Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method
Part II: A linear scheme

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Abstract

A linear stabilized Lagrange–Galerkin scheme for the Oseen-type Peterlin viscoelastic model is presented. Error estimates with the optimal convergence order are proved under a mild stability condition in two and three space dimensions. The scheme consists of the method of characteristics and Brezzi–Pitkäranta's stabilization method for the conforming linear elements, which lead to an efficient computation with a small number of degrees of freedom.

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1 Introduction

We study numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method. In our previous paper [17], Part I, we have presented a nonlinear scheme for the diffusive and the non-diffusive model. Here, in Part II, we present a linear scheme for the diffusive model and establish error estimates with the optimal convergence order.

In the daily life we encounter many biological, industrial or geological fluids that do not satisfy the Newtonian assumption, i.e., the linear dependence between the stress tensor and the deformation tensor. These fluids belong to the class of the non-Newtonian fluids. In order to describe such complex fluids the stress tensor is represented as a sum of the viscous (Newtonian) part and the extra stress due to the polymer contribution.

In the literature we can find several models that are employed to describe various aspects of complex viscoelastic fluids. One of the well-known viscoelastic models is the Oldroyd-B model, which is derived from the Hookean dumbbell model with a linear spring force law. The model is a system of equations for the velocity, the pressure and the extra stress tensor, cf., e.g., [27, 28].

Numerical schemes for the Oldroyd-B type models have been studied by many authors. For example, we can find a finite difference scheme based on the reformulation of the equation for the extra stress tensor by using the log-conformation representation in Fattal and Kupferman [11, 12], free energy dissipative Lagrange–Galerkin schemes with or without the log-conformation representation in Boyaval et al. [4], finite element schemes using the idea of the generalized Lie derivative in Lee and Xu [15] and Lee et al. [16], and further related numerical schemes and computations in [1,3,9,14,19,20,22,33] and references therein. To the best of our knowledge, however, there are no results on error estimates of numerical schemes for the Oldroyd-B model. As for the simplified Oldroyd-B model with no convection terms Picasso and Rappaz [26] and Bonito et al. [2] have given error estimates for stationary and non-stationary problems, respectively. The development of stable and convergent numerical methods for the Oldroyd-B type models, especially in the elasticity-dominated case, is still an active research area.

In this paper, we consider the so-called Peterlin viscoelastic model, which is derived from the dumbbell model with a nonlinear spring force law $F(R) = \gamma(|R|^2)R$ and the Peterlin approximation where $\gamma(|R|^2)$ is replaced by a function $\gamma(\operatorname{tr}\mathbf{C})$. Here $\mathbf{C}$ is the so-called conformation tensor and R is the vector connecting the beads. It is a system of the flow equations and an equation for the conformation tensor, cf. [27, 28]. The diffusive Peterlin viscoelastic model has been studied analytically in our recent paper by Lukáčová-Medviďová et al. [18], where the global existence of weak solutions and the uniqueness of regular solutions have been proved. For the details of the derivation of the diffusive Peterlin model we refer to [18, 21, 29, 30]. Let us mention that, even when the velocity field is given, the equation for the conformation tensor in the Peterlin model is still nonlinear, while the Oldroyd-B model is linear with respect to the extra stress tensor. Hence, we can say that the nonlinearity of the Peterlin model is stronger than that of the Oldroyd-B model. As a starting point of the numerical analysis of the Peterlin model, we consider the Oseen-type model, where the velocity of the material derivative is replaced by a known one, in order to concentrate on the treatment of the stronger nonlinearity.

Our aim is to develop a stabilized Lagrange–Galerkin method for the Peterlin viscoelastic model. It consists of the method of characteristics and Brezzi–Pitkäranta's stabilization method [7] for the conforming linear elements. The method of characteristics derives the robustness in convection-dominated flow problems, and the stabilization method reduces the number of degrees of freedom in computation. In our recent works by Notsu and Tabata [23–25] the stabilized Lagrange–Galerkin method has been applied successfully for the Oseen, Navier–Stokes and natural convection problems and optimal error estimates have been proved. We extend the numerical analysis of the stabilized Lagrange–Galerkin method to the Oseen-type Peterlin model. In this paper, a linear stabilized Lagrange–Galerkin scheme for the diffusive Peterlin model is presented and error estimates with the optimal convergence order are proved under a mild stability condition.

This paper is organized as follows. In Section 2 the mathematical model for the Peterlin viscoelastic fluid is described. In Section 3 a linear stabilized Lagrange–Galerkin scheme is presented. The main result on the convergence with optimal error estimates is stated in Section 4, and proved in Section 5.

2 The Oseen-type Peterlin viscoelastic model

The function spaces and the notation to be used throughout the paper are as follows. Let Ω be a bounded domain in $\mathbb{R}^d$ for $d = 2$ or 3, $\Gamma := \partial\Omega$ the boundary of Ω, and T a positive constant. For $m \in \mathbb{N}\cup\{0\}$ and $p \in [1,\infty]$ we use the Sobolev spaces $W^{m,p}(\Omega)$, $W_0^{1,\infty}(\Omega)$, $H^m(\Omega)\,(= W^{m,2}(\Omega))$, $H_0^1(\Omega)$ and $L_0^2(\Omega) := \{q \in L^2(\Omega); \int_\Omega q\, dx = 0\}$. Furthermore, we employ function spaces $H^m_{sym}(\Omega) := \{\mathbf{D} \in H^m(\Omega)^{d\times d};\ \mathbf{D} = \mathbf{D}^T\}$ and $C^m_{sym}(\bar\Omega) := C^m(\bar\Omega)^{d\times d} \cap H^m_{sym}(\Omega)$, where the superscript T stands for the transposition. For any normed space S with norm $\|\cdot\|_S$, we define function spaces $H^m(0,T;S)$ and $C([0,T];S)$ consisting of S-valued functions in $H^m(0,T)$ and $C([0,T])$, respectively. We use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. The dual pairing between S and the dual space S' is denoted by $\langle\cdot,\cdot\rangle$. The norms on $W^{m,p}(\Omega)$ and $H^m(\Omega)$ and their seminorms are simply denoted by $\|\cdot\|_{m,p}$ and $\|\cdot\|_m\,(= \|\cdot\|_{m,2})$ and by $|\cdot|_{m,p}$ and $|\cdot|_m\,(= |\cdot|_{m,2})$, respectively. The notations $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$, $\|\cdot\|_m$ and $|\cdot|_m$ are employed not only for scalar-valued functions but also for vector- and matrix-valued ones. We also denote the norm on $H^{-1}(\Omega)^2$ by $\|\cdot\|_{-1}$. For t_0 and $t_1 \in \mathbb{R}$ we introduce the function space,

$$Z^m(t_0,t_1) := \left\{\psi \in H^j(t_0,t_1;H^{m-j}(\Omega));\ j = 0,\dots,m,\ \|\psi\|_{Z^m(t_0,t_1)} < \infty\right\}$$

with the norm

$$\|\psi\|_{Z^m(t_0,t_1)} := \left\{\sum_{j=0}^m \|\psi\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega))}\right\}^{1/2},$$

and set $Z^m := Z^m(0,T)$. We often omit $[0,T]$, Ω, and the superscripts d and $d\times d$ for the vector and the matrix if there is no confusion, e.g., we shall write $C(L^\infty)$ in place of $C([0,T];L^\infty(\Omega)^{d\times d})$. For square matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{d\times d}$ we use the notation $\mathbf{A}:\mathbf{B} := \operatorname{tr}(\mathbf{A}\mathbf{B}^T) = \sum_{i,j} A_{ij}B_{ij}$.

We consider the system of equations describing the unsteady motion of an incompressible viscoelastic fluid,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \operatorname{div}\,(2\nu \mathrm{D}(\mathbf{u})) + \nabla p = \operatorname{div}\,[(\operatorname{tr}\mathbf{C})\mathbf{C}] + \mathbf{f} \qquad \text{in } \Omega\times(0,T), \tag{1a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega\times(0,T), \tag{1b}$$

$$\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t} - \varepsilon\Delta\mathbf{C} = (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T - (\operatorname{tr}\mathbf{C})^2\,\mathbf{C} + (\operatorname{tr}\mathbf{C})\mathbf{I} + \mathbf{F} \qquad \text{in } \Omega\times(0,T), \tag{1c}$$

$$
\mathbf{u} = \mathbf{0}, \quad \frac{\partial \mathbf{C}}{\partial \mathbf{n}} = \mathbf{0}, \qquad \text{on } \Gamma \times (0,T), \tag{1d}
$$

$$
\mathbf{u} = \mathbf{u}^0, \quad \mathbf{C} = \mathbf{C}^0, \qquad \text{in } \Omega, \text{ at } t = 0, \tag{1e}
$$

where $(\mathbf{u}, p, \mathbf{C}) : \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d\times d}_{sym}$ are the unknown velocity, pressure and conformation tensor, $\nu > 0$ is a fluid viscosity, $\varepsilon > 0$ is an elastic stress viscosity, $(\mathbf{f}, \mathbf{F}) : \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym}$ is a pair of given external forces, $\mathrm{D}(\mathbf{u}) := (1/2)[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$ is the symmetric part of the velocity gradient, $\mathbf{I}$ is the identity matrix, $\mathbf{n} : \Gamma \to \mathbb{R}^d$ is the outward unit normal, $(\mathbf{u}^0, \mathbf{C}^0) : \Omega \to \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym}$ is a pair of given initial functions, and $\mathrm{D}/\mathrm{D}t$ is the material derivative defined by

$$
\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla,
$$

where $\mathbf{w} : \Omega \times (0,T) \to \mathbb{R}^d$ is a given velocity.

