int_0^T B_{h^{(k)}}(\boldsymbol{u}^{(k)},\boldsymbol{\psi},\boldsymbol{u}^{(k)})$ is bounded, which leads to [3]

$$\int_0^T b_{h^{(k)}}(\boldsymbol{u}^{(k)},\boldsymbol{u}^{(k)},\boldsymbol{\psi})\le C(K,\alpha)\|\boldsymbol{\psi}\|_{L^p(0,T;W^{1,p}(D))} \text{ for } p\in(2,\infty). \tag{4.21}$$

Further estimates of the remaining terms on the right hand side conclude the proof of $(4.20)_8$.

In what follows we have to verify, that $\mathcal{F}(\delta^{(k)}\to\mathcal{F}(\delta)$ and that the limit η from (4.20) is the weak solution asociated with δ, and thus $\mathcal{F}(\delta)=\eta$.

[3]For $p=2$ this estimate is valid for $\boldsymbol{\psi}\in L^p(0,T;\boldsymbol{V})\cap L^4((0,T)\times D)$, cf. [6].

4.2.1. Limiting process $k \to \infty$. Now let us consider (4.1) with $\boldsymbol{u}^{(k)}$ instead of $\boldsymbol{u}$, $h^{(k)}$ instead of h and $\sigma^{(k)} = \partial_t \eta^{(k)}$ instead of σ.

First of all we have to realize, that due to the solenoidal property, which depends on $h^{(k)}$, the test functions are also implicitly dependent on k. This fact presents a difficulty when we pass with $k \to \infty$. Nevertheless we can construct sufficiently smooth test functions $\tilde{\boldsymbol{\psi}}(y,t) = \tilde{\boldsymbol{\varphi}}(x,t)$, which are independent on k and divergence free in $\Omega(h)$ (i.e. $\mathrm{div}_h \tilde{\boldsymbol{\psi}} = 0$). They are also well defined on infinitely many approximate domains $\Omega(h^{(k)})$ and dense in the space of admissible test functions $L^p(0,T;X)$, cf. (3.5). Such a test functions $\tilde{\boldsymbol{\varphi}}$ can be constructed on $(0,T) \times B_M$ as algebraic sum, see [4, Remark 3]

$$\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_1,$$

where $\boldsymbol{\varphi}_0$ is a smooth function with compact support in $\Omega(h)$, $\mathrm{div}\boldsymbol{\varphi}_0 = 0$ on $\Omega(h)$ and $\boldsymbol{\varphi}_0$ is extended by 0 to $(0,T) \times B_M$. Further, having $\xi \in H^1(0,T;H_0^2(0,L))$ we define $\boldsymbol{\varphi}_1 \stackrel{\text{def}}{=} (0, \xi(x_1)/E)$ on $B_M \backslash B_\alpha$, $B_\alpha = (0,L) \times (0,\alpha) \in \mathbb{R}^2$, the constant E comes from (1.15). Note that $\mathrm{div}\boldsymbol{\varphi}_1 = 0$ on $B_M \backslash B_\alpha$. Moreover, $\boldsymbol{\varphi}_1$ such that $\int_{\partial B_\alpha} \boldsymbol{\varphi}_1 \cdot n = \int_0^\alpha \boldsymbol{\varphi}_1^1(L,x_2,t) - \boldsymbol{\varphi}_1^1(0,x_2,t)dx_2 + \int_0^L \frac{\xi}{E}(x_1,t)dx_1 = 0$ can be extended into B_α by a divergence-free extension, whereas remaining boundary conditions on Γ_{in}, Γ_{out}, Γ_c are preserved, see e.g., [7, p.144]. Note, that due to the uniform convergence of $\eta^{(k)}$ we have that $\sup \boldsymbol{\varphi}_0 \subset \Omega(h^{(N)})$ for sufficiently large N and the function $\boldsymbol{\varphi}_0$ is well defined on each $\Omega(h^{(N)})$ for such N. Moreover $\boldsymbol{\varphi}_1$ is defined on $\Omega(h^{(k)})$ for each k. For more details on this construction we refer a reader to [3, Section 7, pp. 35-36], compare [4].

