Convergence of a mixed finite element–finite volume scheme for the isentropic Navier-Stokes system via dissipative measure-valued solutions

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Abstract

We study convergence of a mixed finite element–finite volume numerical scheme for the isentropic Navier-Stokes system under the full range of the adiabatic exponent. We establish suitable stability and consistency estimates and show that the Young measure generated by numerical solutions represents a dissipative measure-valued solutions of the limit system. In particular, using the recently established weak–strong uniqueness principle in the class of dissipative measure-valued solutions we show that the numerical solutions converge strongly to a strong solutions of the limit system as long as the latter exists.

Keywords: Compressible Navier–Stokes system, finite volume scheme, finite element scheme, stability, convergence, measure-valued solution

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Contents

1 Introduction

Time evolution of the density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible barotropic viscous fluid can be described by the Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}. \tag{1.3}$$

We assume the fluid is confined to a bounded physical domain $\Omega \subset R^3$, where the velocity satisfies the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{1.4}$$

For the sake of simplicity, we ignore the effect of external forces in the momentum equation (1.2).

In the literature there is a large variety of efficient numerical methods developed for the compressible Euler and Navier-Stokes equations. The most classical of them are the finite volume methods, see, e.g., [11], [28], [36], the methods based on a suitable combination of the finite volume and finite element methods [4], [12], [13], [19], [20], or the discontinous Galerkin schemes, e.g. [14], [15] and the references therein. Although these methods are frequently used for many physical or engineering applications, there are only partial theoretical results available concerning their analysis for the compressible Euler or Navier-Stokes systems. We refer to the works of Tadmor et al. [16], [34], [35] for entropy stability in the context of hyperbolic balance laws and to the works of Gallouët et al. [19], [20] for the stability analysis of the mixed finite volume–finite element methods based on the Crouzeix-Raviart elements for compressible viscous flows. In [23] Jovanović and Rohde obtained the error estimate for entropy dissipative finite volume methods applied to nonlinear hyperbolic balance laws under (a rather restrictive) assumption of the global existence of a bounded, smooth exact solution.

Our goal in this paper is to study convergence of solutions to the numerical scheme proposed originally by Karlsen and Karper [24], [25], [26], [27] to solve problem (1.1–1.4) in polygonal (numerical) domains, and later modified in [7] to accommodate approximations of smooth physical domains. The scheme is implicit and of mixed type, where the convective terms are approximated via upwind operators, while the viscous stress is handled by means of the Crouzeix–Raviart finite element method. As shown by Karper [27] and in [7], the scheme provides a family of numerical solutions containing a sequence that converges to a weak solution of the Navier-Stokes system as the discretization parameters tend to zero and $\gamma > 3$. Recently, Gallouët et al. [21] established rigorous error estimates on condition that the limit problem admits a smooth solution if $\gamma > 3/2$. Numerical experiments illustrating theoretical predictions have been performed in our recent paper [9]. Our numerical experiments indicate that the restriction on γ imposed in the above theoretical studies is possibly only technical and convergence takes place in the full physically relevant range of adiabatic exponent. As a matter of fact, any *uniformly bounded* sequence of numerical solutions converges unconditionally to the (unique) strong solution provided the initial data are smooth enough, see [6] based on the theoretical results of Sun, Wang, and Zhang [32].

We consider the problem under physically realistic assumptions, where theoretical results are still in short supply. In particular, our results cover completely the *isentropic* pressure–density state equation

$$p(\varrho) = a\varrho^\gamma, \ 1 < \gamma < 2. \tag{1.5}$$

Note that the assumption $\gamma < 2$ is not restrictive in this context as the largest physically relevant exponent is $\gamma = \frac{5}{3}$. Let us remark that the available theoretical results concerning global-in-time existence of *weak* solutions cover only the case $\gamma > \frac{3}{2}$ [10]. The main stumbling block is the lack of suitable *a priori bounds* to keep the convective term under control. Indeed, the expected regularity of the velocity field being $W^{1,2} \hookrightarrow L^6$ (for $N = 3$) in the space variable yields the integrability of the convective term $\varrho \mathbf{u} \otimes \mathbf{u}$ as soon as $\varrho \in L^p$, $p \geq 3/2$. Recently, Plotnikov and Weigant [31] overcome this fundamental problem for the borderline case in the 2D setting. Note that the error estimates obtained by Gallouët et al. [21] provide *conditional* convergence yielding explicit convergence rates for $\gamma > \frac{3}{2}$ and mere boundedness of the numerical solutions in the limit case $\gamma = \frac{3}{2}$. Again, $\gamma = \frac{3}{2}$ is the critical exponent.

Our goal is to establish convergence of the numerical solutions in the full range of the adiabatic exponent γ specified in (1.5). To the best of our knowledge, this is the first convergence result for $1 < \gamma \leq 3/2$. The main idea is to use the concept of *dissipative measure-valued solution* to problem (1.1–1.4) introduced recently in [5], [22]. These are, roughly speaking, measure-valued solutions satisfying, in addition, an energy inequality in which the dissipation defect measure dominates the concentration remainder in the equations. Although very general, a dissipative measure-valued solution coincides with the strong solution of the same initial-value problem as long as the latter exists, see [5]. Our approach is based on the following steps:

- We recall the numerical energy balance identified in Karper's original paper.
- We use the energy estimates to show stability of the numerical method.
- A consistency formulation of the problem is derived involving numerical solutions and error terms vanishing with the time step Δt and the spatial discretization parameter h approaching zero.
- We show that the family of numerical solutions generates a dissipative measure-valued solution of the problem. Such a result is, of course, of independent interest. As claimed recently by Fjordholm et al. [17], [18] the dissipative measure-valued solutions yield, at least in the context of *hyperbolic* conservation laws, a more appropriate solution concept than the weak entropy solutions.
- Finally, using the weak–strong uniqueness principle established in [5], we infer that the numerical solutions converge (a.a.) pointwise to the smooth solution of the limit problem as long as the latter exists.

The paper is organized as follows. The numerical scheme is introduced in Section 2. In Section 3, we recall the numerical counterpart of the energy balance and derive stability estimates. In Section 4, we introduce a consistency formulation of the problem and estimate the numerical errors. Finally, we show that the numerical scheme generates a dissipative measure-valued solution to the compressible Navier–Stokes system and state our main convergence results in Section 5.

2 Numerical scheme

To begin, we introduce the notation necessary to formulate our numerical method.

2.1 Spatial domain, mesh

We suppose that $\Omega \subset R^3$ is a bounded domain. We consider a polyhedral approximation Ω_h, where Ω_h is a polygonal domain,

$$\overline{\Omega}_h = \cup_{E^j \in E_h} E^j, \ \mathrm{int}[E^i] \cap \mathrm{int}[E^j] = \emptyset \text{ for } i \neq j,$$

where each $E^j \in E_h$ is a closed tetrahedron that can be obtained via the affine transformation

$$E^j = h\mathbb{A}_{E^j}\tilde{E} + \mathbf{a}_{E^j}, \ \mathbb{A}_{E^j} \in R^{3\times 3}, \ \mathbf{a}_{E^j} \in R^3,$$

where $\tilde{E}$ is the reference element

$$\tilde{E} = \mathrm{co}\left\{[0,0,0],[1,0,0],[0,1,0],[0,0,1]\right\},$$

and where all eigenvalues of the matrix $\mathbb{A}_{E^j}$ are bounded above and below away from zero uniformly for $h \to 0$. The family E_h of all tetrahedra covering Ω_h is called *mesh*, the positive number h is the parameter of spatial discretization. We write

$$\begin{aligned}
&a \stackrel{<}{\sim} b \Leftrightarrow a \leq cb, \ c > 0 \text{ independent of } h,\\
&a \stackrel{>}{\sim} b \Leftrightarrow a \geq cb, \ c > 0 \text{ independent of } h,\\
&a = b \Leftrightarrow a \stackrel{<}{\sim} b \text{ and } a \stackrel{>}{\sim} b.
\end{aligned}$$

Furthermore, we suppose that:

- a non-empty intersection of two elements E^j, E^i is their common face, edge, or vertex;
- for all compact sets $K_i \subset \Omega$, $K_e \subset R^3 \setminus \overline{\Omega}$ there is $h_0 > 0$ such that

$$K_i \subset \Omega_h, \ K_e \subset R^3 \setminus \overline{\Omega}_h \text{ for all } 0 < h < h_0.$$

The symbol Γ_h denotes the set of all faces in the mesh. We distinguish exterior and interior faces:

$$\Gamma_h = \Gamma_{h,\mathrm{int}} \cup \Gamma_{h,\mathrm{ext}}, \ \Gamma_{h,\mathrm{ext}} = \left\{ \Gamma \in \Gamma_h \ \middle| \ \Gamma \subset \partial\Omega_h \right\}, \ \Gamma_{h,\mathrm{int}} = \Gamma_h \setminus \Gamma_{h,\mathrm{ext}}.$$

2.2 Function spaces

Our scheme utilizes spaces of piecewise smooth functions, for which we define the traces

$$v^{\mathrm{out}} = \lim_{\delta\to 0} v(x + \delta\mathbf{n}_\Gamma), \ v^{\mathrm{in}} = \lim_{\delta\to 0} v(x - \delta\mathbf{n}_\Gamma), \ x \in \Gamma, \ \Gamma \in \Gamma_{h,\mathrm{int}},$$

where $\mathbf{n}_\Gamma$ denotes the outer normal vector to the face $\Gamma \subset \partial E$. Analogously, we define v^{in} for $\Gamma \subset \Gamma_{h,\mathrm{ext}}$. We simply write v for v^{in} if no confusion arises. We also define

$$[[v]] = v^{\mathrm{out}} - v^{\mathrm{in}}, \ \langle v \rangle_\Gamma = \frac{v^{\mathrm{out}} + v^{\mathrm{in}}}{2}, \ \langle v \rangle_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma v \ \mathrm{dS}_x.$$