Remark 1. *The model* (1) *is the* Oseen *approximation to the fully nonlinear problem, where the material derivative terms,*

$$
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}, \quad \frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{C}
$$

exist in place of $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$ *and* $\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}$ *in equations* (1a) *and* (1c)*. The existence of weak solutions and the uniqueness of regular solutions to the fully nonlinear model have been proved in* Lukáčová-Medvid'ová *et al. [18, Theorems 1 and 3]. The corresponding results are obtained under regularity condition on* $\mathbf{w}$ *to the model* (1)*, which is simpler than the fully nonlinear model. Numerical analysis of the fully nonlinear problem is a future work.*

We set an assumption for the given velocity $\mathbf{w}$.

Hypothesis 1. *The function* $\mathbf{w}$ *satisfies* $\mathbf{w} \in C([0,T]; W^{1,\infty}_0(\Omega)^d)$.

Let $V := H^1_0(\Omega)^d$, $Q := L^2_0(\Omega)$ and $W := H^1_{sym}(\Omega)$. We define the bilinear forms a_u on $V \times V$, b on $V \times Q$, $\mathcal{A}$ on $(V \times Q) \times (V \times Q)$ and a_c on $W \times W$ by

$$
a_u(\mathbf{u}, \mathbf{v}) := 2\big(\mathrm{D}(\mathbf{u}), \mathrm{D}(\mathbf{v})\big), \qquad b(\mathbf{u}, q) := -(\operatorname{div} \mathbf{u}, q), \qquad \mathcal{A}\big((\mathbf{u}, p), (\mathbf{v}, q)\big) := \nu a_u(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, q) + b(\mathbf{v}, p),
$$
$$
a_c(\mathbf{C}, \mathbf{D}) := (\nabla \mathbf{C}, \nabla \mathbf{D}),
$$

respectively. We present the weak formulation of the problem (1); find $(\mathbf{u}, p, \mathbf{C}) : (0,T) \to V \times Q \times W$ such that for $t \in (0,T)$

$$
\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(t), \mathbf{v}\right) + \mathcal{A}\big((\mathbf{u}, p)(t), (\mathbf{v}, q)\big) = -\left(\operatorname{tr} \mathbf{C}(t)\, \mathbf{C}(t), \nabla \mathbf{v}\right) + \left(\mathbf{f}(t), \mathbf{v}\right), \tag{2a}
$$

$$
\left(\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}(t), \mathbf{D}\right) + \varepsilon a_c\big(\mathbf{C}(t), \mathbf{D}\big) = 2\big((\nabla \mathbf{u}(t))\mathbf{C}(t), \mathbf{D}\big) - \big((\operatorname{tr} \mathbf{C}(t))^2 \mathbf{C}(t), \mathbf{D}\big) + \left(\operatorname{tr} \mathbf{C}(t) \mathbf{I}, \mathbf{D}\right) + \left(\mathbf{F}(t), \mathbf{D}\right), \tag{2b}
$$

$$
\forall (\mathbf{v}, q, \mathbf{D}) \in V \times Q \times W,
$$

with $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}^0, \mathbf{C}^0)$.

3 A linear stabilized Lagrange–Galerkin scheme

The aim of this section is to present a linear stabilized Lagrange–Galerkin scheme for the model (1).

Let Δt be a time increment, $N_T := \lfloor T/\Delta t \rfloor$ the total number of time steps and $t^n := n\Delta t$ for $n = 0, \ldots, N_T$. Let $\mathbf{g}$ be a function defined in $\Omega \times (0,T)$ and $\mathbf{g}^n := \mathbf{g}(\cdot, t^n)$. For the approximation of the material derivative we employ the first-order characteristics method,

$$
\frac{\mathrm{D}\mathbf{g}}{\mathrm{D}t}(x, t^n) = \frac{\mathbf{g}^n(x) - \big(\mathbf{g}^{n-1} \circ X_1^n\big)(x)}{\Delta t} + O(\Delta t), \tag{3}
$$

where $X_1^n : \Omega \to \mathbb{R}^d$ is a mapping defined by

$$
X_1^n(x) := x - \mathbf{w}^n(x)\Delta t,
$$

and the symbol $\circ$ means the composition of functions,

$$(\mathbf{g}^{n-1} \circ X_1^n)(x) := \mathbf{g}^{n-1}(X_1^n(x)).$$

For the details on deriving the approximation (3) of $\mathrm{D}\mathbf{g}/\mathrm{Dt}$, see, e.g., [24]. The point $X_1^n(x)$ is called the upwind point of x with respect to $\mathbf{w}^n$. The next proposition, which is a direct consequence of [31] and [32], presents sufficient conditions to ensure that all upwind points defined by X_1^n are in Ω and that its Jacobian $J^n := \det(\partial X_1^n/\partial x)$ is around 1.

Proposition 1. *Suppose Hypothesis 1 holds. Then, we have the following for* $n \in \{0, \dots, N_T\}$.
(i) Under the condition

$$\Delta t |\mathbf{w}|_{C(W^{1,\infty})} < 1, \tag{4}$$

$X_1^n : \Omega \to \Omega$ *is bijective.*
(ii) Furthermore, under the condition

$$\Delta t |\mathbf{w}|_{C(W^{1,\infty})} \le 1/4, \tag{5}$$

the estimate $1/2 \le J^n \le 3/2$ *holds.*

For the sake of simplicity we suppose that Ω is a polygonal domain. Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K\in\mathcal{T}_h} K)$, h_K the diameter of $K \in \mathcal{T}_h$ and $h := \max_{K\in\mathcal{T}_h} h_K$ the maximum element size. We consider a regular family of subdivisions $\{\mathcal{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption [8], i.e., there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \le \alpha_0, \quad \forall K \in \mathcal{T}_h, \ \forall h.$$

We define the discrete function spaces X_h, V_h, M_h, Q_h and W_h by

$$\begin{aligned}
X_h &:= \left\{\mathbf{v}_h \in C(\bar{\Omega})^d;\ \mathbf{v}_{h|K} \in P_1(K)^d, \forall K \in \mathcal{T}_h\right\}, & V_h &:= X_h \cap V,\\
M_h &:= \left\{q_h \in C(\bar{\Omega});\ q_{h|K} \in P_1(K), \forall K \in \mathcal{T}_h\right\}, & Q_h &:= M_h \cap Q,\\
W_h &:= \left\{\mathbf{D}_h \in C_{sym}(\bar{\Omega});\ \mathbf{D}_{h|K} \in P_1(K)^{d\times d}, \forall K \in \mathcal{T}_h\right\},
\end{aligned}$$

respectively, where $P_1(K)$ is the polynomial space of linear functions on $K \in \mathcal{T}_h$.

Let δ_0 be a small positive constant fixed arbitrarily and $(\cdot,\cdot)_K$ the $L^2(K)^d$ inner product. We define the bilinear forms $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ and $\mathcal{S}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\mathcal{A}_h\left((\mathbf{u}, p), (\mathbf{v}, q)\right) := \nu a_u\left(\mathbf{u}, \mathbf{v}\right) + b(\mathbf{u}, q) + b(\mathbf{v}, p) - \mathcal{S}_h(p, q), \qquad \mathcal{S}_h(p, q) := \delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K.$$

Let $(\mathbf{f}_h, \mathbf{F}_h) := (\{\mathbf{f}_h^n\}_{n=1}^{N_T}, \{\mathbf{F}_h^n\}_{n=1}^{N_T}) \subset L^2(\Omega)^d \times L^2(\Omega)^{d\times d}$ and $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ be given. A linear stabilized Lagrange–Galerkin scheme for (1) is to find $(\mathbf{u}_h, p_h, \mathbf{C}_h) := \{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ such that, for $n = 1, \dots, N_T$,

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\operatorname{tr} \mathbf{C}_h^n)\mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h\big) + (\mathbf{f}_h^n, \mathbf{v}_h), \tag{6a}$$

$$\begin{aligned}
\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon a_c(\mathbf{C}_h^n, \mathbf{D}_h) &= 2\big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1}, \mathbf{D}_h\big) - \big((\operatorname{tr} \mathbf{C}_h^{n-1})^2 \mathbf{C}_h^n, \mathbf{D}_h\big)\\
&\quad + \big((\operatorname{tr} \mathbf{C}_h^{n-1})\mathbf{I}, \mathbf{D}_h\big) + (\mathbf{F}_h^n, \mathbf{D}_h),
\end{aligned} \tag{6b}$$

$$\forall (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h.$$

4 The main result

In this section we state the main result on error estimates with the optimal convergence order of scheme (6), which is proved in the next section.

We use c to represent a generic positive constant independent of the discretization parameters h and Δt. We also use constants c_w and c_s independent of h and Δt but dependent on $\mathbf{w}$ and the solution $(\mathbf{u}, p, \mathbf{C})$ of (2), respectively, and c_s often depends on $\mathbf{w}$ additionally. Furthermore, c may depend on ν and ε but neither c_w nor c_s depends on them. The symbol "$\prime$ (prime)" is sometimes used in order to distinguish two constants, e.g., c_s and c_s', from each other. We use the following notation for the norms and seminorms, $\|\cdot\|_V = \|\cdot\|_{V_h} := \|\cdot\|_1$, $\|\cdot\|_Q = \|\cdot\|_{Q_h} := \|\cdot\|_0$,

$$\|(\mathbf{u}, \mathbf{C})\|_{Z^2(t_0,t_1)} := \left\{ \|\mathbf{u}\|_{Z^2(t_0,t_1)}^2 + \|\mathbf{C}\|_{Z^2(t_0,t_1)}^2 \right\}^{1/2},$$

$$\|\mathbf{u}\|_{\ell^\infty(X)} := \max_{n=0,\dots,N_T} \|\mathbf{u}^n\|_X, \quad \|\mathbf{u}\|_{\ell^2(X)} := \left\{ \Delta t \sum_{n=1}^{N_T} \|\mathbf{u}^n\|_X^2 \right\}^{1/2},$$

$$|p|_h := \left\{ \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla p)_K \right\}^{1/2}, \quad |p|_{\ell^2(|\cdot|_h)} := \left\{ \Delta t \sum_{n=1}^{N_T} |p^n|_h^2 \right\}^{1/2},$$

for $X = L^2(\Omega)$ or $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by $\overline{D}_{\Delta t} u^n := (u^n - u^{n-1})/\Delta t$.

