Asymptotic preserving error estimates for numerical solutions of compressible Navier-Stokes equations in the low Mach number regime

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Abstract

We study the convergence of numerical solutions of the compressible Navier-Stokes system to its incompressible limit. The numerical solution is obtained by a combined finite element-finite volume method based on the linear Crouzeix-Raviart finite element for the velocity and piecewise constant approximation for the density. The convective terms are approximated using upwinding. The distance between a numerical solution of the compressible problem and the strong solution of the incompressible Navier-Stokes equations is measured by means of a relative energy functional. For barotropic pressure exponent $\gamma \geq 3/2$ and for well-prepared initial data we obtain uniform convergence of order $\mathcal{O}(\sqrt{\Delta t}, h^a, \varepsilon)$, $a = \min\left\{\frac{2\gamma-3}{\gamma}, 1\right\}$. Extensive numerical simulations confirm that the numerical solution of the compressible problem converges to the solution of the incompressible Navier-Stokes equations as the discretization parameters Δt, h and the Mach number ε tend to zero.

Key words: Navier-Stokes system, finite element numerical method, finite volume numerical method, error estimates, asymptotic preserving schemes, low Mach number

AMS classification 35Q30, 65N12, 65N30, 76N10, 76N15, 76M10, 76M12

1 Introduction

We consider the scaled *compressible Navier-Stokes system* in the low Mach number regime:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0, \tag{1.1}$$

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$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}), \tag{1.2}$$

where ϱ and $\boldsymbol{u} \in R^3$ are the unknown density and velocity fields, while $\mathbb{S}$ and p stand for the viscous stress and pressure characterizing the material properties of the fluid via the constitutive relations

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \frac{2}{3}\mathrm{div}_x \boldsymbol{u}\mathbb{I} \right) + \eta \mathrm{div}\boldsymbol{u}\mathbb{I},\ \mu > 0,\ \eta \geq 0, \tag{1.3}$$

$$p \in C^2(0,\infty) \cap C^1[0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0, \tag{1.4}$$
$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0, \quad \lim_{\varrho \to 0+} \frac{p'(\varrho)}{\varrho^\alpha} = p_0 > 0$$

where $\gamma \geq 1$ and $\alpha \leq 1$. The (small) number $\varepsilon > 0$ is the Mach number. We notice that the assumptions (1.4) are compatible with the isentropic pressure law $p(\varrho) = \varrho^\gamma$ provided $1 \leq \gamma \leq 2$. To avoid problems with physical boundaries, we suppose that the fluid motion is space-periodic, specifically

$$\varrho = \varrho(t,x),\ \boldsymbol{u} = \boldsymbol{u}(t,x),\ t \in [0,T),\ x \in \mathbb{T}^3,\ \text{where } \mathbb{T}^3 = \Big([0,1]|_{\{0,1\}}\Big)^3.$$

The problem is completed by imposing the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \varrho_0 > 0 \text{ in } \mathbb{T}^3. \tag{1.5}$$

When $\varepsilon \to 0$, solutions of (1.1)–(1.5) are expected to approach solutions of the *incompressible Navier-Stokes system*

$$\overline{\varrho}\Big(\partial_t \boldsymbol{V} + \boldsymbol{V} \cdot \nabla_x \boldsymbol{V}\Big) + \nabla_x \Pi = \mu \Delta \boldsymbol{V},\ \mathrm{div}_x \boldsymbol{V} = 0,\ \overline{\varrho} > 0 \tag{1.6}$$

for suitable initial data

$$\boldsymbol{V}(0) = \boldsymbol{V}_0, \tag{1.7}$$

see e.g. the seminal work by Kleinerman and Majda [30]. Problem (1.6), (1.7) admits a regular solution in the class

$$\partial_t^l \boldsymbol{V} \in C^l([0,T]; W^{k-l,2}(\mathbb{T}^3; R^3)),\ l = 0,1,2,\ \partial_t^j \Pi \in C^j([0,T]; W^{k-1-j}(\mathbb{T}^3)),\ j = 0,1,\ k \geq 4 \tag{1.8}$$

defined on some maximal time interval $[0, T_{\max})$ provided

$$\boldsymbol{V}_0 \in W^{k,2}(\mathbb{T}^3; R^3),\ \mathrm{div}_x \boldsymbol{V}_0 = 0,$$

see e.g. [30]. The norm of $(\boldsymbol{V}, \Pi)$ in the function spaces (1.8) will be denoted by $\|\boldsymbol{V}, \Pi\|_{\mathcal{X}^k_{T,\mathbb{T}^3}}$.

Our goal is to estimate the distance between a discrete solution of a (convenient) numerical approximation of the compressible problem (1.1)–(1.5) and the exact solution $\boldsymbol{V}$ of the system (1.6), (1.7) in terms of the numerical mesh size h, the time step Δt, the Mach number ε, in a convenient discrete function space with a convenient projection to the discrete space of the unique strong solution of problem (1.6), (1.7) and a suitable norm of the difference of the initial data $(\varrho_0, \boldsymbol{u}_0)$ of the discrete compressible problem and $(\overline{\varrho}, \boldsymbol{V}_0)$ of the incompressible target problem. The multiplicative constant in this estimate must be independent of the specific numerical solution (and of course of h, Δt and ε); it may however depend on the norm of the strong solution $\boldsymbol{V}$. In particular, we shall not require any other extra restrictions on the numerical solution than those imposed by the numerical scheme and in this sense our error analysis results are *unconditional*.