Having $\tilde{\boldsymbol{\psi}}(y,t) = \tilde{\boldsymbol{\psi}}(x_1, \frac{x_2}{h(x_1,t)}, t) = \tilde{\boldsymbol{\varphi}}(x,t)$, $x \in \Omega(h)$, $y \in D$, let us construct the set of admissible test functions $\boldsymbol{\psi}^{(k)}$ by transformation of $\tilde{\boldsymbol{\varphi}}$ from $\Omega(h^{(k)})$ into D,

$$\boldsymbol{\psi}^{(k)}(y_1,y_2,t) := \tilde{\boldsymbol{\psi}}(x_1, \frac{x_2}{h^{(k)}(x_1,t)}, t) = \tilde{\boldsymbol{\varphi}}(x_1,x_2,t), \qquad x \in \Omega(h^{(k)}),\ y \in D. \tag{4.22}$$

The test functions (4.22) have the following property

$$\boldsymbol{\psi}^{(k)} : D \to \mathbb{R}^2; \qquad \mathrm{div}_{h^{(k)}} \boldsymbol{\psi}^{(k)} = 0, \quad E\boldsymbol{\psi}_2^{(k)}(y_1,1,t) = \xi(y_1,t), \quad \text{and}$$
$$\left.\begin{array}{l} \boldsymbol{\psi}^{(k)} \to \tilde{\boldsymbol{\psi}}, \\ \hat{e}_{h^{(k)}}(\boldsymbol{\psi}^{(k)}) \to \hat{e}(\tilde{\boldsymbol{\psi}}) \end{array}\right\} \quad \text{uniformly on } (0,T) \times D.$$

This property follows from the special construction of $\tilde{\boldsymbol{\varphi}}$, the property (4.24) below and the uniform convergence of $h^{(k)}$ and $\partial_{y_1} h^{(k)}$ that follows from (4.19). The test functions (4.22) satisfy the boundary conditions on S_{in}, S_{out}, S_c, cf. (2.9) as well.

Thus it is enough to consider test functions $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}}$, which are independent on k and smooth enough. The limiting process in the test functions follows afterwards using the uniform convergence $\boldsymbol{\psi}^{(k)}$ and $\hat{e}(\boldsymbol{\psi}^{(k)})$.

In the following lines we will present the limiting process for $k \to \infty$ in chosen nonlinear terms. Let us first consider the convective term and show

$$\int_0^T \Big(b_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)}, \boldsymbol{\psi}) - b_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})\Big)\, dt \to 0.$$

Recalling (2.15), the following terms appear in the above expression

$$\int_0^T B_h(\boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{\psi}) + B_{h^{(k)}}(\boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)}, \boldsymbol{\psi}) + B_{(h^{(k)}-h)}(\boldsymbol{u}, \boldsymbol{u}^{(k)}, \boldsymbol{\psi}) dt.$$

To show the convergence of above integrals, we restrict ourselves only to the terms containing $\partial y_1 h^{(k)}$, convergence of terms with $h^{(k)}$ is analogous. Let us consider

$$\int_0^T \int_D \left(\frac{\partial \boldsymbol{u}^{(k)}}{\partial y_2} - \frac{\partial \boldsymbol{u}}{\partial y_2}\right) \cdot \boldsymbol{\psi} u_1 \frac{\partial h}{\partial y_1} + \frac{\partial \boldsymbol{u}^{(k)}}{\partial y_2} \cdot \boldsymbol{\psi} \left(u_1^{(k)} - u_1\right) \frac{\partial h^{(k)}}{\partial y_1}$$
$$\frac{\partial \boldsymbol{u}^{(k)}}{\partial y_2} \cdot \boldsymbol{\psi} \left(\frac{\partial h^{(k)}}{\partial y_1} - \frac{\partial h}{\partial y_1}\right) u_1^{(k)} dy\, dt.$$

The convergence of the first term is obvious due to the weak convergence of $\boldsymbol{u}^{(k)}$ in $L^p(0, T; W^{1,p}(D))$. The strong convergence of $\boldsymbol{u}^{(k)}$ in $L^{p'}((0, T) \times (D))$ and the uniform convergence of $\partial_{y_1} h^{(k)}$ imply the convergence in the remaining two terms.

Now we denote $\hat{e}^{(k)} := \hat{e}_{h^{(k)}}$, $\hat{e} := \hat{e}_h$, cf. (2.11) and Remark 2.2. The limiting process in the viscous term will be realized as follows.