The existence and uniqueness of the solution of scheme (6) are ensured by the following proposition, which is also proved in the next section.

Proposition 2 (existence and uniqueness)**.** *Suppose Hypothesis 1 holds. Then, for any h and Δt satisfying* (4) *there exists a unique solution $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ of scheme* (6)*.*

We state the main results after preparing a projection and a hypothesis.

Definition 1 (Stokes–Poisson projection)**.** *For $(\mathbf{u}, p, \mathbf{C}) \in V \times Q \times W$ we define the* Stokes–Poisson *projection $(\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\mathbf{C}}_h) \in V_h \times Q_h \times W_h$ of $(\mathbf{u}, p, \mathbf{C})$ by*

$$\mathcal{A}_h\left((\hat{\mathbf{u}}_h, \hat{p}_h), (\mathbf{v}_h, q_h)\right) + \varepsilon a_c(\hat{\mathbf{C}}_h, \mathbf{D}_h) + (\hat{\mathbf{C}}_h, \mathbf{D}_h) = \mathcal{A}\left((\mathbf{u}, p), (\mathbf{v}_h, q_h)\right) + \varepsilon a_c(\mathbf{C}, \mathbf{D}_h) + (\mathbf{C}, \mathbf{D}_h),$$
$$\forall (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h. \qquad (7)$$

The Stokes–Poisson projection derives an operator $\Pi_h^{\rm SP} : V \times Q \times W \to V_h \times Q_h \times W_h$ defined by $\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C}) := (\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\mathbf{C}}_h)$. We denote the i-th component of $\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})$ by $[\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_i$ for $i = 1, 2, 3$ and the pair of the first and third components $(\hat{\mathbf{u}}_h, \hat{\mathbf{C}}_h) = ([\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_1, [\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_3)$ by $[\Pi_h^{\rm SP}(\mathbf{u}, p, \mathbf{C})]_{1,3}$ simply.

Remark 2. *The identity* (7) *can be decoupled into the* Stokes *projection and the* Poisson *projection. For the simplicity of the notation we use* (7) *in the sequel. Since the Neumann boundary condition* (1d) *is imposed on $\mathbf{C}$, we use the* Poisson *projection corresponding to the operator $-\varepsilon\Delta + I$ for the unique solvability.*

Hypothesis 2. *The solution $(\mathbf{u}, p, \mathbf{C})$ of* (2) *satisfies $\mathbf{u} \in Z^2(0,T)^d \cap H^1(0,T; V \cap H^2(\Omega)^d) \cap C([0,T]; W^{1,\infty}(\Omega)^d)$, $p \in H^1(0,T; Q \cap H^1(\Omega))$ and $\mathbf{C} \in Z^2(0,T)^{d\times d} \cap H^1(0,T; W \cap H^2(\Omega)^{d\times d})$.*

We now impose the conditions

$$(\mathbf{u}_h^0, \mathbf{C}_h^0) = [\Pi_h^{\rm SP}(\mathbf{u}^0, 0, \mathbf{C}^0)]_{1,3}, \quad (\mathbf{f}_h, \mathbf{F}_h) = (\mathbf{f}, \mathbf{F}). \qquad (8)$$

Theorem 1 (error estimates)**.** *Suppose Hypotheses 1 and 2 hold. Then, there exist positive constants h_0, c_0 and $c_\dagger$ such that, for any pair $(h, \Delta t)$ satisfying*

$$h \in (0, h_0], \quad \Delta t \le c_0 \times \begin{cases} (1 + |\log h|)^{-1/2} & (d = 2), \\ h^{1/2} & (d = 3), \end{cases} \qquad (9)$$

the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (6) *with* (8) *is estimated as follows.*

$$\|\mathbf{C}_h\|_{\ell^\infty(L^\infty)} \le \|\mathbf{C}\|_{C(L^\infty)} + 1, \qquad (10)$$

$$\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)}, \quad \|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^1)}, \quad |p_h - p|_{\ell^2(|\cdot|_h)}, \quad \|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(H^1)}, \quad \left\| \overline{D}_{\Delta t}\mathbf{C}_h - \frac{\partial \mathbf{C}}{\partial t} \right\|_{\ell^2(L^2)} \le c_\dagger(\Delta t + h). \qquad (11)$$

5 Proofs

In what follows we prove Proposition 2 and Theorem 1.

5.1 Preliminaries

Let us list lemmas employed directly in the proofs below. In the lemmas, α_i, $i = 1, \ldots, 4$, are numerical constants, which are independent of h, Δt, ν and ε but may be dependent on Ω.

Lemma 1 ([10]). *Let Ω be a bounded domain with a* Lipschitz-*continuous boundary. Then, the following inequalities hold.*

$$\|\mathrm{D}(\mathbf{v})\|_0 \le \|\mathbf{v}\|_1 \le \alpha_1 \|\mathrm{D}(\mathbf{v})\|_0, \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d.$$

Let $\Pi_h : C(\bar{\Omega}) \to M_h$ be the Lagrange interpolation operator. The operators defined on $C(\bar{\Omega})^d$ and $C(\bar{\Omega})^{d\times d}$ are also denoted by the same symbol Π_h. We introduce the function

$$D(h) := \begin{cases} (1 + |\log h|)^{1/2} & (d = 2), \\ h^{-1/2} & (d = 3), \end{cases} \tag{12}$$

which is used in the sequel.

Lemma 2 ([5, 8]). *The following inequalities hold.*

$$\begin{aligned} \|\Pi_h \mathbf{D}\|_{0,\infty} &\le \|\mathbf{D}\|_{0,\infty}\,, && \forall \mathbf{D} \in C(\bar{\Omega})^{d\times d}, \\ \|\Pi_h \mathbf{D} - \mathbf{D}\|_1 &\le \alpha_{20} h \,\|\mathbf{D}\|_2\,, && \forall \mathbf{D} \in H^2(\Omega)^{d\times d}, \\ \|\mathbf{D}_h\|_{0,\infty} &\le \alpha_{21} D(h)\, \|\mathbf{D}_h\|_1\,, && \forall \mathbf{D}_h \in W_h. \end{aligned}$$

The next lemma is obtained by combining the error estimates for the Stokes and the Poisson problems, see, e.g., [6, 8, 13] for the proof.

Lemma 3. *Assume $(\mathbf{u}, p, \mathbf{C}) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega)) \times (W \cap H^2(\Omega)^{d\times d})$. Let $(\hat{\mathbf{u}}_h, \hat{p}_h, \hat{\mathbf{C}}_h) \in V_h \times Q_h \times W_h$ be the* Stokes-Poisson *projection of $(\mathbf{u}, p, \mathbf{C})$ by (7). Then, the following inequalities hold.*

$$\|\hat{\mathbf{u}}_h - \mathbf{u}\|_1\,,\ \|\hat{p}_h - p\|_0\,,\ |\hat{p}_h - p|_h \le \alpha_{31} h\, \|(\mathbf{u}, p)\|_{H^2\times H^1}\,, \qquad \|\hat{\mathbf{C}}_h - \mathbf{C}\|_1 \le \alpha_{32} h \|\mathbf{C}\|_2.$$

Lemma 4 ([24, 31]). *Under Hypothesis 1 and the condition (5) the following inequalities hold for any $n \in \{0, \ldots, N_T\}$.*

$$\begin{aligned} \|\mathbf{g} \circ X_1^n\|_0 &\le (1 + \alpha_{40} |\mathbf{w}^n|_{1,\infty} \Delta t)\, \|\mathbf{g}\|_0\,, && \forall \mathbf{g} \in L^2(\Omega)^s, \\ \|\mathbf{g} - \mathbf{g} \circ X_1^n\|_0 &\le \alpha_{41} \|\mathbf{w}^n\|_{0,\infty} \Delta t\, \|\mathbf{g}\|_1\,, && \forall \mathbf{g} \in H^1(\Omega)^s, \end{aligned}$$

where $s = d$ or $d \times d$.

Proof. We prove only the former estimate, since the latter is a direct consequence of [24, Lemma 6]. Let $n \in \{0, \ldots, N_T\}$ be fixed arbitrarily. By changing the variable from x to $y := X_1^n(x)$, we have

$$\|\mathbf{g} \circ X_1^n\|_0^2 = \int_\Omega \mathbf{g}\left(X_1^n(x)\right)^2 dx = \int_\Omega \mathbf{g}(y)^2 \frac{1}{J^n}\, dy \le (1 + c|\mathbf{w}^n|_{1,\infty} \Delta t)^2\, \|\mathbf{g}\|_0^2,$$

where J^n is the Jacobian $\det(\partial y/\partial x)$. Here we have used the estimate,

$$\frac{1}{J^n} \le \frac{1}{1 - |1 - J^n|} \le 1 + 2|1 - J^n| \le 1 + 2c|\mathbf{w}^n|_{1,\infty} \Delta t \le (1 + c|\mathbf{w}^n|_{1,\infty} \Delta t)^2,$$

which is derived from Proposition 1-(ii) and $1/(1 - s) \le 1 + 2s$ ($s \in [0, 1/2]$). Thus we obtain the result by setting $\alpha_{40} = c$. $\square$