Taking into account the low Mach number regime, this type of error estimates are in particular referred to as asymptotic preserving. The concept of *asymptotic preserving schemes* has been introduced by Jin et al., see [25], [28] and the references therein. A numerical scheme is called asymptotic preserving, if it is uniformly consistent with respect to a singular parameter - the Mach number ε in our case. In particular, the scheme reduces to a consistent approximation of the limit equation. In [2], [3], [4], [8], [27], [37] asymptotic preserving properties, in the sense of uniform consistency with respect to ε, has been studied for the Euler equations of gas dynamics and some related problems, see also [1], [17], [18], [31], [35], [36], [39] for some well-known low Mach number schemes. We would like to point out that these schemes are usually based on a suitable splitting between the stiff and non-stiff parts of the underlying PDE and the corresponding semi-implicit or implicit-explicit time discretization. To the best of our knowledge, asymptotic error estimates uniform with respect to all three parameters ε, Δt, h have not yet been obtained for any discrete scheme approximating fully nonlinear Euler or Navier-Stokes equations. On the other hand, in the case of a linear kinetic transport equation, such asymptotic error estimates in a diffusive scaling have been recently presented in [27], [34].

In spite of the importance of this property for applications, the mathematical literature on this subject is in short supply, mostly due to the complexity of the problem. In this paper we present an *unconditional and uniform* result for the compressible Navier-Stokes equations providing quantitatively *a uniform convergence rate* in terms of the space-time discretization parameters $(h, \Delta t)$ and the Mach number ε. In particular, our scheme is asymptotic preserving in the sense of Jin, cf. [28].

The relative energy method introduced on the continuous level in [11], [16], [15] and its numerical counterpart developed in Gallouët et al. [22] seems to provide a convenient strategy to achieve this goal. It is worth mentioning the recent work of Fischer [19], which can be seen as a counterpart of our approach. Specifically, he uses the relative energy functional to rigorously estimate the difference between an approximate (numerical) solution to the incompressible Navier-Stokes equations and any weak solution to the compressible Navier-Stokes system in the low Mach number regime.

We shall apply the relative energy technique to the rather academic numerical scheme suggested in Karper [29] with fully implicit time discretization, where the convergence to a weak solution was proved for $\gamma > 3$. Using the concept of the relative energy method the error estimates could be obtained for $\gamma > 3/2$ by Gallouët et al. [22]. The convergence result has been recently extended by Feireisl and Lukáčová [13] via dissipative measure valued solution to the physically relevant range $\gamma \in (1, 2)$. The application to other less academic numerical schemes, as mentioned above, is in the course of investigation.

2 Preliminaries

Before formulating our main result, it seems convenient to introduce the basic notation and known facts concerning the numerical apparatus used in the text.

2.1 Physical domain, mesh approximation

We assume that

$$\mathbb{T}^3 = \cup_{K \in \mathcal{T}_h} K, \tag{2.1}$$

where $\mathcal{T}_h$ is a family of closed tetrahedra (called *triangulation* of $\mathbb{T}^3$) having the following properties: If $K \cap L \neq \emptyset$, $K \neq L$, then $K \cap L$ is either a common face, or a common edge, or a common vertex. By $\mathcal{E}(K)$, we denote the set of the faces σ of the element $K \in \mathcal{T}_h$. The set of all faces of the mesh is denoted by $\mathcal{E}$. For each face of the mesh $\sigma = K|L$, $\boldsymbol{n}_{\sigma,K}$ stands for the normal vector of σ, oriented from K to L (so that $\boldsymbol{n}_{\sigma,K} = -\boldsymbol{n}_{\sigma,L}$), where $K|L$ denotes a common face. We denote by $|K|$ and $|\sigma|$ the (3 and 2 dimensional) Lebesgue measure of the tetrahedron K and of the face σ respectively, and by h_K and h_σ the diameter of K and σ respectively. We measure the regularity of the mesh by the parameter θ_h defined by

$$\theta_h = \inf \left\{ \frac{\xi_K}{h_K}, K \in \mathcal{T}_h \right\} \tag{2.2}$$

where ξ_K stands for the diameter of the largest ball included in K. We denote by h the maximal size of the mesh,

$$h = \max_{K\in\mathcal{T}} h_K. \tag{2.3}$$

The triangulation $\mathcal{T}_h$ is supposed to be regular and quasi-uniform, meaning that there exists positive real numbers c_0 and θ_0 independent of h such that

$$\theta_h \geq \theta_0 \text{ and } c_0 h \leq h_K, \tag{2.4}$$

respectively. In the sequel, we frequently omit the subscript h and write simply $\mathcal{T}$ instead of $\mathcal{T}_h$.