$$\begin{aligned}
&\int_0^T ((\boldsymbol{u}^{(k)}, \boldsymbol{\psi}))_{h^{(k)}} - ((\boldsymbol{u}, \boldsymbol{\psi}))_h dt \qquad (4.23)\\
&= \int_0^T \int_D h \left[\tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)}))\hat{e}^{(k)}_{ij}(\boldsymbol{\psi}) - \tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{\psi})\right]\\
&\qquad + \left[h^{(k)} - h\right] \tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)}))\hat{e}^{(k)}_{ij}(\boldsymbol{\psi})\, dy\, dt\\
&= \underbrace{\int_0^T \int_D h\, \tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)})) \left[\hat{e}^{(k)}_{ij}(\boldsymbol{\psi}) - \hat{e}_{ij}(\boldsymbol{\psi})\right] dy\, dt}_{\text{(I)}}\\
&+ \underbrace{\int_0^T \int_D h \left[\tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)})) - \tau_{ij}(\hat{e}(\boldsymbol{u}))\right] \hat{e}_{ij}(\boldsymbol{\psi})\, dy\, dt}_{\text{(II)}}\\
&+ \underbrace{\int_0^T \int_D \left[h^{(k)} - h\right] \tau_{ij}(\hat{e}^{(k)}(\boldsymbol{u}^{(k)}))\hat{e}^{(k)}_{ij}(\boldsymbol{\psi})\, dy\, dt}_{\text{(III)}}.
\end{aligned}$$

It is easy to see that the term (III) goes to zero. Using the fact that

$$\hat{e}_h(\boldsymbol{u}) = \nabla \boldsymbol{u} F(h) + (\nabla \boldsymbol{u} F(h))^T \in \mathbb{R}^{2\times 2}_{sym}; \quad F(h) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ \frac{-y_2}{h}\frac{\partial h}{\partial y_1} & \frac{1}{h} \end{bmatrix} \qquad (4.24)$$

and due to the uniform convergence of $h^{(k)}$ in $C(0,T;C^1[0,L])$ the convergence in all components of F is obvious and we obtain that (I) $\to 0$.

In order to show the convergence in the term (II), we will use the Minty-Trick. Let us denote for better readability $\xi^k := \hat{e}_{h^{(k)}}(\boldsymbol{u}^{(k)})$ $\xi := \hat{e}_h(\boldsymbol{u})$ and $\phi := \hat{e}(\boldsymbol{\psi})$. Define the operator $\mathcal{A}$, $\mathcal{A} : L^p((0,T)\times D) \to L^{p'}((0,T)\times D)$,

$$\langle \mathcal{A}(\xi^k), \phi \rangle := \int_0^T \int_D h\, \tau_{ij}(\xi^k)\phi\, dy\, dt.$$

Lemma 2.4 implies the monotonicity of operator $\mathcal{A}$, $\langle \mathcal{A}(\xi^k) - \mathcal{A}(\xi), \xi^k - \xi \rangle \geq 0$. Further, from assumptions (2.1) on $h^{(k)}$ and Lemma 2.5 we obtain for $\boldsymbol{u}, \boldsymbol{\psi} \in L^p(0,T;W^{1,p}(D))$ and for any k

$$\left|\langle A(\xi^k), \phi \rangle\right| \leq c(K,\alpha)\|\phi\|_{L^p(0,T;L^p(D))}.$$

Thus $\mathcal{A}$ is bounded in $L^{p'}((0,T)\times D)$ and consequently $\mathcal{A}(\xi^k) \rightharpoonup f$. Moreover, from the weak convergence of $\nabla \boldsymbol{u}^{(k)}$ and the uniform convergence of $h^{(k)}$ in $C(0,T;C^1[0,L])$ we obtain the weak convergence $\xi^k \rightharpoonup \xi$.

Now we prove that $\lim_{k\to\infty} \langle A(\xi^k), \xi^k \rangle = \langle f, \xi \rangle$, i.e., we show that $\lim_{k\to\infty} \langle A(\xi^k), \xi^k - \xi \rangle = 0$. To this end, we limit in the rest terms of the weak formulation (4.1) with $\boldsymbol{u}^{(k)}$, $h^{(k)}$, $\sigma^{(k)}$ instead of $\boldsymbol{u}$, h, σ using test functions $\boldsymbol{\psi} = \boldsymbol{u}^{(k)} - \boldsymbol{u}$. In what follows we present the limiting process in the nonlinear convetive term. Recalling (2.15) we can write

$$\int_0^T b_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)} - \boldsymbol{u})\, dt = \frac{1}{2}\int_0^T B_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)} - \boldsymbol{u}) - B_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)})\, dt. \quad (4.25)$$