5.2 Proof of Proposition 2

For each time step n scheme (6) can be rewritten as

$$\Big(\frac{\mathbf{u}_h^n}{\Delta t},\mathbf{v}_h\Big)+\nu a_u(\mathbf{u}_h^n,\mathbf{v}_h)+b(\mathbf{v}_h,p_h^n)+((\mathrm{tr}\,\mathbf{C}_h^n)\mathbf{C}_h^{n-1},\nabla\mathbf{v}_h)=(\mathbf{g}_h^n,\mathbf{v}_h),\qquad\forall\mathbf{v}_h\in V_h,\tag{13a}$$

$$b(\mathbf{u}_h^n,q_h)-\mathcal{S}_h(p_h^n,q_h)=0,\qquad\forall q_h\in Q_h,\tag{13b}$$

$$\Big(\frac{\mathbf{C}_h^n}{\Delta t},\mathbf{D}_h\Big)+\varepsilon a_c(\mathbf{C}_h^n,\mathbf{D}_h)-2\left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1},\mathbf{D}_h\right)+\left((\mathrm{tr}\,\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n,\mathbf{D}_h\right)=(\mathbf{G}_h^n,\mathbf{D}_h),\qquad\forall\mathbf{D}_h\in W_h,\tag{13c}$$

where $\mathbf{g}_h^n:=(1/\Delta t)(\mathbf{u}_h^{n-1}\circ X_1^n)+\mathbf{f}_h^n$ and $\mathbf{G}_h^n:=(1/\Delta t)(\mathbf{C}_h^{n-1}\circ X_1^n)+(\mathrm{tr}\,\mathbf{C}_h^{n-1})\mathbf{I}+\mathbf{F}_h^n$. Selecting specific bases of V_h, Q_h and W_h and expanding $\mathbf{u}_h^n$, p_h^n and $\mathbf{C}_h^n$ in terms of the associated basis functions, we can derive the system of linear equations from (13). The result, i.e., existence and uniqueness, is equivalent to the invertibility of the coefficient matrix of the system, which is obtained by proving $(\mathbf{u}_h^n,p_h^n,\mathbf{C}_h^n)=(\mathbf{0},0,\mathbf{0})$ below when $(\mathbf{g}_h^n,\mathbf{G}_h^n)=(\mathbf{0},\mathbf{0})$. Substituting $(\mathbf{u}_h^n,-p_h^n,\frac{1}{2}(\mathrm{tr}\,\mathbf{C}_h^n)\mathbf{I})$ into $(\mathbf{v}_h,q_h,\mathbf{D}_h)$ in (13) and adding (13b) to (13a), we have

$$\frac{1}{\Delta t}\|\mathbf{u}_h^n\|_0^2+2\nu\|\mathrm{D}(\mathbf{u}_h^n)\|_0^2+\delta_0|p_h^n|_h^2+\left((\mathrm{tr}\,\mathbf{C}_h^n)\mathbf{C}_h^{n-1},\nabla\mathbf{u}_h^n\right)=0,\tag{14a}$$

$$\frac{1}{2\Delta t}\|\mathrm{tr}\,\mathbf{C}_h^n\|_0^2+\frac{\varepsilon}{2}\|\nabla\mathrm{tr}\,\mathbf{C}_h^n\|_0^2-\left(\mathrm{tr}[(\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1}],\mathrm{tr}\,\mathbf{C}_h^n\right)+\frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathrm{tr}\,\mathbf{C}_h^n\right\|_0^2=0.\tag{14b}$$

By the identity

$$\left((\mathrm{tr}\,\mathbf{C}_h^n)\mathbf{C}_h^{n-1},\nabla\mathbf{u}_h^n\right)-\left(\mathrm{tr}[(\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1}],\mathrm{tr}\,\mathbf{C}_h^n\right)=0,$$

the sum of (14a) and (14b) yields

$$\frac{1}{\Delta t}\|\mathbf{u}_h^n\|_0^2+2\nu\|\mathrm{D}(\mathbf{u}_h^n)\|_0^2+\delta_0|p_h^n|_h^2+\frac{1}{2\Delta t}\|\mathrm{tr}\,\mathbf{C}_h^n\|_0^2+\frac{\varepsilon}{2}\|\nabla\mathrm{tr}\,\mathbf{C}_h^n\|_0^2+\frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathrm{tr}\,\mathbf{C}_h^n\right\|_0^2=0.$$

Hence, we have $(\mathbf{u}_h^n,p_h^n)=(\mathbf{0},0)$. Substituting $\mathbf{C}_h^n$ into $\mathbf{D}_h$ in (13c) and noting that $\mathbf{u}_h^n=\mathbf{0}$, we obtain

$$\frac{1}{\Delta t}\|\mathbf{C}_h^n\|_0^2+\varepsilon\|\nabla\mathbf{C}_h^n\|_0^2+\left\|(\mathrm{tr}\,\mathbf{C}_h^{n-1})\mathbf{C}_h^n\right\|_0^2=0,$$

which implies $\mathbf{C}_h^n=0$. Thus, we get $(\mathbf{u}_h^n,p_h^n,\mathbf{C}_h^n)=(\mathbf{0},0,\mathbf{0})$, which completes the proof. □

5.3 An estimate at each time step

In this subsection we present a proposition which is employed in the proof of Theorem 1.

Let $(\hat{\mathbf{u}}_h,\hat{p}_h,\hat{\mathbf{C}}_h)(t):=\Pi_h^{\mathrm{SP}}(\mathbf{u},p,\mathbf{C})(t)\in V_h\times Q_h\times W_h$ for $t\in[0,T]$ and let

$$\mathbf{e}_h^n:=\mathbf{u}_h^n-\hat{\mathbf{u}}_h^n,\qquad\epsilon_h^n:=p_h^n-\hat{p}_h^n,\qquad\mathbf{E}_h^n:=\mathbf{C}_h^n-\hat{\mathbf{C}}_h^n,\qquad\boldsymbol{\eta}(t):=(\mathbf{u}-\hat{\mathbf{u}}_h)(t),\qquad\boldsymbol{\Xi}(t):=(\mathbf{C}-\hat{\mathbf{C}}_h)(t).$$

Then, from (6), (7) and (2), we have for $n\geq 1$

$$\left(\frac{\mathbf{e}_h^n-\mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right)+\mathcal{A}_h\big((\mathbf{e}_h^n,\epsilon_h^n),(\mathbf{v}_h,q_h)\big)=\langle\mathbf{r}_h^n,\mathbf{v}_h\rangle,\qquad\forall(\mathbf{v}_h,q_h)\in V_h\times Q_h,\tag{15a}$$

$$\left(\frac{\mathbf{E}_h^n-\mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right)+\varepsilon a_c(\mathbf{E}_h^n,\mathbf{D}_h)=\langle\mathbf{R}_h^n,\mathbf{D}_h\rangle,\qquad\forall\mathbf{D}_h\in W_h,\tag{15b}$$

where

$$\mathbf{r}_h^n:=\sum_{i=1}^{4}\mathbf{r}_{hi}^n\in V_h',\qquad\mathbf{R}_h^n:=\sum_{i=1}^{11}\mathbf{R}_{hi}^n\in W_h',\tag{16}$$

$$\langle\mathbf{r}_{h1}^n,\mathbf{v}_h\rangle:=\left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t}-\frac{\mathbf{u}^n-\mathbf{u}^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right),$$

$$\langle\mathbf{r}_{h2}^n,\mathbf{v}_h\rangle:=\frac{1}{\Delta t}\left(\boldsymbol{\eta}^n-\boldsymbol{\eta}^{n-1}\circ X_1^n,\mathbf{v}_h\right),$$

$$
\begin{aligned}
\langle \mathbf{r}_{h3}^n, \mathbf{v}_h \rangle &:= \big((\operatorname{tr} \mathbf{C}^n)(\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1}), \nabla \mathbf{v}_h \big), \\
\langle \mathbf{r}_{h4}^n, \mathbf{v}_h \rangle &:= \big([\operatorname{tr} (\boldsymbol{\Xi}^n - \mathbf{E}_h^n)] \mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h \big), \\
\langle \mathbf{R}_{h1}^n, \mathbf{D}_h \rangle &:= \left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h2}^n, \mathbf{D}_h \rangle &:= \frac{1}{\Delta t} \left(\boldsymbol{\Xi}^n - \boldsymbol{\Xi}^{n-1} \circ X_1^n, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h3}^n, \mathbf{D}_h \rangle &:= -(\boldsymbol{\Xi}^n, \mathbf{D}_h), \\
\langle \mathbf{R}_{h4}^n, \mathbf{D}_h \rangle &:= 2 \left((\nabla \mathbf{e}_h^n) \mathbf{C}_h^{n-1}, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h5}^n, \mathbf{D}_h \rangle &:= -2 \left((\nabla \boldsymbol{\eta}^n) \mathbf{C}_h^{n-1}, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h6}^n, \mathbf{D}_h \rangle &:= -2 \left((\nabla \mathbf{u}^n)(\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1}), \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h7}^n, \mathbf{D}_h \rangle &:= \left((\operatorname{tr} \mathbf{C}_h^{n-1})^2 (\boldsymbol{\Xi}^n - \mathbf{E}_h^n), \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h8}^n, \mathbf{D}_h \rangle &:= -\big([\operatorname{tr} (\mathbf{C}_h^{n-1} + \hat{\mathbf{C}}_h^{n-1})](\operatorname{tr} \mathbf{E}_h^{n-1}) \mathbf{C}^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h9}^n, \mathbf{D}_h \rangle &:= \big([\operatorname{tr} (\mathbf{C}^{n-1} + \hat{\mathbf{C}}_h^{n-1})](\operatorname{tr} \boldsymbol{\Xi}^{n-1}) \mathbf{C}^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h10}^n, \mathbf{D}_h \rangle &:= \left([\operatorname{tr} (\mathbf{C}^n + \mathbf{C}^{n-1})][\operatorname{tr} (\mathbf{C}^n - \mathbf{C}^{n-1})] \mathbf{C}^n, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h11}^n, \mathbf{D}_h \rangle &:= -\left([\operatorname{tr} (\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1})] \mathbf{I}, \mathbf{D}_h \right).
\end{aligned}
$$

We note that

$$
(\mathbf{e}_h^0, \mathbf{E}_h^0) = (\mathbf{u}_h^0, \mathbf{C}_h^0) - (\hat{\mathbf{u}}_h^0, \hat{\mathbf{C}}_h^0) = [\Pi_h^{\mathrm{SP}}(0, -p^0, 0)]_{1,3}. \tag{17}
$$

In the following we use the constants α_i defined in Lemma i, $i = 1, \ldots, 4$, and the notation $\mathbb{H}^2 := H^2(\Omega)^2 \times H^1(\Omega) \times H^2(\Omega)^{2\times 2}$.