Next, we introduce the space of piecewise constant functions

$$Q_h(\Omega_h) = \left\{ v \in L^1(\Omega_h) \ \middle|\ v|_E = \text{const} \in R \text{ for any } E \in E_h \right\},$$

with the associated projection

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h(\Omega_h),\ \Pi_h^Q[v] = \langle v \rangle_E = \frac{1}{|E|} \int_E v \ \mathrm{d}x,\ E \in E_h.$$

We shall occasionally write

$$\Pi_h^Q[v] = \langle v \rangle .$$

Finally, we introduce the Crouzeix–Raviart finite element spaces

$$V_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \ \middle|\ v|_E = \text{affine function on } E \in E_h,\ \int_\Gamma v^{\mathrm{in}} \ \mathrm{dS}_x = \int_\Gamma v^{\mathrm{out}} \ \mathrm{dS}_x \text{ for } \Gamma \in \Gamma_{h,\mathrm{int}} \right\},$$

$$V_{0,h}(\Omega_h) = \left\{ v \in V_h(\Omega_h) \ \middle|\ \int_\Gamma v^{\mathrm{in}} \ \mathrm{dS}_x = 0 \text{ for } \Gamma \in \Gamma_{h,\mathrm{ext}} \right\},$$

along with the associated projection

$$\Pi_h^V : W^{1,1}(\Omega_h) \to V_h(\Omega_h),\ \int_\Gamma \Pi_h^V[v] \ \mathrm{dS}_x = \int_\Gamma v \ \mathrm{dS}_x \text{ for any } \Gamma \in \Gamma_h.$$

We denote by $\nabla_h v$, $\mathrm{div}_h v$ the piecewise constant functions resulting from the action of the corresponding differential operator on v on each fixed element in E_h,

$$\nabla_h v \in Q_h(\Omega_h; R^3),\ \nabla_h v = \nabla_x v \text{ for } E \in E_h,\ \mathrm{div}_h \mathbf{v} \in Q_h(\Omega_h),\ \mathrm{div}_h v = \mathrm{div}_x v \text{ for } E \in E_h.$$

2.3 Discrete time derivative, dissipative upwind

For a given time step $\Delta t > 0$ and the (already known) value of the numerical solution v_h^{k-1} at a given time level $t_{k-1} = (k-1)\Delta t$, we introduce the discrete time derivative

$$D_t v_h = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

to compute the numerical approximation v_h^k at the level $t_k = t_{k-1} + \Delta t$.

To approximate the convective terms, we use the dissipative upwind operators introduced in [7] (see also [8]), specifically,

$$\begin{aligned} \mathrm{Up}[r_h, \mathbf{u}_h] &= \underbrace{\{r_h\} \langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}_{\text{convective part}} - \frac{1}{2} \underbrace{\max\{h^\alpha; |\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma|\} [[r_h]]}_{\text{dissipative part}} \\ &= \underbrace{r_h^{\mathrm{out}} [\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma]^- + r_h^{\mathrm{in}} [\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma]^+}_{\text{standard upwind}} - \frac{h^\alpha}{2} [[r_h]] \chi \left(\frac{\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right), \end{aligned} \tag{2.1}$$

where

$$\chi(z) = \begin{cases} 0 \text{ for } z < -1, \\ z+1 \text{ if } -1 \leq z \leq 0, \\ 1-z \text{ if } 0 < z \leq 1, \\ 0 \text{ for } z > 1. \end{cases}$$

2.4 Numerical scheme

Given the initial data

$$\varrho_h^0 \in Q_h(\Omega_h),\ \mathbf{u}_h^0 \in V_{0,h}(\Omega_h; R^3), \tag{2.2}$$

and the numerical solution

$$\varrho_h^{k-1} \in Q_h(\Omega_h),\ \mathbf{u}_h^{k-1} \in V_{0,h}(\Omega_h; R^3),\ k \geq 1,$$

the value $[\varrho_h^k, \mathbf{u}_h^k] \in Q_h(\Omega_h) \times V_{0,h}(\Omega_h; R^3)$ is obtained as a solution of the following system of equations:

$$\int_{\Omega_h} D_t \varrho_h^k \phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k]\, [[\phi]] \ \mathrm{dS}_x = 0 \tag{2.3}$$

for any $\phi \in Q_h(\Omega_h)$;

$$\begin{aligned} \int_{\Omega_h} D_t \left(\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle \right) \cdot \boldsymbol{\phi} \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle, \mathbf{u}_h^k] \cdot [[\langle \boldsymbol{\phi} \rangle]] \ \mathrm{dS}_x - \int_{\Omega_h} p(\varrho_h^k) \mathrm{div}_h \boldsymbol{\phi} \ \mathrm{d}x \\ + \mu \int_{\Omega_h} \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\phi} \ \mathrm{d}x + \left(\frac{\mu}{3} + \eta \right) \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\phi} \ \mathrm{d}x = 0 \end{aligned} \tag{2.4}$$

for any $\boldsymbol{\phi} \in V_{0,h}(\Omega_h; R^3)$. The specific form of the viscous stress in (2.4) reflects the fact that the viscosity coefficients are constant.

It was shown in [27] (see also [8, Part II]) that system (2.3), (2.4) is solvable for any choice of the initial data (2.2). In addition, $\varrho_h^k > 0$ whenever $\varrho_h^0 > 0$. In general, the solution $[\varrho_h^k, \mathbf{u}_h^k]$ may not be uniquely determined by $[\varrho_h^{k-1}, \mathbf{u}_h^{k-1}]$ unless the time step Δt is conveniently adjusted by a CFL type condition. We make more comments on this option in Remark 4.3 below.

As shown in [7] (see also [8, Part II]), the family of numerical solutions converges, up to a suitable subsequence, to a weak solution of the Navier-Stokes system (1.1–1.4) as $h \to 0$ if

- the time step is adjusted so that $\Delta t \approx h$;
- the viscosity coefficients satisfy $\mu > 0$, $\eta \geq 0$,
- the pressure satisfies

$$p(\varrho) = a\varrho^\gamma + b\varrho,\ a, b > 0,\ \gamma > 3.$$

If the limit solution of the Navier–Stokes system is smooth, then qualitative error estimates can be derived on condition that p satisfies (1.5) with $\gamma \geq 3/2$, see Gallouët et al. [21]. Unfortunately, many real world applications correspond to smaller adiabatic exponents, the most popular among them is the air with $\gamma = 7/5$. It is therefore of great interest to discuss convergence of the scheme in the physically relevant range $1 < \gamma < 2$.

3 Stability - energy estimates

It is crucial for our analysis that the numerical scheme (2.2–2.4) admits a certain form of total energy balance. For the pressure potential

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma},\ P''(\varrho) = \frac{p'(\varrho)}{\varrho} = a\gamma\varrho^{\gamma-2},$$

the *total energy balance* reads

$$\begin{aligned}
&\int_{\Omega_h} D_t \left[\frac{1}{2}\varrho_h^k |\left\langle \mathbf{u}_h^k \right\rangle|^2 + P(\varrho_h^k)\right] \ \mathrm{d}x + \int_{\Omega_h} \left[\mu|\nabla_h \mathbf{u}_h^k|^2 + (\mu/3+\eta)|\mathrm{div}_h \mathbf{u}_h^k|^2\right] \ \mathrm{d}x \\
&= -\frac{1}{2}\int_{\Omega_h} P''(s_h^k)\frac{\left(\varrho_h^k - \varrho_h^{k-1}\right)^2}{\Delta t} \ \mathrm{d}x - \int_{\Omega_h} \frac{\Delta t}{2}\varrho_h^{k-1}\left|\frac{\left\langle \mathbf{u}_h^k \right\rangle - \left\langle \mathbf{u}_h^{k-1} \right\rangle}{\Delta t}\right|^2 \ \mathrm{d}x \\
&- \frac{h^\alpha}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[\varrho_h^k]]\,[[P'(\varrho_h^k)]]\,\chi\left(\frac{\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma}{h^\alpha}\right) \ \mathrm{dS}_x \\
&- \frac{1}{2}\sum_{\Gamma\in\Gamma_h}\int_\Gamma P''(z_h^k)\,[[\varrho_h^k]]^2\,|\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma|\ \mathrm{dS}_x \\
&- \frac{h^\alpha}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \{\varrho_h^k\}\cdot[[\left\langle \mathbf{u}_h^k\right\rangle]]^2\,\chi\left(\frac{\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma}{h^\alpha}\right) \ \mathrm{dS}_x \\
&- \frac{1}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \left((\varrho_h^k)^{\mathrm{in}}[\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma]^+ - (\varrho_h^k)^{\mathrm{out}}[\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma]^-\right)[[\left\langle \mathbf{u}_h^k\right\rangle]]^2 \ \mathrm{dS}_x,
\end{aligned} \tag{3.1}$$

with

$$s_h^k \in \mathrm{co}\{\varrho_h^k, \varrho_h^{k-1}\},\ z_h^k \in \mathrm{co}\{(\varrho^k)^{\mathrm{in}}, (\varrho_h^k)^{\mathrm{out}}\},$$

see [8, Chapter 7, Section 7.5.4]. As the numerical densities are positive, all terms on the right-hand side of (3.1) representing numerical dissipation are non-positive. For completeness, we remark that the scheme conserves the total mass, specifically,

$$\int_{\Omega_h} \varrho_h^k \ \mathrm{d}x = \int_{\Omega_h} \varrho_h^0 \ \mathrm{d}x,\ k = 1, 2, \ldots \tag{3.2}$$

3.1 Dissipative terms and the pressure growth

It is easy to check that

$$P''(z)(\varrho_1 - \varrho_2)^2 \geq a\gamma(\varrho_1^{\gamma/2} - \varrho_2^{\gamma/2})^2 \text{ whenever } z \in \mathrm{co}\{\varrho_1, \varrho_2\},\ \varrho_1, \varrho_2 > 0,\ 1 < \gamma < 2. \tag{3.3}$$

Indeed it is enough to assume $0 < \varrho_1 \leq z \leq \varrho_2$; whence

$$P''(z)(\varrho_1 - \varrho_2)^2 \geq a\gamma\varrho_2^{\gamma-2}(\varrho_1 - \varrho_2)^2,$$

and (3.3) reduces to showing

$$\varrho_2^{\gamma/2-1}(\varrho_2 - \varrho_1) \geq (\varrho_2^{\gamma/2} - \varrho_1^{\gamma/2}) \text{ or, equivalently, } \varrho_1\varrho_2^{\gamma/2-1} \leq \varrho_1^{\gamma/2},$$

where the last inequality follows immediately as $\varrho_1 \leq \varrho_2$, $1 < \gamma < 2$.