We can estimate the first term on the right hand side using the Hölder and the interpolation inequality $\|\boldsymbol{\varphi}\|_4 \leq c\|\boldsymbol{\varphi}\|_{1,2}^{1/2}\|\boldsymbol{\varphi}\|_2^{1/2}$, cf. [9, Lemma 3.1]

$$\int_0^T B_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)} - \boldsymbol{u})dt \leq C(K,\alpha)\|\boldsymbol{u}^{(k)} - \boldsymbol{u}\|_{L^2(L^2)}\left(\|\boldsymbol{u}^{(k)}\|^{\frac{3}{2}}_{L^2(W^{1,2})} + \|\boldsymbol{u}^{(k)}\|_{L^2(W^{1,2})}\|\boldsymbol{u}\|^{\frac{1}{2}}_{L^2(W^{1,2})}\right),$$

here $L^2(W^{1,2}) := L^2(0,T;W^{1,2}(D))$. Thus, the strong convergence of $\boldsymbol{u}^{(k)}$ in $L^2((0,T)\times D)$ implies, that the first term on the right hand side of (4.25) tends to 0. Further, we can rewrite the second term as

$$B_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)}) = B_h(\boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}) + B_{(h^{(k)}-h)}(\boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}) + B_{h^{(k)}}(\boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)}) + B_{h^{(k)}}(\boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)} - \boldsymbol{u}).$$

Due to the weak convergence of $\nabla \boldsymbol{u}^{(k)}$, uniform convergence of $h^{(k)}$ and the strong convergence of $\boldsymbol{u}^{(k)}$, cf. (4.20) we obtain also for the second term $\int_0^T B_{h^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{u}^{(k)} - \boldsymbol{u}, \boldsymbol{u}^{(k)})dt \to 0$. This concludes the proof of convergence in the convective term (4.25). The limiting process in the remaining terms of (4.1) is obvious and we omit it here.

Consequently, we have obtained $\lim_{k\to\infty}\left\langle A(\xi^k),\xi^k\right\rangle = \langle f,\xi\rangle$ and the Minty-Trick argument implies that $f=\mathcal{A}(\xi)$, i.e.

$$\mathcal{A}(\xi^k) \rightharpoonup \mathcal{A}(\xi) \quad \text{and thus} \quad \left\langle \mathcal{A}(\xi^k),\phi\right\rangle \to \langle \mathcal{A}(\xi),\phi\rangle$$

for any $\phi \in L^p((0,T)\times D)$ as $k\to\infty$.

This concludes the limiting process in (4.1) and the Section 4.2. We have found out that $\mathcal{F}(\delta^{(k)}) \to \mathcal{F}(\delta)$ as $k\to\infty$ and that η is the weak solution of (4.1) associated with the limit δ, $(h = R_0+\delta)$, thus $\mathcal{F}(\delta)=\eta$.

Finally, using the continuity of the mapping $\mathcal{F}$, its relative compactness in Y and the property $\mathcal{F}(B_{\alpha,K})\subset B_{\alpha,K}$ we deduce from the Schauder fixed point theorem, that there exists at least one fixed point of the mapping $\mathcal{F}$ defined by the weak formulation (4.1), $\mathcal{F}(\eta)=\eta$. Thus, we obtain the existence of at least one weak solution (1.14) of the original unsteady fluid-structure interaction problem (1.1) – (1.13). The proof of the Theorem 1.2 is now completed. □

Remark on the global existence result. Let us point out that we have obtained the existence of weak solution until some time T^*. We remind that this time is obtained in order to achieve the fixed point of the mapping $\mathcal{F}$ and to avoid the contact of the elastic boundary $\Gamma_w(t)$ with the fixed boundary for given data P_{in}, P_{out}, P_w, R_0 and α, K. Similarly as in [4, Grandmont et al.], we can prolongate the solution in time and even obtain the global existence until the contact with the solid bottom.

Indeed, we can construct a non-decreasing sequence of times $\{T^* = T^*_1,\dots,T^*_{m-1},T^*_m,\dots\}$, such that for given α, K, $\alpha \le \min\{R_{min}, \frac{1}{R_{min}+R_{max}}\}$, starting from initial data in time T^*_{m-1}, we have the existence of weak solution for some time $T^*_{m-1}+T := T^*_m$. We distinguish between two situations. Either $\sup T^*_m = \infty$, which means, that the contact with the solid bottom never happens and we obtain global existence. Otherwise $\sup T^*_m := T^{**} < \infty$ for given α. In this case we can decrease α. If the time interval of the existence cannot be prolongated for chosen α, we have to decrease α again. This is repeated until α reaches 0. The later represents the contact with the solid boundary at some time $T^{**}+\bar{T}$, where $\bar{T}\ge 0$.

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Anna Hundertmark, Mária Lukáčová-Medviďová
Institute for Mathematics
Johannes Gutenberg University,
Staudingerweg 9, Mainz
Germany
e-mail: hundertm@uni-mainz.de, lukacova@mathematik.uni-mainz.de

Šárka Nečasová
Academy of Sciences of Czech Republic
Žitná 25, Praha
Czech Republic
e-mail: matus@math.cas.cz