Proposition 3. *Suppose that Hypotheses 1 and 2 hold and assume* (5). *Let* M_0 *be a positive constant independent of* h *and* Δt. *Let* $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ *be the solution of scheme* (6) *with* (8). *Suppose that for an* $n \in \{1, \ldots, N_T\}$

$$
\|\mathbf{C}_h^{n-1}\|_{0,\infty} \le M_0. \tag{18}
$$

Then, there exist positive constants c_1 *and* c_2, *dependent on* M_0 *but independent of* h *and* Δt, *such that*

$$
\begin{aligned}
&\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2\Big) + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\gamma_0}{2\varepsilon}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2 \\
&\le c_1\Big(\frac{1}{2}\|\mathbf{e}_h^{n-1}\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^{n-1}\|_1^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2\Big) \\
&\quad + c_2\Big[\Delta t\|(\mathbf{u}, \mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big(\frac{1}{\Delta t}\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(t^{n-1},t^n;\mathbb{H}^2)}^2 + 1\Big)\Big],
\end{aligned} \tag{19}
$$

where $\gamma_0 := \nu\varepsilon/\{32\alpha_1^2(\varepsilon+1)M_0^2\}$.

For the proof we use the next lemma, which is proved in Appendix.

Lemma 5. *Suppose Hypotheses 1 and 2 hold. Let* $n \in \{1, \ldots, N_T\}$ *be any fixed number. Then, under the condition* (5) *it holds that*

$$
\|\mathbf{r}_{h1}^n\|_0 \le c_w\sqrt{\Delta t}\|\mathbf{u}\|_{Z^2(t^{n-1},t^n)}, \tag{20a}
$$

$$
\|\mathbf{r}_{h2}^n\|_0 \le \frac{c_w h}{\sqrt{\Delta t}}\|(\mathbf{u}, p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}, \tag{20b}
$$

$$
\|\mathbf{r}_{h3}^n\|_{-1} \le c_s\big(\|\mathbf{E}_h^{n-1}\|_0 + \sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)} + h\big), \tag{20c}
$$

$$
\|\mathbf{r}_{h4}^n\|_{-1} \le c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}\left(\|\mathbf{E}_h^n\|_0 + h\right), \tag{20d}
$$

$$
\|\mathbf{R}_{h1}^n\|_0 \le c_w\sqrt{\Delta t}\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}, \tag{20e}
$$

$$
\|\mathbf{R}_{h2}^n\|_0 \le \frac{c_w h}{\sqrt{\Delta t}}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)}, \tag{20f}
$$

$$
\|\mathbf{R}_{h3}^n\|_0 \le c_s h, \tag{20g}
$$

$$\|\mathbf{R}_{h4}^n\|_0 \le 4\|\mathbf{C}_h^{n-1}\|_{0,\infty}\|\mathbf{e}_h^n\|_1, \tag{20h}$$
$$\|\mathbf{R}_{h5}^n\|_0 \le c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}h, \tag{20i}$$
$$\|\mathbf{R}_{h6}^n\|_0 \le c_s\big(\|\mathbf{E}_h^{n-1}\|_0 + \sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)} + h\big), \tag{20j}$$
$$\|\mathbf{R}_{h7}^n\|_0 \le c_s\|\mathbf{C}_h^{n-1}\|_{0,\infty}^2(\|\mathbf{E}_h^n\|_0 + h), \tag{20k}$$
$$\|\mathbf{R}_{h8}^n\|_0 \le c_s(\|\mathbf{C}_h^{n-1}\|_{0,\infty} + 1)\|\mathbf{E}_h^{n-1}\|_0, \tag{20l}$$
$$\|\mathbf{R}_{h9}^n\|_0 \le c_s h, \tag{20m}$$
$$\|\mathbf{R}_{h10}^n\|_0 \le c_s\sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)}, \tag{20n}$$
$$\|\mathbf{R}_{h11}^n\|_0 \le c_s(\|\mathbf{E}_h^{n-1}\|_0 + \sqrt{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;L^2)} + h). \tag{20o}$$

Proof of Proposition 3. Substituting $(\mathbf{e}_h^n, -\epsilon_h^n)$ into $(\mathbf{v}_h, q_h)$ in (15a) and noting that

$$\begin{aligned}
\Big(\frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{e}_h^n\Big) &\ge \frac{1}{2\Delta t}\big(\|\mathbf{e}_h^n\|_0^2 - \|\mathbf{e}_h^{n-1}\circ X_1^n\|_0^2\big) \ge \frac{1}{2\Delta t}\Big[\|\mathbf{e}_h^n\|_0^2 - (1+\alpha_{40}|\mathbf{w}^n|_{1,\infty}\Delta t)^2\|\mathbf{e}_h^{n-1}\|_0^2\Big] \\
&\ge \overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2\Big) - c_w\|\mathbf{e}_h^{n-1}\|_0^2, \\
\mathcal{A}_h\big((\mathbf{e}_h^n,\epsilon_h^n),(\mathbf{e}_h^n,-\epsilon_h^n)\big) &\ge \frac{2\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|p_h^n|_h^2, \\
\langle \mathbf{r}_h^n, \mathbf{e}_h^n\rangle &\le \|\mathbf{r}_h^n\|_{-1}\|\mathbf{e}_h^n\|_1 \le \frac{\alpha_1^2}{4\nu}\|\mathbf{r}_h^n\|_{-1}^2 + \frac{\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2,
\end{aligned}$$

we have

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2\Big) + \frac{\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 \le \frac{\alpha_1^2}{4\nu}\|\mathbf{r}_h^n\|_{-1}^2 + c_w\|\mathbf{e}_h^{n-1}\|_0^2. \tag{21}$$

Similarly, substituting $\mathbf{E}_h^n$ and $\overline{D}_{\Delta t}\mathbf{E}_h^n$ into $\mathbf{D}_h$ in (15b) and noting that

$$\begin{aligned}
\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{E}_h^n\Big) &\ge \overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{E}_h^n\|_0^2\Big) - c_w\|\mathbf{E}_h^{n-1}\|_0^2, \\
\varepsilon a_c(\mathbf{E}_h^n, \mathbf{E}_h^n) &= \varepsilon|\mathbf{E}_h^n|_1^2 \ge 0, \\
\langle \mathbf{R}_h^n, \mathbf{E}_h^n\rangle &\le \|\mathbf{R}_h^n\|_0\|\mathbf{E}_h^n\|_0 \le \|\mathbf{R}_h^n\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2, \\
\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t}, \overline{D}_{\Delta t}\mathbf{E}_h^n\Big) &= \Big(\overline{D}_{\Delta t}\mathbf{E}_h^n + \frac{\mathbf{E}_h^{n-1} - \mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t}, \overline{D}_{\Delta t}\mathbf{E}_h^n\Big) \\
&\ge \|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2 - \alpha_{41}\|\mathbf{w}^n\|_{0,\infty}\|\mathbf{E}_h^{n-1}\|_1\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0, \\
&\ge \|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2 - c_w\|\mathbf{E}_h^{n-1}\|_1^2 - \frac{1}{4}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2, \\
&= \frac{3}{4}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2 - c_w\|\mathbf{E}_h^{n-1}\|_1^2, \\
\varepsilon a_c\big(\mathbf{E}_h^n, \overline{D}_{\Delta t}\mathbf{E}_h^n\big) &\ge \overline{D}_{\Delta t}\Big(\frac{\varepsilon}{2}|\mathbf{E}_h^n|_1^2\Big), \\
\big\langle \mathbf{R}_h^n, \overline{D}_{\Delta t}\mathbf{E}_h^n\big\rangle &\le \|\mathbf{R}_h^n\|_0\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0 \le \|\mathbf{R}_h^n\|_0^2 + \frac{1}{4}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2,
\end{aligned}$$

we have the following two inequalities,

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{E}_h^n\|_0^2\Big) \le \|\mathbf{R}_h^n\|_0^2 + c_w(\|\mathbf{E}_h^n\|_0^2 + \|\mathbf{E}_h^{n-1}\|_0^2), \tag{22a}$$
$$\overline{D}_{\Delta t}\Big(\frac{\varepsilon}{2}|\mathbf{E}_h^n|_1^2\Big) + \frac{1}{2}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2 \le \|\mathbf{R}_h^n\|_0^2 + c_w\|\mathbf{E}_h^{n-1}\|_1^2. \tag{22b}$$

Lemma 5, (16) and (18) imply that

$$\|\mathbf{r}_h^n\|_{-1}^2 \le c_s\big(M_0^2\|\mathbf{E}_h^n\|_0^2 + \|\mathbf{E}_h^{n-1}\|_0^2\big) + c_s'\Big[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big(\frac{1}{\Delta t}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + M_0^2 + 1\Big)\Big], \tag{23a}$$

$$\|\mathbf{R}_h^n\|_0^2 \le c_s\Big[M_0^4\|\mathbf{E}_h^n\|_0^2 + (M_0^2+1)\|\mathbf{E}_h^{n-1}\|_0^2\Big] + c_s'\Big[\Delta t\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big(\frac{1}{\Delta t}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)}^2 + M_0^4 + M_0^2 + 1\Big)\Big]$$
$$+ 16M_0^2\|\mathbf{e}_h^n\|_1^2. \tag{23b}$$

Multiplying (22a) by γ_0 and (22b) by γ_0/ε, adding them to (21) and using (23) and $16M_0^2\gamma_0(\varepsilon+1)/\varepsilon = \nu/(2\alpha_1^2)$, we get

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2\Big) + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\gamma_0}{2\varepsilon}\|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0^2 \le p_1(M_0)\Big(\frac{1}{2}\|\mathbf{e}_h^{n-1}\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^{n-1}\|_1^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2\Big)$$
$$+p_2(M_0)\Big[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big(\frac{1}{\Delta t}\|(\mathbf{u},p,\mathbf{C})\|_{H^1(t^{n-1},t^n;\mathbb{H}^2)}^2 + 1\Big)\Big],$$

where $p_i(\xi)$, $i=1,2$, are polynomials in ξ with non-negative coefficients independent of h and Δt. By taking $c_i = p_i(M_0)$, $i=1,2$, we finally obtain (19). $\square$

5.4 Proof of Theorem 1

We prove Theorem 1 through three steps, where the function $D(h)$ defined in (12) is often used.