Consequently, the terms on the right-hand side of (3.1) representing the numerical dissipation and containing P'' satisfy

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_h} P''(s_h^k)\frac{\left(\varrho_h^k - \varrho_h^{k-1}\right)^2}{\Delta t}\ \mathrm{d}x &\geq \frac{a\gamma}{2}\int_{\Omega_h}\frac{\left((\varrho_h^k)^{\gamma/2} - (\varrho_h^{k-1})^{\gamma/2}\right)^2}{\Delta t}\ \mathrm{d}x,\\
\frac{h^\alpha}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[\varrho_h^k]]\,[[P'(\varrho_h^k)]]\,\chi\left(\frac{\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma}{h^\alpha}\right)\ \mathrm{dS}_x &\geq \frac{a\gamma h^\alpha}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[(\varrho_h^k)^{\gamma/2}]]^2\,\chi\left(\frac{\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma}{h^\alpha}\right)\ \mathrm{dS}_x,\\
\frac{1}{2}\sum_{\Gamma\in\Gamma_h}\int_\Gamma P''(z_h^k)\,[[\varrho_h^k]]^2\,|\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma|\ \mathrm{dS}_x &\geq \frac{a\gamma}{2}\sum_{\Gamma\in\Gamma_h}\int_\Gamma [[(\varrho_h^k)^{\gamma/2}]]^2\,|\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma|\ \mathrm{dS}_x.
\end{aligned} \tag{3.4}$$

In particular, the energy balance (3.1) gives rise to

$$\begin{aligned}
&\int_{\Omega_h} D_t\left[\frac{1}{2}\varrho_h^k|\left\langle \mathbf{u}_h^k\right\rangle|^2 + P(\varrho_h^k)\right]\ \mathrm{d}x + \int_{\Omega_h}\left[\mu|\nabla_h\mathbf{u}_h^k|^2 + (\mu/3+\eta)|\mathrm{div}_h\mathbf{u}_h^k|^2\right]\ \mathrm{d}x\\
&+ a\int_{\Omega_h}\frac{\left((\varrho_h^k)^{\gamma/2} - (\varrho_h^{k-1})^{\gamma/2}\right)^2}{\Delta t}\ \mathrm{d}x + \Delta t\int_{\Omega_h}\varrho_h^{k-1}\left|\frac{\left\langle \mathbf{u}_h^k\right\rangle - \left\langle \mathbf{u}_h^{k-1}\right\rangle}{\Delta t}\right|^2\ \mathrm{d}x\\
&+ a\sum_{\Gamma\in\Gamma_h}\int_\Gamma [[(\varrho_h^k)^{\gamma/2}]]^2\max\left\{h^\alpha; |\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma|\right\}\ \mathrm{dS}_x\\
&+ ah^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \left\{\varrho_h^k\right\}\cdot[[\left\langle \mathbf{u}_h^k\right\rangle]]^2\,\chi\left(\frac{\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma}{h^\alpha}\right)\ \mathrm{dS}_x\\
&+ \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left((\varrho_h^k)^{\mathrm{in}}[\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma]^+ - (\varrho_h^k)^{\mathrm{out}}[\left\langle \mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma]^-\right)[[\left\langle \mathbf{u}_h^k\right\rangle]]^2\ \mathrm{dS}_x \lesssim 0.
\end{aligned} \tag{3.5}$$

4 Consistency

Our goal is to derive a consistency formulation for the discrete solutions satisfying (2.3), (2.4). To this end, it is convenient to deal with quantities defined on $R \times \Omega_h$. Accordingly, we introduce

$$\varrho_h(t,\cdot) = \varrho_h^0 \text{ for } t < \Delta t,\ \varrho_h(t,\cdot) = \varrho_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1,2,\dots, \tag{4.1}$$

$$\mathbf{u}_h(t,\cdot) = \mathbf{u}_h^0 \text{ for } t < \Delta t,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1,2,\dots, \tag{4.2}$$

and

$$D_t v_h = \frac{v(t,\cdot) - v(t-\Delta t,\cdot)}{\Delta t},\ t > 0. \tag{4.3}$$

For the sake of simplicity, we keep the time step Δt constant, however, a similar ansatz obviously works also for $\Delta t = \Delta t_k$ adjusted at each level of iteration.

A suitable consistency formulation of equation (2.3) reads

$$-\int_{\Omega_h} \varrho_h^0 \varphi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_{\Omega_h} \left[\varrho_h \partial_t \varphi + \varrho_h \mathbf{u}_h \cdot \nabla_x \varphi\right]\ \mathrm{d}x\,\mathrm{d}t + \mathcal{O}(h^\beta),\ \beta > 0, \tag{4.4}$$

for any test function $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}_h)$, where β denotes a generic positive exponent, and, accordingly, the remainder term $\mathcal{O}(h^\beta)$, that may depend also on the test function φ, tends to zero as $h \to 0$. Formula (4.4) as well as the specific form of the remainder will be deduced in Section 4.2. Note that (4.4) mimicks the standard weak formulation of (1.1).

Similarly, we want to rewrite (2.4) in the form

$$\begin{aligned}
-\int_{\Omega_h} \varrho_h^0 \left\langle \mathbf{u}_h^0 \right\rangle \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x &= \int_0^T \int_{\Omega_h} \Big[\varrho_h \left\langle \mathbf{u}_h \right\rangle \partial_t \boldsymbol{\varphi} + \varrho_h \left\langle \mathbf{u}_h \right\rangle \otimes \mathbf{u}_h : \nabla_x \boldsymbol{\varphi} + p(\varrho_h) \mathrm{div}_x \boldsymbol{\varphi}\Big]\ \mathrm{d}x\ \mathrm{d}t \\
&\quad - \int_0^T \int_{\Omega_h} \Big[\mu \nabla_h \mathbf{u}_h : \nabla_x \boldsymbol{\varphi} + (\mu/3 + \eta) \mathrm{div}_h \mathbf{u}_h \cdot \mathrm{div}_x \boldsymbol{\varphi}\Big]\ \mathrm{d}x\,\mathrm{d}t + \mathcal{O}(h^\beta)
\end{aligned} \tag{4.5}$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,\infty) \times \Omega_h; R^3)$, see Section 4.3.

4.1 Preliminaries, some useful estimates

We collect certain well-known estimates used in the subsequent analysis. They are obtained by rescaling the standard estimates from the reference element. We refer to [8, Part II, Chapters 8,9] for the proofs.

4.1.1 Discrete negative and trace estimates for piecewise smooth functions

The following inverse inequality

$$\|v\|_{L^p(\Omega_h)} \lesssim h^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|v\|_{L^q(\Omega_h)},\ 1 \le q \le p \le \infty, \tag{4.6}$$

holds for any $v \in Q_h(\Omega_h)$.

The trace estimates read

$$\|v\|_{L^p(\Gamma)} \lesssim h^{1/p} \|v\|_{L^p(E)} \text{ whenever } \Gamma \subset \partial E,\ 1 \leq p \leq \infty \tag{4.7}$$

for any $v \in Q_h(\Omega_h)$.

Finally, we report a discrete version of Poincaré's inequality

$$\|v - \langle v \rangle\|_{L^2(E)} \equiv \|v - \Pi_h^Q[v]\|_{L^2(E)} \lesssim h \|\nabla_h v\|_{L^2(E)} \text{ for any } v \in V_h(\Omega_h). \tag{4.8}$$

4.1.2 Sobolev estimates for broken norms

We have

$$\|v\|^2_{L^6(\Omega_h)} \lesssim \sum_{\Gamma_{h,\text{int}}} \int_\Gamma \frac{[[v]]^2}{h} \, \mathrm{dS}_x + \|v\|^2_{L^2(\Omega_h)} \tag{4.9}$$

for any $v \in Q_h(\Omega_h)$. In particular, we may combine the negative estimates (4.6) with (4.9) to obtain

$$\begin{aligned} \|\varrho_h\|_{L^\infty(\Omega_h)} &= \left(\left\| \varrho_h^{\gamma/2} \right\|_{L^\infty(\Omega_h)} \right)^{2/\gamma} \lesssim h^{-1/\gamma} \left(\left\| \varrho_h^{\gamma/2} \right\|^2_{L^6(\Omega_h)} \right)^{1/\gamma} \\ &\lesssim h^{-1/\gamma} \left(\sum_{\Gamma_{h,\text{int}}} \int_\Gamma \frac{\left[\left[\varrho^{\gamma/2}\right]\right]^2}{h} \, \mathrm{dS}_x \right)^{1/\gamma} + h^{-1/\gamma} \left(\left\| \varrho^{\gamma/2} \right\|^2_{L^2(\Omega_h)} \right)^{1/\gamma} \\ &\lesssim h^{-\frac{2+\alpha}{\gamma}} \left(\sum_{\Gamma_{h,\text{int}}} \int_\Gamma h^\alpha \left[\left[\varrho^{\gamma/2}\right]\right]^2 \, \mathrm{dS}_x \right)^{1/\gamma} + h^{-1/\gamma} \|\varrho\|_{L^\gamma(\Omega_h)} \end{aligned} \tag{4.10}$$

Next, we have the discrete variant of Sobolev's inequality

$$\|v\|^2_{L^6(\Omega_h)} \lesssim \sum_{E \in E_h} \|\nabla_h v\|^2_{L^2(E;R^3)} \equiv \|\nabla_h v\|^2_{L^2(\Omega_h;R^3)} \tag{4.11}$$

for any $v \in V_{0,h}(\Omega_h)$.