Step 1 (Setting c_0 and h_0): From (8) and (17) we have

$$\|\mathbf{e}_h^0\|_0 \le \|\mathbf{u}_h^0 - \mathbf{u}^0\|_1 + \|\mathbf{u}^0 - \hat{\mathbf{u}}_h^0\|_1 \le 2\alpha_{31}h\|(u,p)^0\|_{H^2\times H^1} = \sqrt{2}c_I h \tag{24}$$

for $c_I := \sqrt{2}\alpha_{31}\|(u,p)^0\|_{H^2\times H^1}$. The constants c_1 and c_2 in Proposition 3 depend on M_0. Now, we take $M_0 = \|\mathbf{C}\|_{C(L^\infty)} + 1$. Then, c_1 and c_2 are fixed. Let c_3 and c_* be constants defined by

$$c_3 := \exp\Big(\frac{3c_1T}{2}\Big)\max\Big\{\sqrt{c_2}\|(\mathbf{u},\mathbf{C})\|_{Z^2}, \sqrt{c_2}\big(\|(\mathbf{u},p,\mathbf{C})\|_{H^1(\mathbb{H}^2)} + \sqrt{T}\big) + c_I\Big\}.$$

and $c_* := c_3\sqrt{2/\gamma_0}$. We can choose sufficiently small positive constants c_0 and h_0 such that

$$\alpha_{21}\big[c_*\{c_0 + h_0D(h_0)\} + (\alpha_{20}+\alpha_{32})h_0D(h_0)\|\mathbf{C}\|_{C(H^2)}\big] \le 1, \tag{25a}$$

$$(\Delta t \le)\quad \frac{c_0}{D(h_0)} \le \frac{1}{2c_1}, \tag{25b}$$

$$(\Delta t|\mathbf{w}|_{1,\infty} \le)\quad \frac{c_0|\mathbf{w}|_{1,\infty}}{D(h_0)} \le \frac{1}{4}, \tag{25c}$$

since $hD(h)$ and $1/D(h)$ tend to zero as h tends to zero.

Let $(h,\Delta t)$ be any pair satisfying (9). Since condition (4) is satisfied, Proposition 2 ensures the existence and uniqueness of the solution $(\mathbf{u}_h, p_h,\ \mathbf{C}_h) = \{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h\times Q_h\times W_h$ of scheme (6) with (8).

Step 2 (Induction): By induction we show that the following property $\mathrm{P}(n)$ holds for $n\in\{0,\dots,N_T\}$,

$$\mathrm{P}(n)\text{:}\begin{cases}\text{(a) } \dfrac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \dfrac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2 + \dfrac{\nu}{2\alpha_1^2}\|\mathbf{e}_h\|_{\ell_n^2(H^1)}^2 + \delta_0|\epsilon_h|_{\ell_n^2(|\cdot|_h)}^2 + \dfrac{\gamma_0}{2\varepsilon}\|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell_n^2(L^2)}^2 \\ \qquad \le \exp(3c_1n\Delta t)\Big[\dfrac{1}{2}\|\mathbf{e}_h^0\|_0^2 + \dfrac{\gamma_0}{2}\|\mathbf{E}_h^0\|_1^2 + c_2\Big\{\Delta t^2\|(\mathbf{u},\mathbf{C})\|_{Z^2(0,t^n)}^2 + h^2\big(\|(\mathbf{u},p,\mathbf{C})\|_{H^1(0,t^n;\mathbb{H}^2)}^2 + n\Delta t\big)\Big\}\Big], \\ \text{(b) } \|\mathbf{C}_h^n\|_{0,\infty} \le \|\mathbf{C}\|_{C(L^\infty)} + 1,\end{cases}$$

where $\|\mathbf{e}_h\|_{\ell_n^2(H^1)} = |\epsilon_h|_{\ell_n^2(|\cdot|_h)} = \|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell_n^2(L^2)} = 0$ for $n=0$.

$\mathrm{P}(n)$-(a) can be rewritten as

$$x_n + \Delta t\sum_{i=1}^n y_i \le \exp(3c_1n\Delta t)\Big(x_0 + \Delta t\sum_{i=1}^n b_i\Big), \tag{26}$$

where

$$x_n := \frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2, \qquad y_i := \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^i\|_1^2 + \delta_0|\epsilon_h^i|_h^2 + \frac{\gamma_0}{2\varepsilon}\|\overline{D}_{\Delta t}\mathbf{E}_h^i\|_0^2,$$

$$b_i := c_2\Big\{\Delta t\|(\mathbf{u},\mathbf{C})\|^2_{Z^2(t^{i-1},t^i)} + h^2\Big(\frac{1}{\Delta t}\|(\mathbf{u},p,\mathbf{C})\|^2_{H^1(t^{i-1},t^i;\mathbb{H}^2)} + 1\Big)\Big\}.$$

We firstly prove the general step in the induction. Supposing that P($n-1$) holds true for an integer $n \in \{1,\dots,N_T\}$, we prove that P(n) also holds. We prove P(n)-(a). Since (5) and (18) with $M_0 = \|\mathbf{C}\|_{C(L^\infty)} + 1$ are satisfied from (25c) and P($n-1$)-(b), respectively, we have (19) from Proposition 3. The inequality (19) implies that

$$\overline{D}_{\Delta t} x_n + y_n \le c_1(x_n + x_{n-1}) + b_n,$$

which leads to

$$x_n + \Delta t y_n \le \exp(3c_1\Delta t)(x_{n-1} + \Delta t b_n) \tag{27}$$

by $(1+c_1\Delta t)/(1-c_1\Delta t) \le (1+c_1\Delta t)(1+2c_1\Delta t) \le \exp(3c_1\Delta t)$, where $c_1\Delta t \le 1/2$ from (25b). From (27) and P($n-1$)-(a) we have

$$\begin{aligned}
x_n + \Delta t\sum_{i=1}^{n} y_i &\le \exp(3c_1\Delta t)(x_{n-1} + \Delta t b_n) + \Delta t\sum_{i=1}^{n-1} y_i \le \exp(3c_1\Delta t)\Big(x_{n-1} + \Delta t\sum_{i=1}^{n-1} y_i + \Delta t b_n\Big)\\
&\le \exp(3c_1\Delta t)\Big[\exp\big\{3c_1(n-1)\Delta t\big\}\Big(x_0 + \Delta t\sum_{i=1}^{n-1} b_i\Big) + \Delta t b_n\Big]\\
&\le \exp(3c_1 n\Delta t)\Big(x_0 + \Delta t\sum_{i=1}^{n} b_i\Big).
\end{aligned}$$

Thus, we obtain P(n)-(a).

For the proof of P(n)-(b) we prepare the estimate of $\|\mathbf{E}_h^n\|_1$. We have

$$x_0 = \frac{1}{2}\|\mathbf{e}_h^0\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^0\|_1^2 = \frac{1}{2}\|\mathbf{e}_h^0\|_0^2 \le c_I^2 h^2 \tag{28}$$

from (24). P(n)-(a) with (28) implies that

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{\gamma_0}{2}\|\mathbf{E}_h^n\|_1^2 + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h\|^2_{\ell^2_n(H^1)} + \delta_0|\epsilon_h|^2_{\ell^2_n(|\cdot|_h)} + \frac{\gamma_0}{2\varepsilon}\|\overline{D}_{\Delta t}\mathbf{E}_h\|^2_{\ell^2_n(L^2)}\\
&\quad\le \exp(3c_1 T)\Big[c_I^2 h^2 + c_2\Big\{\Delta t^2\|(\mathbf{u},\mathbf{C})\|^2_{Z^2} + h^2\big(\|(\mathbf{u},p,\mathbf{C})\|^2_{H^1(\mathbb{H}^2)} + T\big)\Big\}\Big]\\
&\quad\le \exp(3c_1 T)\Big[c_2\Delta t^2\|(\mathbf{u},\mathbf{C})\|^2_{Z^2} + h^2\Big\{c_2\big(\|(\mathbf{u},p,\mathbf{C})\|^2_{H^1(\mathbb{H}^2)} + T\big) + c_I^2\Big\}\Big]\\
&\quad\le \big\{c_3(\Delta t + h)\big\}^2,
\end{aligned}\tag{29}$$

which yields

$$\|\mathbf{E}_h^n\|_1 \le \sqrt{\frac{2}{\gamma_0}}c_3(\Delta t + h) = c_*(\Delta t + h). \tag{30}$$

We prove P(n)-(b) as follows:

$$\begin{aligned}
\|\mathbf{C}_h^n\|_{0,\infty} &\le \|\mathbf{C}_h^n - \mathit{\Pi}_h\mathbf{C}^n\|_{0,\infty} + \|\mathit{\Pi}_h\mathbf{C}^n\|_{0,\infty} \le \alpha_{21}D(h)\|\mathbf{C}_h^n - \mathit{\Pi}_h\mathbf{C}^n\|_1 + \|\mathit{\Pi}_h\mathbf{C}^n\|_{0,\infty}\\
&\le \alpha_{21}D(h)\big(\|\mathbf{C}_h^n - \hat{\mathbf{C}}_h^n\|_1 + \|\hat{\mathbf{C}}_h^n - \mathbf{C}^n\|_1 + \|\mathbf{C}^n - \mathit{\Pi}_h\mathbf{C}^n\|_1\big) + \|\mathit{\Pi}_h\mathbf{C}^n\|_{0,\infty}\\
&\le \alpha_{21}D(h)\big[c_*(\Delta t + h) + \alpha_{32}h\|\mathbf{C}^n\|_2 + \alpha_{20}h\|\mathbf{C}^n\|_2\big] + \|\mathbf{C}^n\|_{0,\infty}\\
&\le \alpha_{21}\big[c_*\{c_0 + h_0D(h_0)\} + (\alpha_{20}+\alpha_{32})h_0D(h_0)\|\mathbf{C}\|_{C(H^2)}\big] + \|\mathbf{C}\|_{C(L^\infty)}\\
&\le 1 + \|\mathbf{C}\|_{C(L^\infty)},
\end{aligned}$$

from (30), (9) and (25a). Therefore, P(n) holds true.

The proof of P(0) is easier than that of the general step. P(0)-(a) obviously holds with equality. P(0)-(b) is obtained as follows:

$$\|\mathbf{C}_h^0\|_{0,\infty} \le \|\mathbf{C}_h^0 - \mathit{\Pi}_h\mathbf{C}^0\|_{0,\infty} + \|\mathit{\Pi}_h\mathbf{C}^0\|_{0,\infty} \le \alpha_{21}D(h)(\|\mathbf{C}_h^0 - \mathbf{C}^0\|_1 + \|\mathbf{C}^0 - \mathit{\Pi}_h\mathbf{C}^0\|_1) + \|\mathit{\Pi}_h\mathbf{C}^0\|_{0,\infty}$$

$$
\begin{aligned}
&\le \alpha_{21}(\alpha_{20}+\alpha_{32})hD(h)\|\mathbf{C}^0\|_2 + \|\mathbf{C}^0\|_{0,\infty} \\
&\le 1 + \|\mathbf{C}\|_{C(L^\infty)}.
\end{aligned}
$$

Thus, the induction is completed.

Step 3: Finally we derive (10) and (11). Since $\mathrm{P}(N_T)$ holds true, we have (10) and

$$
\|\mathbf{e}_h\|_{\ell^\infty(L^2)\cap\ell^2(H^1)}, \quad |\epsilon_h|_{\ell^2(|\cdot|_h)}, \quad \|\overline{D}_{\Delta t}\mathbf{E}_h\|_{\ell^2(L^2)} \le cc_s(\Delta t + h) \tag{31}
$$

from (29). Combining (31) and the estimates

$$
\begin{aligned}
\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)} &\le \|\mathbf{e}_h\|_{\ell^\infty(L^2)} + \|\boldsymbol{\eta}\|_{\ell^\infty(L^2)} \le \|\mathbf{e}_h\|_{\ell^\infty(L^2)} + \alpha_{31}h\|(\mathbf{u},p)\|_{C(H^2\times H^1)}, \\
\Big\|\overline{D}_{\Delta t}\mathbf{C}_h^n - \frac{\partial \mathbf{C}^n}{\partial t}\Big\|_0 &\le \|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0 + \|\overline{D}_{\Delta t}\boldsymbol{\Xi}^n\|_0 + \Big\|\overline{D}_{\Delta t}\mathbf{C}^n - \frac{\partial \mathbf{C}^n}{\partial t}\Big\|_0 \\
&\le \|\overline{D}_{\Delta t}\mathbf{E}_h^n\|_0 + \frac{\alpha_{32}h}{\sqrt{\Delta t}}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^2)} + \sqrt{\frac{\Delta t}{3}}\Big\|\frac{\partial^2 \mathbf{C}}{\partial t^2}\Big\|_{L^2(t^{n-1},t^n;L^2)},
\end{aligned}
$$

we can obtain the first and the last inequalities of (11) with a positive constant $c_\dagger$ independent of h and Δt. The other inequalities of (11) are similarly proved by using (30) and (31). □

6 Conclusions

In this paper we have presented a linear stabilized Lagrange–Galerkin scheme (6) for the Oseen-type diffusive Peterlin viscoelastic model. The scheme employs the conforming linear finite elements for all unknowns, velocity, pressure and conformation tensor, together with Brezzi–Pitkäranta's stabilization method. In Theorem 1 we have established error estimates with the optimal convergence order under a mild condition $\Delta t = O(1/\sqrt{1+|\log h|})$ in two and three space dimensions. Although we have treated the stabilized scheme to reduce the number of degrees of freedom, the extension of the result to the combination of stable pairs for $(\mathbf{u}, p)$ and conventional elements for $\mathbf{C}$ is straightforwards, e.g., P2/P1/P2 element. In future we will extend this work to the Peterlin viscoelastic model with the nonlinear convective terms.

We have studied a nonlinear stabilized Lagrange–Galerkin scheme in our previous paper [17], Part I, where essentially unconditional stability and error estimates with the optimal convergence order in two space dimensions are proved including the case $\varepsilon = 0$. Numerical results by the linear and the nonlinear schemes have been presented in [21], see also our forthcoming paper for further details.

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Appendix : Proof of Lemma 5

We prove only (20a)–(20d), (20h) and (20l), since the other estimates are similarly obtained. Let $t(s) := t^{n-1} + s\Delta t$ $(s \in [0,1])$ and $y(x,s) := x - (1-s)\mathbf{w}^n(x)\Delta t$.

We prove (20a). We have that

$$
\begin{aligned}
\mathbf{r}_{h1}^n(x) &= \Big\{\Big(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\Big)\mathbf{u}\Big\}(x,t^n) - \frac{1}{\Delta t}\Big[\mathbf{u}\big(y(x,s),t(s)\big)\Big]_{s=0}^1 \\
&= \Big\{\Big(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\Big)\mathbf{u}\Big\}(x,t^n) - \int_0^1 \Big\{\Big(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\Big)\mathbf{u}\Big\}\big(y(x,s),t(s)\big)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta t \int_0^1 ds \int_s^1 \Big\{ \Big(\frac{\partial}{\partial t} + \mathbf{w}^n(x) \cdot \nabla \Big)^2 \mathbf{u} \Big\} \big(y(x, s_1), t(s_1) \big) ds_1 \\
&= \Delta t \int_0^1 s_1 \Big\{ \Big(\frac{\partial}{\partial t} + \mathbf{w}^n(x) \cdot \nabla \Big)^2 \mathbf{u} \Big\} \big(y(x, s_1), t(s_1) \big) ds_1,
\end{aligned}
$$

which implies

$$
\|\mathbf{r}_{h1}^n\|_0 \le \Delta t \int_0^1 s_1 \Big\| \Big\{ \Big(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot) \cdot \nabla \Big)^2 \mathbf{u} \Big\} \big(y(\cdot, s_1), t(s_1) \big) \Big\|_0 ds_1 \le c_w \sqrt{\Delta t} \|\mathbf{u}\|_{Z^2(t^{n-1}, t^n)},
$$

where for the last inequality we have changed the variable from x to y and used the evaluation $\det(\partial y(x, s_1)/\partial x) \ge 1/2$ ($\forall s_1 \in [0, 1]$) from Proposition 1-(ii).

We prove (20b). Since we have that

$$
\mathbf{r}_{h2}^n = \frac{1}{\Delta t} \Big[\boldsymbol{\eta}\big(y(\cdot, s), t(s)\big) \Big]_{s=0}^1 = \int_0^1 \Big\{ \Big(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot) \cdot \nabla \Big) \boldsymbol{\eta} \Big\} \big(y(\cdot, s), t(s) \big) ds,
$$

we also have

$$
\begin{aligned}
\|\mathbf{r}_{h2}^n\|_0 &\le \int_0^1 \Big\| \Big\{ \Big(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot) \cdot \nabla \Big) \boldsymbol{\eta} \Big\} \big(y(\cdot, s), t(s) \big) \Big\|_0 ds \le \int_0^1 \Big(\Big\| \frac{\partial \boldsymbol{\eta}}{\partial t} \big(y(\cdot, s), t(s) \big) \Big\|_0 + c_w \big\| \nabla \boldsymbol{\eta} \big(y(\cdot, s), t(s) \big) \big\|_0 \Big) ds \\
&\le \sqrt{2} \int_0^1 \Big\{ \Big\| \frac{\partial \boldsymbol{\eta}}{\partial t} \big(\cdot, t(s) \big) \Big\|_0 + c_w \big\| \nabla \boldsymbol{\eta} \big(\cdot, t(s) \big) \big\|_0 \Big\} ds \le \sqrt{\frac{2}{\Delta t}} \Big(\Big\| \frac{\partial \boldsymbol{\eta}}{\partial t} \Big\|_{L^2(t^{n-1}, t^n; L^2)} + c_w \big\| \nabla \boldsymbol{\eta} \big\|_{L^2(t^{n-1}, t^n; L^2)} \Big) \\
&\le \sqrt{\frac{2}{\Delta t}} \alpha_{31} h (1 + c_w) \|(\mathbf{u}, p)\|_{H^1(t^{n-1}, t^n; H^2 \times H^1)},
\end{aligned}
$$

which implies (20b), where Proposition 1-(ii) has been used for the third inequality.