Finally, we recall the projection estimates for the Crouzeix–Raviart spaces

$$\left\|\Pi_h^V[v] - v\right\|_{L^q(\Omega_h)} + h \left\|\nabla_h \Pi_h^V[v] - \nabla_x v\right\|_{L^q(\Omega_h;R^3)} \lesssim h^j \|\nabla^j v\|_{L^q(\Omega_h;R^{3j})},\ j = 1, 2,\ 1 \leq q \leq \infty. \tag{4.12}$$

4.1.3 Upwind consistency formula

We report the universal formula

$$
\begin{aligned}
\int_{\Omega_h} r\mathbf{u}\cdot\nabla_x\phi \, \mathrm{d}x = & \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[r,\mathbf{u}]\,[[F]] \, \mathrm{dS}_x \\
& + \frac{h^\alpha}{2} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[r]]\,[[F]]\,\chi\left(\frac{\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma}{h^\alpha}\right) \, \mathrm{dS}_x \\
& + \sum_{E\in E_h} \sum_{\Gamma_E\subset\partial E} \int_{\Gamma_E} (F-\phi)\,[[r]]\,[\langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma]^- \, \mathrm{dS}_x \\
& + \sum_{E\in E_h} \sum_{\Gamma_E\subset\partial E} \int_{\Gamma_E} \phi r\Big(\mathbf{u}\cdot\mathbf{n} - \langle\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma\Big) \, \mathrm{dS}_x + \int_{\Omega_h} r(F-\phi)\mathrm{div}_h\mathbf{u} \, \mathrm{d}x
\end{aligned}
\tag{4.13}
$$

for any $r, F \in Q_h(\Omega_h)$, $\mathbf{u}\in V_{0,h}(\Omega_h;R^3)$, $\phi\in C^1(\Omega_h)$, see [8, Chapter 9, Lemma 7].

4.2 Consistency formulation of the continuity method

Our goal is to derive the consistency formulation (4.4) of the discrete equation of continuity (2.3).

4.2.1 Time derivative

We consider test functions of the form $\psi(t)\phi(x)$ to obtain

$$
\begin{aligned}
\int_0^T \int_{\Omega_h} D_t(\varrho_h)\,\langle\psi\phi\rangle \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \psi \int_{\Omega_h} D_t(\varrho_h)\phi \, \mathrm{d}x \, \mathrm{d}t \\
&= -\int_0^T \int_{\Omega_h} \frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}\varrho_h\phi \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{\Delta t}\int_{-\Delta t}^0 \int_{\Omega_h} \varrho_h^0\psi(t+\Delta t)\phi \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

whenever the function $\psi\in C_c^\infty[0,T)$ and Δt is small enough so that the interval $[T-\Delta t,\infty)$ is not included in the support of ψ. By means of the mean-value theorem we get that

$$
\int_0^T \int_{\Omega_h} D_t(\varrho_h)\,\langle\psi\phi\rangle \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_{\Omega_h} \partial_t\psi\varrho_h\phi \, \mathrm{d}x \, \mathrm{d}t - \int_{\Omega_h} \varrho_h^0\psi(0)\phi \, \mathrm{d}x + \mathcal{O}(h^\beta) \tag{4.14}
$$

for any $\phi\in C(\Omega_h)$, $\psi\in C_c^\infty[0,T)$. Note that the $\mathcal{O}(h)$ term depends on the second derivative of ψ.

4.2.2 Convective term - upwind

Relation (4.13) evaluated for $r = \varrho_h^k$, $\mathbf{u} = \mathbf{u}_h^k$, $F = \langle \phi \rangle$, $\phi \in C^1(\Omega_h)$ gives rise to

$$
\begin{aligned}
\int_{\Omega_h} \varrho_h^k \mathbf{u}_h^k \cdot \nabla_x \phi \ \mathrm{d}x = & \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \, [[\langle \phi \rangle]] \ \mathrm{dS}_x \\
& + \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[\varrho_h^k\right]\right] [[\langle \phi \rangle]] \, \chi \left(\frac{\left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma}{h^\alpha} \right) \ \mathrm{dS}_x \\
& + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\langle \phi \rangle - \phi) \left[\left[\varrho_h^k\right]\right] [\left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma]^- \ \mathrm{dS}_x \\
& + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \varrho_h^k \Big(\mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \Big) \ \mathrm{dS}_x + \int_{\Omega_h} \varrho_h (\langle \phi \rangle - \phi) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x.
\end{aligned}
\tag{4.15}
$$

Using an elementary inequality

$$
|\varrho_1 - \varrho_2| \leq \left| (\varrho_1)^{\gamma/2} - (\varrho_2)^{\gamma/2} \right| \left| (\varrho_1)^{1-\gamma/2} + (\varrho_2)^{1-\gamma/2} \right|, \ 1 \leq \gamma \leq 2 \tag{4.16}
$$

we get

$$
\begin{aligned}
\frac{h^\alpha}{2} & \left| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[\varrho_h^k\right]\right] [[\langle \phi \rangle]] \, \chi \left(\frac{\left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma}{h^\alpha} \right) \ \mathrm{dS}_x \right| \stackrel{<}{\sim} h^{1+\alpha} \|\phi\|_{C^1(\overline{\Omega}_h)} \left| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[\varrho_h^k\right]\right] \ \mathrm{dS}_x \right| \\
& \stackrel{<}{\sim} h^{1+\alpha} \|\phi\|_{C^1(\overline{\Omega}_h)} \left(\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[(\varrho_h^k)^{\gamma/2} \right]\right]^2 \ \mathrm{dS}_x + \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left\{ (\varrho_h^k)^{1-\gamma/2} \right\}^2 \ \mathrm{dS}_x \right),
\end{aligned}
$$

where, by virtue of (3.5), there exists a sequence $\{g_k\}_{k=1}^\infty$ such that

$$
h^{1+\alpha} \|\phi\|_{C^1(\overline{\Omega}_h)} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[(\varrho_h^k)^{\gamma/2} \right]\right]^2 \ \mathrm{dS}_x \leq c(\phi) h g_k, \ \text{and } \Delta t \sum_k g_k < \infty.
$$

In accordance with (3.2) and the trace estimates (4.7),

$$
h^{1+\alpha} \|\phi\|_{C^1(\overline{\Omega}_h)} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left\{ (\varrho_h^k)^{1-\gamma/2} \right\}^2 \ \mathrm{dS}_x \stackrel{<}{\sim} h^\alpha c(\phi) \sum_{E \in E_h} \int_E (\varrho_h^k)^{2-\gamma} \ \mathrm{d}x \stackrel{<}{\sim} h^\alpha.
$$

We may infer that

$$
\frac{h^\alpha}{2} \left\| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h]] \, [[\langle \phi \rangle]] \, \chi \left(\frac{\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right) \ \mathrm{dS}_x \right\|_{L^1(0,T)} = \mathcal{O}(h^\beta), \beta > 0 \text{ whenever } \alpha > 0. \tag{4.17}
$$

Next, using (3.5) again, we deduce

$$
\begin{aligned}
&\left| \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\langle \phi \rangle - \phi) \left[\left[\varrho_h^k \right]\right] \left[\left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right]^- \ \mathrm{dS}_x \right| \\
&\lesssim h \|\phi\|_{C^1(\overline{\Omega}_h)} \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} | \left[\left[(\varrho_h^k)^{\gamma/2} \right]\right] | \, | \left\{ (\varrho_h^k)^{1-\gamma/2} \right\} | \, | \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma | \ \mathrm{dS}_x \\
&\lesssim h \left(\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left[\left[(\varrho_h^k)^{\gamma/2} \right]\right]^2 | \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma | \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^{2-\gamma} |\mathbf{u}_h^k| \ \mathrm{dS}_x \right)^{1/2} \\
&\lesssim h^{1/2} \left(\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left[\left[(\varrho_h^k)^{\gamma/2} \right]\right]^2 | \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma | \ \mathrm{dS}_x \right)^{1/2} \left(\sum_{E\in E_h} \int_E (\varrho_h^k)^{2-\gamma} | \langle \mathbf{u}_h \rangle | \ \mathrm{d}x; \right)^{1/2}
\end{aligned}
$$

whence, using (4.10) to control the last term, we conclude

$$
\left\| \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\langle \phi \rangle - \phi) \left[\left[\varrho_h \right]\right] \left[\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma \right]^- \ \mathrm{dS}_x \right\|_{L^2(0,T)} = \mathcal{O}(h^\beta). \tag{4.18}
$$

Furthermore,

$$
\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \varrho_h^k \Big(\mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \Big) \ \mathrm{dS}_x = \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\phi - \langle \phi \rangle_\Gamma)\, \varrho_h^k \Big(\mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \Big) \ \mathrm{dS}_x,
$$

where, by virtue of Poincaré's inequality and the trace estimates (4.7),

$$
\begin{aligned}
&\left| \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\phi - \langle \phi \rangle_\Gamma)\, \varrho_h^k \Big(\mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \Big) \ \mathrm{dS}_x \right| \\
&\lesssim h \|\nabla_x \phi\|_{L^\infty(\Omega_h)} \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \varrho_h^k \left| \mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right| \ \mathrm{dS}_x \lesssim \sum_{E\in E_h} \int_E \varrho_h^k \left| \mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right| \ \mathrm{d}x \\
&\lesssim h \sum_{E\in E_h} \|\nabla_h \mathbf{u}_h^k\|_{L^2(E)} \|\varrho_h^k\|_{L^2(E)} \lesssim h \|\nabla_h \mathbf{u}_h^k\|_{L^2(\Omega_h)} \|\varrho_h^k\|_{L^2(\Omega_h)} \lesssim h \|\nabla_h \mathbf{u}_h^k\|_{L^2(\Omega_h)} \|\varrho_h^k\|^{1/2}_{L^\infty(\Omega_h)}.
\end{aligned}
$$

Going back to (4.10) we observe that the right-hand side is controlled as soon as

$$
1 - \frac{2+\alpha}{2\gamma} > 0 \text{ meaning } \alpha < 2(\gamma - 1). \tag{4.19}
$$

Finally, it is easy to check that the last integral in (4.15) can be handled in the same way. Thus we conclude that the consistency formulation (4.4) holds for any test function $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}_h)$ as long as $\alpha > 0$, $\gamma > 1$ are interrelated through (4.19).

4.3 Consistency formulation of the momentum method

Our goal is to take $\Pi_h^V[\boldsymbol{\phi}]$, $\boldsymbol{\phi} \in C_c^\infty(\Omega_h; R^3)$ as a test function in the momentum scheme (2.4). To begin, observe that

$$\int_{\Omega_h} \nabla_h \mathbf{u}_h : \nabla_h \Pi_h^V[\boldsymbol{\phi}] \ \mathrm{d}x = \int_{\Omega_h} \nabla_h \mathbf{u}_h : \nabla_x \boldsymbol{\phi} \ \mathrm{d}x, \ \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h \mathrm{div}_h \Pi_h^V[\boldsymbol{\phi}] \ \mathrm{d}x = \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \boldsymbol{\phi} \ \mathrm{d}x$$

$$\int_{\Omega_h} p(\varrho_h) \mathrm{div}_h \Pi_h^V[\boldsymbol{\phi}] \ \mathrm{d}x = \int_{\Omega_h} p(\varrho_h) \mathrm{div}_x \boldsymbol{\phi} \ \mathrm{d}x,$$

see [8, Chapter 9, Lemma 8].

4.3.1 Time derivative

We compute

$$\begin{aligned} \int_{\Omega_h} D_t(\varrho_h^k \langle \mathbf{u}_h^k \rangle) \cdot \boldsymbol{\phi} \ \mathrm{d}x = & \int_{\Omega_h} D_t(\varrho_h^k \langle \mathbf{u}_h^k \rangle) \cdot \Pi_h^V[\boldsymbol{\phi}] \ \mathrm{d}x \\ & + \int_{\Omega_h} \varrho_h^{k-1} \frac{\langle \mathbf{u}_h^k \rangle - \langle \mathbf{u}_h^{k-1} \rangle}{\Delta t} \cdot \left(\boldsymbol{\phi} - \Pi_h^V[\boldsymbol{\phi}] \right) \ \mathrm{d}x \\ & + \int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \langle \mathbf{u}_h^k \rangle \cdot \left(\boldsymbol{\phi} - \Pi_h^V[\boldsymbol{\phi}] \right) \ \mathrm{d}x, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \left| \int_{\Omega_h} \varrho_h^{k-1} \frac{\langle \mathbf{u}_h^k \rangle - \langle \mathbf{u}_h^{k-1} \rangle}{\Delta t} \cdot \left(\boldsymbol{\phi} - \Pi_h^V[\boldsymbol{\phi}] \right) \ \mathrm{d}x \right| & \lesssim h^2 \|\phi\|_{C^2(\overline{\Omega}_h)} \int_{\Omega_h} \varrho_h^{k-1} \left| \frac{\langle \mathbf{u}_h^k \rangle - \langle \mathbf{u}_h^{k-1} \rangle}{\Delta t} \right| \ \mathrm{d}x \\ & \lesssim h^2 \left(\int_{\Omega_h} \varrho_h^{k-1} \ \mathrm{d}x \right)^{1/2} \left(\int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\langle \mathbf{u}_h^k \rangle - \langle \mathbf{u}_h^{k-1} \rangle}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \\ & \lesssim h^2 (\Delta t)^{-1/2} \left(\Delta t \int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\langle \mathbf{u}_h^k \rangle - \langle \mathbf{u}_h^{k-1} \rangle}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2}, \end{aligned}$$

where the most right integral is controlled in $L^2(0,T)$ by the numerical dissipation in (3.5).

As for the remaining integral, we may use inequality (4.6) to obtain

$$\begin{aligned} & \left| \int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \langle \mathbf{u}_h^k \rangle \cdot \left(\boldsymbol{\phi} - \Pi_h^V[\boldsymbol{\phi}] \right) \ \mathrm{d}x \right| \lesssim h^2 \int_{\Omega_h} \frac{|\varrho_h^k - \varrho_h^{k-1}|}{\Delta t} | \langle \mathbf{u}_h^k \rangle | \ \mathrm{d}x \\ & \lesssim h^2 (\Delta t)^{-1} \|\mathbf{u}_h^k\|_{L^6(\Omega_h; R^3)} \sup_k \|\varrho_h^k\|_{L^{6/5}(\Omega_h)} \lesssim h^2 (\Delta t)^{-1} h^{-1/2} \|\mathbf{u}_h^k\|_{L^6(\Omega_h; R^3)} \sup_k \|\varrho_h^k\|_{L^1(\Omega_h)} \end{aligned}$$

Finally, we may repeat the same argument as in Section 4.2.1 to conclude that

$$\begin{aligned}
&\int_0^T \int_{\Omega_h} \psi D_t(\varrho_h \langle \mathbf{u}_h \rangle) \Pi_h^V[\boldsymbol{\phi}] \ \mathrm{d}x \ \mathrm{d}t \\
&\quad = -\int_0^T \int_{\Omega_h} \varrho_h \langle \mathbf{u}_h \rangle \cdot \boldsymbol{\phi} \partial_t \psi \ \mathrm{d}x \ \mathrm{d}t - \int_{\Omega_h} \psi(0) \varrho_h^0 \langle \mathbf{u}_h^0 \rangle \cdot \phi \ \mathrm{d}x + \mathcal{O}(h^\beta)
\end{aligned} \tag{4.21}$$

provided $\psi \in C_c^\infty[0,T)$, $\boldsymbol{\phi} \in C_c^\infty(\Omega_h; R^3)$.

4.3.2 Convective term - upwind

Applying formula (4.13) we obtain

$$\begin{aligned}
&\int_{\Omega_h} \varrho_h^k \left(\langle \mathbf{u}_h^k \rangle \otimes \mathbf{u}_h^k \right) : \nabla_x \boldsymbol{\phi} \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \langle \mathbf{u}_h^k \rangle, \mathbf{u}_h^k] \cdot [[\langle \Pi_h^V[\boldsymbol{\phi}] \rangle]] \ \mathrm{dS}_x \\
&\quad = \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k \langle \mathbf{u}_h^k \rangle]] \cdot [[\langle \Pi_h^V[\boldsymbol{\phi}] \rangle]] \, \chi \left(\frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right) \ \mathrm{dS}_x \\
&\quad + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\langle \Pi_h^V[\boldsymbol{\phi}] \rangle - \boldsymbol{\phi} \right) \cdot [[\varrho_h^k \langle \mathbf{u}_h^k \rangle]] \, [\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma]^- \ \mathrm{dS}_x \\
&\quad + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \varrho_h^k \boldsymbol{\phi} \cdot \langle \mathbf{u}_h^k \rangle \left(\mathbf{u}_h^k \cdot \mathbf{n} - \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma \right) \mathrm{dS}_x \\
&\quad + \int_{\Omega_h} \varrho_h^k \langle \mathbf{u}_h^k \rangle \cdot \left(\langle \Pi_h^V[\boldsymbol{\phi}] \rangle - \boldsymbol{\phi} \right) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x.
\end{aligned} \tag{4.22}$$

We proceed in several steps.

Step 1

Applying (4.12) we get

$$\begin{aligned}
&\left| \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k \langle \mathbf{u}_h^k \rangle]] \cdot [[\langle \Pi_h^V[\boldsymbol{\phi}] \rangle]] \, \chi \left(\frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right) \ \mathrm{dS}_x \right| \\
&\lesssim h^{1+\alpha} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left| \ [[\varrho_h^k \langle \mathbf{u}_h^k \rangle]] \right| \ \chi \left(\frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right) \ \mathrm{dS}_x,
\end{aligned}$$

where

$$[[\varrho_h^k \langle \mathbf{u}_h^k \rangle]] = (\varrho_h^k)^{\mathrm{out}} \, [[\langle \mathbf{u}_h^k \rangle]] + \langle \mathbf{u}_h^k \rangle \, [[\varrho_h^k]] \, . \tag{4.23}$$

Indeed we have the triangle inequality along with the observation

$$\|\Pi_h^V[\boldsymbol{\phi}] - \boldsymbol{\phi}\|_{L^\infty(\Omega_h;R^3)} \lesssim h\|\nabla_x \boldsymbol{\phi}\|_{L^\infty}$$

as $\boldsymbol{\phi}$ is smooth.

Consequently,

$$\begin{aligned}
&\left| \frac{h^\alpha}{2} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k \langle \mathbf{u}_h^k \rangle]] \cdot [[\langle \Pi_h^V[\boldsymbol{\phi}] \rangle]]\, \chi\left(\frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right) \,\mathrm{dS}_x \right| \\
&\lesssim h^{1+\alpha} \left(\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \{\varrho_h^k\}\, [[\langle \mathbf{u}_h^k \rangle]]^2 \chi\left(\frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right) \,\mathrm{dS}_x \right)^{1/2} \left(\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \varrho_h^k \,\mathrm{dS}_x \right)^{1/2} \\
&+ h^{1+\alpha} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \left| \langle \mathbf{u}_h^k \rangle\, [[\varrho_h^k]] \right| \,\mathrm{dS}_x,
\end{aligned}$$

where the first integral on the right-hand side is controlled by the numerical dissipation in (3.5) and the trace estimates.