(20c), (20d) and (20h) are obtained as follows:

$$
\begin{aligned}
\|\mathbf{r}_{h3}^n\|_{-1} &\le c \|(\mathrm{tr}\, \mathbf{C}^n)(\mathbf{C}^n - \mathbf{C}^{n-1} + \boldsymbol{\Xi}^{n-1} - \mathbf{E}_h^{n-1})\|_0 \le c_s \big(\|\mathbf{C}^n - \mathbf{C}^{n-1}\|_0 + \|\boldsymbol{\Xi}^{n-1}\|_0 + \|\mathbf{E}_h^{n-1}\|_0 \big) \\
&\le c_s \big(\sqrt{\Delta t} \|\mathbf{C}\|_{H^1(t^{n-1}, t^n; L^2)} + \alpha_{32} h \|\mathbf{C}^{n-1}\|_2 + \|\mathbf{E}_h^{n-1}\|_0 \big) \\
&\le c_s' \big(\|\mathbf{E}_h^{n-1}\|_0 + \sqrt{\Delta t} \|\mathbf{C}\|_{H^1(t^{n-1}, t^n; L^2)} + h \big), \\
\|\mathbf{r}_{h4}^n\|_{-1} &\le c \|[\mathrm{tr}\, (\boldsymbol{\Xi}^n - \mathbf{E}_h^n)] \mathbf{C}_h^{n-1}\|_0 \le c \|\mathbf{C}_h^{n-1}\|_{0,\infty} \|\mathrm{tr}\, (\boldsymbol{\Xi}^n - \mathbf{E}_h^n)\|_0 \\
&\le c \|\mathbf{C}_h^{n-1}\|_{0,\infty} (\|\boldsymbol{\Xi}^n\|_0 + \|\mathbf{E}_h^n\|_0) \le c \|\mathbf{C}_h^{n-1}\|_{0,\infty} (\alpha_{32} h \|\mathbf{C}^n\|_2 + \|\mathbf{E}_h^n\|_0) \\
&\le c_s \|\mathbf{C}_h^{n-1}\|_{0,\infty} (\|\mathbf{E}_h^n\|_0 + h), \\
\|\mathbf{R}_{h4}^n\|_0 &= 2 \|(\nabla \mathbf{e}_h^n) \mathbf{C}_h^{n-1}\|_0 \le 4 \|\mathbf{C}_h^{n-1}\|_{0,\infty} \|\nabla \mathbf{e}_h^n\|_0 \le 4 \|\mathbf{C}_h^{n-1}\|_{0,\infty} \|\mathbf{e}_h^n\|_1,
\end{aligned}
$$

where in the estimate of $\|\mathbf{R}_{h4}^n\|_0$ the inequality $\|AB\|_0 \le 2\|A\|_{0,\infty}\|B\|_0$ for $A \in L^\infty(\Omega)^{2\times 2}$ and $B \in L^2(\Omega)^{2\times 2}$ has been employed.

Finally, (20l) is proved as

$$
\|\mathbf{R}_{h8}^n\|_0 = \|[\mathrm{tr}\, (\mathbf{C}_h^{n-1} + \hat{\mathbf{C}}_h^{n-1})](\mathrm{tr}\, \mathbf{E}_h^{n-1}) \mathbf{C}^n\|_0 \le c_s (\|\mathbf{C}_h^{n-1}\|_{0,\infty} + \|\hat{\mathbf{C}}_h^{n-1}\|_{0,\infty}) \|\mathbf{E}_h^{n-1}\|_0 \le c_s' (\|\mathbf{C}_h^{n-1}\|_{0,\infty} + 1) \|\mathbf{E}_h^{n-1}\|_0,
$$

where for the last inequality we have used the boundedness of $\|\hat{\mathbf{C}}_h^{n-1}\|_{0,\infty}$ obtained by the estimate

$$
\begin{aligned}
\|\hat{\mathbf{C}}_h^{n-1}\|_{0,\infty} &\le \|\hat{\mathbf{C}}_h^{n-1} - \Pi_h \mathbf{C}^{n-1}\|_{0,\infty} + \|\Pi_h \mathbf{C}^{n-1}\|_{0,\infty} \le \alpha_{21} D(h) \|\hat{\mathbf{C}}_h^{n-1} - \Pi_h \mathbf{C}^{n-1}\|_1 + \|\mathbf{C}\|_{C(L^\infty)} \\
&\le \alpha_{21} D(h) \big(\|\hat{\mathbf{C}}_h^{n-1} - \mathbf{C}^{n-1}\|_1 + \|\mathbf{C}^{n-1} - \Pi_h \mathbf{C}^{n-1}\|_1 \big) + \|\mathbf{C}\|_{C(L^\infty)} \\
&\le \alpha_{21} D(h) \big(\alpha_{32} h \|\mathbf{C}^{n-1}\|_2 + \alpha_{20} h \|\mathbf{C}^{n-1}\|_2 \big) + \|\mathbf{C}\|_{C(L^\infty)} \\
&\le \alpha_{21} h D(h) (\alpha_{20} + \alpha_{32}) \|\mathbf{C}\|_{C(H^2)} + \|\mathbf{C}\|_{C(L^\infty)} \le c_s. \qquad \square
\end{aligned}
$$

References

[1] M. Aboubacar, H. Matallah, and M.F. Webster. Highly elastic solutions for Oldroyd-B and Phan-Thien/Tanner fluids with a finite volume/element method: planar contraction flows. *Journal of Non-Newtonian Fluid Mechanics*, 103:65–103, 2002.

[2] A. Bonito, P. Clément, and M. Picasso. Mathematical and numerical analysis of a simplified time-dependent viscoelastic flow. *Numerische Mathematik*, 107:213–255, 2007.

[3] A. Bonito, M. Picasso, and M. Laso. Numerical simulation of 3D viscoelastic flows with free surfaces. *Journal of Computational Physics*, 215:691–716, 2006.

[4] S. Boyaval, T. Lelièvre, and C. Mangoubi. Free-energy-dissipative schemes for the Oldroyd-B model. *ESAIM: M2AN*, 43:523–561, 2009.

[5] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer, New York, 3rd edition, 2008.

[6] F. Brezzi and J. Douglas Jr. Stabilized mixed methods for the Stokes problem. *Numerische Mathematik*, 53:225–235, 1988.

[7] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch, editor, *Efficient Solutions of Elliptic Systems*, pages 11–19, Wiesbaden, 1984. Vieweg.

[8] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.

[9] M.J. Crochet and R. Keunings. Finite element analysis of die swell of a highly elastic fluid. *Journal of Non-Newtonian Fluid Mechanics*, 10:339–356, 1982.

[10] G. Duvaut and J.L. Lions. *Inequalities in Mechanics and Physics*. Springer, Berlin, 1976.

[11] R. Fattal and R. Kupferman. Constitutive laws for the matrix-logarithm of the conformation tensor. *Journal of Non-Newtonian Fluid Mechanics*, 123:281–285, 2004.

[12] R. Fattal and R. Kupferman. Time-dependent simulation of viscoelastic flows at high Weissenberg number using the log-conformation representation. *Journal of Non-Newtonian Fluid Mechanics*, 126:23–37, 2005.

[13] L.P. Franca and R. Stenberg. Error analysis of some Galerkin least squares methods for the elasticity equations. *SIAM Journal on Numerical Analysis*, 28:1680–1697, 1991.

[14] R. Keunings. On the high Weissenberg number problem. *Journal of Non-Newtonian Fluid Mechanics*, 20:209–226, 1986.

[15] Y.-J. Lee and J. Xu. New formulations, positivity preserving discretizations and stability analysis for non-Newtonian flow models. *Computer Methods in Applied Mechanics and Engineering*, 195:1180–1206, 2006.

[16] Y.-J. Lee, J. Xu, and C.-S. Zhang. Global existence, uniqueness and optimal solvers of discretized viscoelastic flow models. *Mathematical Models and Methods in Applied Sciences*, 21(8):1713–1732, 2011.

[17] M. Lukáčová-Medviďová, H. Mizerová, H. Notsu, and M. Tabata. Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method, Part I: A nonlinear scheme. arXiv:1603.01339 [math.NA].

[18] M. Lukáčová-Medviďová, H. Mizerová, and Š. Nečasová. Global existence and uniqueness result for the diffusive Peterlin viscoelastic model. *Nonlinear Analysis: Theory, Methods & Applications*, 120:154–170, 2015.

[19] M. Lukáčová-Medviďová, H. Notsu, and B. She. Energy dissipative characteristic schemes for the diffusive Oldroyd-B viscoelastic fluid. *International Journal for Numerical Methods in Fluids*, 2015. Published online. DOI: 10.1002/fld.4195.

[20] J.M. Marchal and M.J. Crochet. A new mixed finite element for calculating viscoelastic flow. *Journal of Non-Newtonian Fluid Mechanics*, 26:77–114, 1987.

[21] H. Mizerová. Analysis and numerical solution of the Peterlin viscoelastic model. 2015. PhD thesis, University of Mainz, Germany.

[22] L. Nadau and A. Sequeira. Numerical simulations of shear-dependent viscoelastic flows with a combined finite element-finite volume method. *Computers & Mathematics with Applications*, 53:547–568, 2007.

[23] H. Notsu and M. Tabata. Error estimates of stable and stabilized Lagrange–Galerkin schemes for natural convection problems. arXiv:1511.01234 [math.NA].

[24] H. Notsu and M. Tabata. Error estimates of a pressure-stabilized characteristics finite element scheme for the Oseen equations. *Journal of Scientific Computing*, 65(3):940–955, 2015.

[25] H. Notsu and M. Tabata. Error estimates of a stabilized Lagrange–Galerkin scheme for the Navier–Stokes equations. *ESAIM: M2AN*, 50(2):361–380, 2016.

[26] M. Picasso and J. Rappaz. Existence, a priori and a posteriori error estimates for a nonlinear three-field problem arising from Oldroyd-B viscoelastic flows. *ESAIM: M2AN*, 35:879–897, 2001.

[27] M. Renardy. *Mathematical Analysis of Viscoelastic Flows*. CBMS-NSF Conference Series in Applied Mathematics 73. SIAM, New York, 2000.

[28] M. Renardy. Mathematical analysis of viscoelastic fluids. In *Handbook of Differential Equations: Evolutionary Equations*, volume 4, pages 229–265, Amsterdam, 2008. North-Holland.

[29] M. Renardy. The mathematics of myth: Yield stress behaviour as a limit of non-monotone constitutive theories. *Journal of Non-Newtonian Fluid Mechanics*, 165:519–526, 2010.

[30] M. Renardy and T. Wang. Large amplitude oscillatory shear flows for a model of a thixotropic yield stress fluid. *Journal of Non-Newtonian Fluid Mechanics*, 222:1–17, 2015.

[31] H. Rui and M. Tabata. A second order characteristic finite element scheme for convection-diffusion problems. *Numerische Mathematik*, 92:161–177, 2002.

[32] M. Tabata and S. Uchiumi. An exactly computable Lagrange–Galerkin scheme for the Navier–Stokes equations and its error estimates. To appear in *Mathematics of Computation*.

[33] P. Wapperom, R. Keunings, and V. Legat. The backward-tracking Lagrangian particle method for transient viscoelastic flows. *Journal of Non-Newtonian Fluid Mechanics*, 91:273–295, 2000.