Finally, applying the inequality (4.6), trace inequality (4.7) and Sobolev's inequality (4.11), we obtain

$$\begin{aligned}
h^{1+\alpha} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \left| \langle \mathbf{u}_h^k \rangle\, [[\varrho_h^k]] \right| \,\mathrm{dS}_x &\lesssim h^{1+\alpha} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma |\langle \mathbf{u}_h^k \rangle| \left| \{\varrho_h^k\}^{1-\gamma/2} \right| \left| [[(\varrho_h^k)^{\gamma/2}]] \right| \,\mathrm{dS}_x \\
&\lesssim h^{1+\alpha} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \left(\int_\Gamma [[(\varrho_h^k)^{\gamma/2}]]^2 \,\mathrm{dS}_x \right)^{1/2} \| \langle \mathbf{u}_h^k \rangle \|_{L^6(\Gamma)} \left\| (\varrho_h^k)^{1-\gamma/2} \right\|_{L^3(\Gamma)} \\
&\lesssim h^{\frac{1+\alpha}{2}} \sum_{E\in E_h} \left(h^\alpha \int_{\partial E} [[(\varrho_h^k)^{\gamma/2}]]^2 \,\mathrm{dS}_x \right)^{1/2} \| \langle \mathbf{u}_h^k \rangle \|_{L^6(E)} \left\| (\varrho_h^k)^{1-\gamma/2} \right\|_{L^3(E)} \\
&\lesssim h^{\frac{1+\alpha}{2}} \|\nabla_h \mathbf{u}_h\|_{L^2(\Omega_h)} \left\| (\varrho_h^k)^{1-\gamma/2} \right\|_{L^3(\Omega_h)},
\end{aligned}$$

where we have used the numerical dissipation in (3.5). Thus, in order to complete the estimates we have to control

$$\left\| (\varrho_h^k)^{1-\gamma/2} \right\|_{L^3(\Omega_h)}$$

uniformly in k. As $1 < \gamma < 2$, it is enough to consider the critical case $\gamma = 1$, for which the inverse inequality (4.6) gives rise to

$$\left\| (\varrho_h^k)^{1/2} \right\|_{L^3(\Omega_h)} = \left\| \varrho_h^k \right\|_{L^{3/2}(\Omega_h)}^{1/2} \lesssim h^{-1/2} \|\varrho_h^k\|_{L^1(\Omega_h)}^{1/2}.$$

Step 2

Using (4.23) we deduce

$$\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\left\langle\Pi_h^V[\boldsymbol{\phi}]\right\rangle-\boldsymbol{\phi}\right)\cdot\left[\left[\varrho_h^k\left\langle\mathbf{u}_h^k\right\rangle\right]\right]\left[\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right]^-\,\mathrm{dS}_x$$
$$=\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\left\langle\Pi_h^V[\boldsymbol{\phi}]\right\rangle-\boldsymbol{\phi}\right)\cdot\Big((\varrho_h^k)^{\mathrm{out}}\left[\left[\left\langle\mathbf{u}_h^k\right\rangle\right]\right]+\left\langle\mathbf{u}_h^k\right\rangle\left[\left[\varrho_h^k\right]\right]\Big)\left[\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right]^-\,\mathrm{dS}_x,$$

where, furthermore,

$$\left|\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\left\langle\Pi_h^V[\boldsymbol{\phi}]\right\rangle-\boldsymbol{\phi}\right)(\varrho_h^k)^{\mathrm{out}}\left[\left[\left\langle\mathbf{u}_h^k\right\rangle\right]\right]\left[\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right]^-\,\mathrm{dS}_x\right|$$
$$\lesssim h^2\|\boldsymbol{\phi}\|_{C^2(\overline{\Omega}_h;R^3)}\left(\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}-(\varrho_h^k)^{\mathrm{out}}\left[\left[\left\langle\mathbf{u}_h^k\right\rangle\right]\right]^2\left[\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right]^-\,\mathrm{dS}_x\right)^{1/2}\times$$
$$\times\left(\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}(\varrho_h^k)^{\mathrm{out}}|\left\langle\mathbf{u}_h^k\right\rangle|\,\mathrm{dS}_x\right)^{1/2},$$

where the former integral in the product on the right-hand is controlled by the numerical dissipation in (3.5), while

$$\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}(\varrho_h^k)^{\mathrm{out}}|\left\langle\mathbf{u}_h^k\right\rangle|\,\mathrm{dS}_x\lesssim h^{-1}\|\mathbf{u}_h^k\|_{L^6(\Omega_h;R^3)}\|\varrho_h^k\|_{L^{6/5}(\Omega_h)},\lesssim h^{-3/2}\|\mathbf{u}_h^k\|_{L^6(\Omega_h;R^3)}\|\varrho_h^k\|_{L^1(\Omega_h)}.$$

Finally,

$$\left|\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\left\langle\Pi_h^V[\boldsymbol{\phi}]\right\rangle-\boldsymbol{\phi}\right)\cdot\left\langle\mathbf{u}_h^k\right\rangle\left[\left[\varrho_h^k\right]\right]\left[\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right]^-\,\mathrm{dS}_x\right|$$
$$\lesssim h^2\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\|\left\langle\mathbf{u}_h^k\right\rangle\|^2_{L^6(\Gamma)}\|\varrho_h^k\|_{L^{3/2}(\Gamma)}\lesssim h\|\mathbf{u}_h^k\|^2_{L^6(\Omega_h)}\|\varrho_h^k\|_{L^{3/2}(\Omega_h)}\lesssim h^{3-3/\gamma}\|\mathbf{u}_h^k\|^2_{L^6(\Omega_h)}\|\varrho_h^k\|_{L^\gamma(\Omega_h)},$$

where the exponent $3-3/\gamma>0$ as soon as $\gamma>1$.

Step 3

We write

$$\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\varrho_h^k\boldsymbol{\phi}\cdot\left\langle\mathbf{u}_h^k\right\rangle\left(\mathbf{u}_h^k\cdot\mathbf{n}-\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right)\,\mathrm{dS}_x$$
$$=\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\varrho_h^k(\boldsymbol{\phi}-\langle\boldsymbol{\phi}\rangle_\Gamma)\cdot\left\langle\mathbf{u}_h^k\right\rangle\left(\mathbf{u}_h^k\cdot\mathbf{n}-\left\langle\mathbf{u}_h^k\cdot\mathbf{n}\right\rangle_\Gamma\right)\,\mathrm{dS}_x,$$

where, by virtue of the trace inequality (4.7) and Poincaré's inequality (4.8),

$$\left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \varrho_h^k (\boldsymbol{\phi} - \langle \boldsymbol{\phi} \rangle_\Gamma) \cdot \left\langle \mathbf{u}_h^k \right\rangle \left(\mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right) \, \mathrm{dS}_x \right|$$
$$\lesssim h \left\| \sqrt{\varrho_h^k} \right\|_{L^\infty(\Omega_h)} \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \sqrt{\varrho_h^k} | \left\langle \mathbf{u}_h^k \right\rangle | \left| \mathbf{u}_h^k \cdot \mathbf{n} - \left\langle \mathbf{u}_h^k \cdot \mathbf{n} \right\rangle_\Gamma \right| \, \mathrm{dS}_x$$
$$\lesssim h \left\| \sqrt{\varrho_h^k} \right\|_{L^\infty(\Omega_h)} \| \sqrt{\varrho_h^k} \left\langle \mathbf{u}_h^k \right\rangle \|_{L^2(\Omega_h)} \| \nabla_h \mathbf{u}_h^k \|_{L^2(\Omega_h; R^3)},$$

where, in view of (4.10)

$$\left\| \sqrt{\varrho_h^k} \right\|_{L^\infty(\Omega_h)} \lesssim h^{-\frac{2+\alpha}{2\gamma}}, \text{ with} \frac{2+\alpha}{2\gamma} < 1 \text{ or } 0 < \alpha < 2(\gamma - 1).$$

Step 4

Finally,

$$\left| \int_{\Omega_h} \varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle \cdot \left(\left\langle \Pi_h^V [\boldsymbol{\phi}] \right\rangle - \boldsymbol{\phi} \right) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x \right|$$
$$\lesssim h^2 \left\| \sqrt{\varrho_h^k} \right\|_{L^\infty(\Omega_h)} \| \sqrt{\varrho_h^k} \left\langle \mathbf{u}_h^k \right\rangle \|_{L^2(\Omega_h)} \| \nabla_h \mathbf{u}_h^k \|_{L^2(\Omega_h; R^3)};$$

whence the rest of the proof follows exactly as in Step 3.

Summing up the previous observations, we obtain the consistency formulation of the momentum method (4.5).

Remark 4.1. As $\boldsymbol{\varphi}$ has compact support, equation (4.5) is satisfied also on the limit domain Ω for all h small enough.

Thus we have shown the following result.

Proposition 4.2. *Let the pressure p satisfy (1.5), with $1 < \gamma < 2$. Suppose that $[\varrho_h, \mathbf{u}_h]$ is a family of numerical solutions given through (4.1), (4.2), where $[\varrho_h^k, \mathbf{u}_h^k]$ satisfy (2.2–2.4), where*

$$\Delta t \approx h, \ 0 < \alpha < 2(\gamma - 1). \tag{4.24}$$

Then

$$- \int_{\Omega_h} \varrho_h^0 \varphi(0, \cdot) \ \mathrm{d}x = \int_0^T \int_{\Omega_h} \left[\varrho_h \partial_t \varphi + \varrho_h \mathbf{u}_h \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t + \mathcal{O}(h^\beta), \ \beta > 0,$$

for any test function $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}_h)$,

$$
\begin{aligned}
-\int_{\Omega_h} \varrho_h^0 \left\langle \mathbf{u}_h^0 \right\rangle \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x = & \int_0^T \int_{\Omega_h} \Big[\varrho_h \left\langle \mathbf{u}_h \right\rangle \partial_t \boldsymbol{\varphi} + \varrho_h \left\langle \mathbf{u}_h \right\rangle \otimes \mathbf{u}_h : \nabla_x \boldsymbol{\varphi} + p(\varrho_h) \mathrm{div}_x \boldsymbol{\varphi} \Big]\ \mathrm{d}x\ \mathrm{d}t \\
& - \int_0^T \int_{\Omega_h} \Big[\mu \nabla_h \mathbf{u}_h : \nabla_x \boldsymbol{\varphi} + (\mu/3 + \eta) \mathrm{div}_h \mathbf{u}_h \cdot \mathrm{div}_x \boldsymbol{\varphi} \Big]\ \mathrm{d}x\, \mathrm{d}t + \mathcal{O}(h^\beta),\ \beta > 0,
\end{aligned}
\tag{4.25}
$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,\infty) \times \Omega_h; R^3)$.

Moreover, the solution satisfies the energy inequality

$$
\begin{aligned}
\int_{\Omega_h} \left[\frac{1}{2} \varrho_h | \left\langle \mathbf{u}_h \right\rangle |^2 + P(\varrho_h) \right] (\tau, \cdot)\ \mathrm{d}x + \int_0^\tau \int_{\Omega_h} \mu |\nabla_h \mathbf{u}_h|^2 + (\mu/3 + \eta) |\mathrm{div}_h \mathbf{u}_h|^2\ \mathrm{d}x\ \mathrm{d}t \\
\leq \int_{\Omega_h} \left[\frac{1}{2} \varrho_h^0 | \left\langle \mathbf{u}_h^0 \right\rangle |^2 + P(\varrho_h^0) \right]\ \mathrm{d}x
\end{aligned}
\tag{4.26}
$$

for a.e. $\tau \in [0,T]$.

Remark 4.3. A close inspection of the previous discussion shows that the same method can be used to handle a variable time step Δt_k adjusted for each step of iteration by means of a CFL-type condition, such as $||\mathbf{u}_h^{k-1} + c_h^{k-1}||_{L^\infty(\Omega)} \Delta t_k / h \leq CFL$. Here $CFL \in (0,1]$ and $c_h^{k-1} \equiv \sqrt{p'(\rho_h^{k-1})}$ denotes the sound speed. Though this condition is necessary for stability of time-explicit numerical schemes, it still may be appropriate even for implicit schemes for areas of high-speed flows. Note that the only part that must be changed in the proof of Proposition 4.2 is Section 4.3.1, where the time derivative in the momentum method is estimated.

5 Measure-valued solutions

Our ultimate goal is to perform the limit $h \to 0$. For the sake of simplicity, we consider the initial data

$$
\varrho_0 \in L^\infty(R^3),\ \varrho^0 \geq \underline{\varrho} > 0 \text{ a.a. in } R^3,\ \mathbf{u}_0 \in L^2(R^3).
$$

With this ansatz, it is easy to find the approximation $[\varrho_h^0, \mathbf{u}_h^0]$ such that

$$
\begin{gathered}
\varrho_h^0 \to \varrho_0 \text{ in } L^\gamma_{\mathrm{loc}}(\Omega),\ \varrho_h^0 > 0,\ \int_{\Omega_h} \varrho_h^0 \phi\ \mathrm{d}x \to \int_\Omega \varrho_0 \phi\ \mathrm{d}x \text{ for any } \phi \in L^\infty(R^3), \\
\varrho_h^0 \left\langle \mathbf{u}_h^0 \right\rangle \to \varrho_0 \mathbf{u}_0 \text{ in } L^2_{\mathrm{loc}}(\Omega; R^3),\ \int_{\Omega_h} \varrho_h^0 \left\langle \mathbf{u}_h^0 \right\rangle \cdot \phi\ \mathrm{d}x \to \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \phi\ \mathrm{d}x \text{ for any } \phi \in L^\infty(R^3; R^3), \\
\int_{\Omega_h} \left[\frac{1}{2} \varrho_h^0 | \left\langle \mathbf{u}_h^0 \right\rangle |^2 + P(\varrho_h^0) \right]\ \mathrm{d}x \to \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right]\ \mathrm{d}x \text{ as } h \to 0.
\end{gathered}
\tag{5.1}
$$

5.1 Weak limit

Extending ϱ_h by $\underline{\varrho} > 0$ and $\mathbf{u}_h$ to be zero outside Ω_h, we may use the energy estimates (4.26) to deduce that, at least for suitable subsequences,

$$\begin{aligned} \varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),\ \varrho \geq 0 \\ \langle \mathbf{u}_h \rangle,\ \mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^3), \\ \text{where } \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega)),\ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}), \\ \varrho_h \langle \mathbf{u}_h \rangle \to \overline{\varrho \mathbf{u}} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)), \end{aligned}$$

see [7] or [8, Part II, Section 10.4].

Remark 5.1. Note that, by virtue of Poincaré's inequality (4.8) and the energy estimates (4.26),

$$\| \mathbf{u}_h - \langle \mathbf{u}_h \rangle \|_{L^2(0,T;L^2(K;R^3))} \stackrel{<}{\sim} h \text{ for any compact } K \in \Omega,$$

in particular, the weak limits of $\mathbf{u}_h$, $\langle \mathbf{u}_h \rangle$ coincide in Ω.

In addition, the limit functions satisfy the equation of continuity in the form

$$-\int_\Omega \varrho_0 \varphi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_\Omega \left[\varrho \partial_t \varphi + \overline{\varrho \mathbf{u}} \cdot \nabla_x \varphi \right]\ \mathrm{d}x\,\mathrm{d}t \tag{5.2}$$

for any test function $\varphi \in C_c^\infty([0,\infty)\times\overline{\Omega})$. It follows from (5.2) that $\varrho \in C_{\mathrm{weak}}([0,T];L^\gamma(\Omega))$; whence (5.2) can be rewritten as

$$\left[\int_\Omega \varrho \varphi(\tau,\cdot)\ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \overline{\varrho \mathbf{u}} \cdot \nabla_x \varphi \right]\ \mathrm{d}x\,\mathrm{d}t \tag{5.3}$$

for any $0 \leq \tau \leq T$ and any $\varphi \in C^\infty([0,T]\times\overline{\Omega})$.

5.2 Young measure generated by numerical solutions

The energy inequality (3.1), along with the consistency (4.4), (4.5) provide a suitable platform for the use of the theory of measure-valued solutions developed in [5]. Consider the family $[\varrho_h, \mathbf{u}_h]$. In accordance with the weak convergence statement derived in the preceding part, this family generates a Young measure - a parameterized measure

$$\nu_{t,x} \in L^\infty((0,T)\times\Omega;\mathcal{P}([0,\infty)\times R^3)) \text{ for a.a. } (t,x)\in(0,T)\times\Omega,$$

such that

$$\langle \nu_{t,x}, g(\varrho,\mathbf{u}) \rangle = \overline{g(\varrho,\mathbf{u})}(t,x) \text{ for a.a. } (t,x)\in(0,T)\times\Omega,$$

whenever $g \in C([0,\infty) \times R^3)$, and

$$g(\varrho_h, \mathbf{u}_h) \to \overline{g(\varrho, \mathbf{u})} \text{ weakly in } L^1((0,T) \times \Omega).$$

Moreover, in view of Remark 5.1, the Young measures generated by $[\varrho_h, \mathbf{u}_h]$ and $[\varrho, \langle \mathbf{u}_h \rangle]$ coincide for a.a. $(t,x) \in (0,T) \times \Omega$.

Accordingly, the equation of continuity (5.3) can be written as

$$\left[\int_\Omega \varrho \varphi(\tau, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \langle \nu_{t,x}, \varrho \mathbf{u} \rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t \tag{5.4}$$

In order to apply a similar treatment to the momentum equation (4.25), we have to replace the expression $\varrho_h \langle \mathbf{u}_h \rangle \otimes \mathbf{u}_h$ in the convective term by $\varrho_h \langle \mathbf{u}_h \rangle \otimes \langle \mathbf{u}_h \rangle$. This is possible as

$$\begin{aligned} &\| \varrho_h \langle \mathbf{u}_h \rangle \otimes \mathbf{u}_h - \varrho_h \langle \mathbf{u}_h \rangle \otimes \langle \mathbf{u}_h \rangle \|_{L^1(\Omega_h; R^{3\times 3})} = \| \varrho_h \langle \mathbf{u}_h \rangle \otimes (\mathbf{u}_h - \langle \mathbf{u}_h \rangle) \|_{L^1(\Omega_h; R^{3\times 3})} \\ &\lesssim h \| \sqrt{\varrho_h} \langle \mathbf{u}_h \rangle \|_{L^2(\Omega_h); R^3)} \| \nabla_h \mathbf{u}_h \|_{L^2(\Omega_h; R^{3\times 3})} \| \sqrt{\varrho_h} \|_{L^\infty(\Omega_h)}, \end{aligned}$$

where, by virtue of (4.10),

$$h \| \sqrt{\varrho_h} \|_{L^\infty(\Omega_h)} \lesssim h^{1 - \frac{2+\alpha}{2\gamma}},$$

where the exponent is positive as soon as (4.24) holds, specifically, $0 < \alpha < 2(\gamma - 1)$. Indeed the two terms on the right-hand side of (4.10) are controlled by means of the energy estimates (3.5).

Moreover, we have

$$\varrho_h \langle \mathbf{u}_h \rangle \otimes \langle \mathbf{u}_h \rangle + p(\varrho_h)\mathbb{I} \to \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho)\mathbb{I} \} \ \text{ weakly-(*) in } \ [L^\infty(0,T; \mathcal{M}(\Omega))]^{3\times 3};$$

whence letting $h \to 0$ in (4.25) gives rise to

$$\begin{aligned} - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x = &\int_0^T \int_\Omega \Big[\langle \nu_{t,x}; \varrho \mathbf{u} \rangle \, \partial_t \boldsymbol{\varphi} + \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho)\mathbb{I} \} : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \ \mathrm{d}t \\ &- \int_0^T \int_{\Omega_h} \Big[\mu \nabla \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\mu/3 + \eta) \mathrm{div} \mathbf{u} \cdot \mathrm{div}_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

or, equivalently,

$$\begin{aligned} \left[\int_\Omega \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = &\int_0^\tau \int_\Omega \Big[\langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \boldsymbol{\varphi} + \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho)\mathbb{I} \} : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \ \mathrm{d}t \\ &- \int_0^\tau \int_\Omega \Big[\mu \nabla \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\mu/3 + \eta) \mathrm{div} \mathbf{u} \cdot \mathrm{div}_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{5.5}$$

for any $0 \le \tau \le T$, $\varphi \in C_c^\infty([0,T] \times \Omega; R^3)$, where we have set

$$\nu_{0,x} = \delta_{[\varrho_0(x), \mathbf{u}_0(x)]}.$$

Finally, we introduce the *concentration remainder*

$$\mathcal{R} = \{\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}\} - \langle\nu_{t,x};\varrho\mathbf{u}\otimes\mathbf{u}+p(\varrho)\mathbb{I}\rangle \in [L^\infty(0,T;\mathcal{M}(\Omega))]^{3\times3}$$

and rewrite (5.5) in the form

$$\begin{aligned}
&\left[\int_\Omega \langle\nu_{t,x};\varrho\mathbf{u}\rangle\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&= \int_0^\tau\int_\Omega \Big[\langle\nu_{t,x};\varrho\mathbf{u}\rangle\cdot\partial_t\boldsymbol{\varphi} + \langle\nu_{t,x};\varrho\mathbf{u}\otimes\mathbf{u}\rangle : \nabla_x\boldsymbol{\varphi} + \langle\nu_{t,x},p(\varrho)\rangle\,\mathrm{div}_x\boldsymbol{\varphi}\Big]\ \mathrm{d}x\ \mathrm{d}t \qquad (5.6)\\
&- \int_0^\tau\int_\Omega \Big[\mu\nabla\mathbf{u}:\nabla_x\boldsymbol{\varphi} + (\mu/3+\eta)\mathrm{div}\mathbf{u}\cdot\mathrm{div}_x\boldsymbol{\varphi}\Big]\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \mathcal{R}:\nabla_x\varphi\ \mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for any $0\le\tau\le T$, $\varphi\in C_c^\infty([0,T]\times\Omega;R^3)$.

Similarly, the energy inequality (4.26) can be written as

$$\begin{aligned}
&\left[\int_\Omega\left[\frac{1}{2}\langle\nu_{t,x};\varrho|\mathbf{u}|^2+P(\varrho)\rangle\right]\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau\int_{\Omega_h}\mu|\nabla\mathbf{u}|^2+(\mu/3+\eta)|\mathrm{div}\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t \qquad (5.7)\\
&+\mathcal{D}(\tau)\le 0
\end{aligned}$$

for a.e. $\tau\in[0,T]$, with the *dissipation defect* $\mathcal{D}$ satisfying

$$\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)}\ \mathrm{d}t \lesssim \int_0^\tau\mathcal{D}(t)\ \mathrm{d}t,\ \mathcal{D}(\tau)\ge\liminf_{h\to\infty}\int_0^\tau\int_{\Omega_h}|\nabla_h\mathbf{u}_h|^2\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega|\nabla_x\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t, \quad (5.8)$$

cf. [5, Lemma 2.1].

At this stage, we recall the concept of *dissipative measure valued solution* introduced in [5]. These are measure–valued solutions of the Navier-Stokes system (1.1–1.4) satisfying the energy inequality (5.7), where the concentration remainder in the momentum equation is dominated by the dissipation defect as stated in (5.8) and the following analogue of Poincaré's inequality holds:

$$\lim_{h\to0}\int_0^\tau\int_{\Omega_h}|\mathbf{u}_h-\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t \le \liminf_{h\to\infty}\int_0^\tau\int_{\Omega_h}|\nabla_h\mathbf{u}_h|^2\ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega|\nabla_x\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t(\le\mathcal{D}(\tau)), \quad (5.9)$$

where $\mathbf{u}$ is a weak limit of $\mathbf{u}_h$, or, equivalently, of $\langle\mathbf{u}_h\rangle$. Consequently, relations (5.4), (5.6–5.8) imply that the Young measure $\{\nu_{t,x}\}_{t,x\in(0,T)\times\Omega}$ represents a dissipative measure-valued solution of the Navier-Stokes system (1.1–1.4) in the sense of [5] as soon as we check (5.9).

By standard Poincaré's inequality in Ω_h we get, on one hand,

$$\int_{\Omega_h}|\mathbf{u}_h-\mathbf{u}|^2\ \mathrm{d}x = \int_{\Omega_h}|\mathbf{u}_h-\Pi_h^V[\mathbf{u}]|^2\ \mathrm{d}x + \int_{\Omega_h}|\Pi_h^V[\mathbf{u}]-\mathbf{u}|^2\ \mathrm{d}x \lesssim \int_{\Omega_h}|\nabla_h\mathbf{u}_h-\nabla_h\Pi_h^V\mathbf{u}|^2\ \mathrm{d}x+\mathcal{O}(h^\beta).$$

On the other hand,

$$\liminf_{h\to\infty}\int_0^\tau\int_{\Omega_h}|\nabla_h\mathbf{u}_h|^2\ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega|\nabla_x\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t = \liminf_{h\to\infty}\int_0^\tau\int_{\Omega_h}|\nabla_h\mathbf{u}_h-\nabla_x\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t.$$

Thus it is enough to observe that, by virtue of (4.12),

$$\nabla_h \Pi_h^V[\mathbf{u}] \to \nabla_x \mathbf{u} \text{ (strongly) in } L^2(\Omega_h; R^3) \text{ whenever } \mathbf{u} \in W_0^{1,2}(\Omega; R^3).$$

Seeing that validity of (5.6) as well as the bound on the dissipation remainder (5.8) can be extended to the class of test functions $\varphi \in C^1([0,T]\times\overline{\Omega}; R^3)$, $\varphi|_{\partial\Omega} = 0$, we have shown the following result.

Theorem 5.2. *Let the pressure p satisfy (1.5), with $1 < \gamma < 2$. Suppose that $[\varrho_h, \mathbf{u}_h]$ is a family of numerical solutions given through (4.1), (4.2), where $[\varrho_h^k, \mathbf{u}_h^k]$ satisfy (2.2–2.4), where*

$$\Delta t \approx h,\ 0 < \alpha < 2(\gamma - 1),$$

and the initial data satisfy (5.1).

Then any Young measure $\{\nu_{t,x}\}_{t,x\in(0,T)\times\Omega}$ generated by $[\varrho_h^k, \mathbf{u}_h^k]$ for $h \to 0$ represents a dissipative measure-valued solution of the Navier-Stokes system (1.1–1.4) in the sense of [5].

Of course, the conclusion of Theorem 5.2 is rather weak, and, in addition, the Young measure need not be unique. On the other hand, however, we may use the weak-strong uniqueness principle established in [5, Theorem 4.1] to obtain our final convergence result.

Theorem 5.3. *In addition to the hypotheses of Theorem 5.2, suppose that the Navier-Stokes system (1.1–1.4) endowed with the initial data $[\varrho_0, \mathbf{u}_0]$ admits a regular solution $[\varrho, \mathbf{u}]$ belonging to the class*

$$\varrho,\ \nabla_x\varrho,\ \mathbf{u}, \nabla_x\mathbf{u} \in C([0,T]\times\overline{\Omega}),\ \partial_t\mathbf{u} \in L^2(0,T; C(\overline{\Omega}; R^3)),\ \varrho > 0,\ \mathbf{u}|_{\partial\Omega} = 0.$$

Then

$$\varrho_h \to \varrho \text{ (strongly) in } L^\gamma((0,T)\times K),\ \mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2((0,T)\times K; R^3)$$

for any compact $K \subset \Omega$.

Indeed, the weak–strong uniqueness implies that the Young measure generated by the family of numerical solutions coincides at each point (t,x) with the Dirac mass supported by the smooth solution of the problem. In particular, the numerical solutions converge strongly and no oscillations occur. Note that the Navier–Stokes system admits local-in-time strong solutions for arbitrary smooth initial data, see e.g. Cho et al. [2] , and even global-in-time smooth solutions for small initial data, see, e.g., Matsumura and Nishida [30], as soon as the physical domain Ω is sufficiently smooth.

6 Conclusions

We have studied the convergence of numerical solutions obtained by the mixed finite element–finite volume scheme applied to the isentropic Navier-Stokes equations. We have used the first order implicit time discretization. This approach yields large discrete nonlinear systems that have to be solved at each time step, which is computationally expensive. For practical applications this may be considered as a drawback. Consequently, the higher order space discretizations and suitable implicit-explicit time discretizations are typically used. In this paper we concentrate on the convergence analysis yielding the generalization to other types of discretization schemes for future research.

The main novelty here is the convergence proof for the case when the existence of a weak solution is still an open problem. We have assumed the isentropic pressure–density state equation $p(\varrho) = a\varrho^\gamma$ with $\gamma \in (1, 2)$. Remind that this assumption is not restrictive, since the largest physically relevant exponent is $\gamma = 5/3$. In order to establish the convergence result we have used the concept of dissipative measure-valued solutions. These are the measure-valued solutions, that, in addition, satisfy an energy inequality in which the dissipation defect measure dominates the concentration remainder in the equations. The energy inequality (3.1), along with the consistency (4.4), (4.5) gave us a suitable framework to apply the theory of measure-valued solutions. As shown in Section 5.2 the numerical solutions $[\varrho_h, \mathbf{u}_h]$ generate a Young measure - a parameterized measure $\{\nu_{t,x}\}_{t,x\in(0,T)\times\Omega}$, that represents a dissipative measure-valued solution of the Navier-Stokes system (1.1–1.4), cf. Theorem 5.2. Finally, using the weak-strong uniqueness principle established in [5, Theorem 4.1] we have obtained the convergence of the numerical solutions to the exact regular solution, as long as the latter exists, cf. Theorem 5.3. The present result is the first convergence result for numerical solutions of three-dimensional compressible isentropic Navier-Stokes equations in the case of full adiabatic exponent $\gamma \in (1, 2)$. An interesting question arises whether or not our numerical scheme converges to a *weak solution* in the range $3/2 < \gamma < 2$. Recall that Karper's convergence result applies only for $\gamma > 3$. The main stumbling block is that the presence of the numerical viscosity is not compatible with the refined pressure estimates as well as with the use of the oscillation defect measure necessary in the proof of convergence to a weak solution, see [8, Chapters 11,12].

In the context of *viscous fluids*, the use of the measure-valued solutions should be seen as a tool for proving the convergence rather than capturing the true behavior of the limit of a numerical scheme. This may change dramatically, however, in the inviscid or even slightly viscous fluids. In the light of very recent results by De Lellis, Székelyhidi and coauthors [1], [3], the set of weak solutions to the compressible Euler system is highly unstable, and, at least in the incompressible case, the weak solutions are in fact "dense" in the set of measure-valued solutions, see [33]. If we accept the vanishing viscosity process to yield admissible solutions to the Euler system, then, possibly, the measure-valued solutions might be the right concept.

To conclude, let us remark that the measure-valued solutions are relevant, even for the viscous fluids, in the case of oscillatory initial data. Indeed, as observed by Lions [29], the density oscillations propagate along characteristics generated by the velocity field in (1.1).

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