2.2 Spaces of discrete functions

The symbol $Q_h(\mathbb{T}^3)$ denotes the space of piecewise constant functions:

$$Q_h(\mathbb{T}^3) = \{q \in L^2(\mathbb{T}^3) \mid \ \forall K \in \mathcal{T},\ q_{|K} \in R\}; \tag{2.5}$$

for v in $C(\mathbb{T}^3)$, we set

$$v_K = \frac{1}{|K|}\int_K v\,\mathrm{d}x \text{ for } K \in \mathcal{T} \text{ and } \Pi_h^Q v(x) = \sum_{K\in\mathcal{T}} v_K 1_K(x),\ x \in \mathbb{T}^3, \tag{2.6}$$

where 1_K is the characteristic function of K.

We introduce the Crouzeix-Raviart finite element spaces:

$$V_h(\mathbb{T}^3) = \{v \in L^2(\mathbb{T}^3),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K),\ \forall\sigma \in \mathcal{E},\ \sigma = K|L,\ \int_\sigma v_{|K}\,\mathrm{d}S = \int_\sigma v_{|L}\,\mathrm{d}S\}, \tag{2.7}$$

where $\mathbb{P}_1(K)$ denotes the set of affine functions (on K), along with the associated projection

$$\Pi_h^V : C(\mathbb{T}^3) \to V_h(\mathbb{T}^3),\ \int_\sigma \Pi_h^V[\phi]\ \mathrm{dS}_x = \int_\sigma \phi\ \mathrm{dS}_x \text{ for all } \sigma \in \mathcal{E}. \tag{2.8}$$

For a Sobolev function $v \in W^{1,1}(\mathbb{T}^3)$ having well defined traces on the faces, we set

$$v_\sigma = \frac{1}{|\sigma|}\int_\sigma v\mathrm{d}S \text{ for } \sigma \in \mathcal{E}. \tag{2.9}$$

We recall that each function $v \in V_h(\mathbb{T}^3)$ can be written in the form

$$v(x) = \sum_{\sigma\in\mathcal{E}} v_\sigma \varphi_\sigma(x), \quad x \in \mathbb{T}^3, \tag{2.10}$$

where the set $\{\varphi_\sigma\}_{\sigma\in\mathcal{E}} \subset V_h(\mathbb{T}^3)$ is the Crouzeix-Raviart basis determined by

$$\forall(\sigma,\sigma') \in \mathcal{E}^2,\ \frac{1}{|\sigma'|}\int_{\sigma'} \varphi_\sigma\,\mathrm{d}S = \delta_{\sigma,\sigma'}. \tag{2.11}$$

Next, we recall the standard properties of the projection Π_h^V summarized in Lemmas 2.1–2.5 below; the relevant proofs can be found in the Appendix of [22]. The reader may consult also the monograph Boffi, Brezzi and Fortin [5], the Crouzeix and Raviart paper [7], or Gallouët, Herbin and Latché [21].

Lemma 2.1. *The following estimates hold true:*

$$\|\Pi_h^V[\phi]\|_{L^\infty(K)} \stackrel{<}{\sim} \|\phi\|_{L^\infty(K)}, \tag{2.12}$$

for all $K \in \mathcal{T}$ and $\phi \in C(K)$;

$$\|\phi - \Pi_h^V[\phi]\|_{L^p(K)} \stackrel{<}{\sim} h^s\|\nabla^s\phi\|_{L^p(K;R^{3^s})},\ s = 1,2,\ 1 \leq p \leq \infty, \tag{2.13}$$

and

$$\|\nabla(\phi - \Pi_h^V[\phi])\|_{L^p(K;R^d)} \leq ch^{s-1}\|\nabla^s\phi\|_{L^p(K;R^{3^s})},\ s = 1,2,\ 1 \leq p \leq \infty, \tag{2.14}$$

for all $K \in \mathcal{T}$ and $\phi \in C^s(K)$.

Remark 2.1. *Here and hereafter the expression $a \lesssim b$ means the inequality $a \le cb$, where the positive number c may depend on the Sobolev and Lebesgue exponents involved in a and b (here s, p), the diameter of $\mathbb{T}^3$, the Lebesgue measure $|\mathbb{T}^3|$ of $\mathbb{T}^3$ and θ_0. In particular, it is independent of θ, h, Δt, ε and $\mathcal{T}_h$.*

Lemma 2.2. *Let $1 \le p < \infty$. Then*

$$h \sum_{\sigma \in \mathcal{E}} |\sigma| \ |v_\sigma|^p \lesssim ||v||^p_{L^p(\mathbb{T}^3)}, \tag{2.15}$$

with any $v \in V_h(\mathbb{T}^3)$.

We shall need the following version of [12, Lemma 2.2] (see also [14, Theorem 10.17]).

Lemma 2.3. *Let $r \ge 0$ be such that*

$$0 < \int_{\mathbb{T}^3} r \ \mathrm{d}x = a, \ \int_{\mathbb{T}^3} r^\gamma \ \mathrm{d}x \le b, \ \text{for } \gamma > 1.$$

Then the following Poincaré–Sobolev type inequality holds true:

$$||v||^2_{L^6(\mathbb{T}^3)} \lesssim \sum_{K \in \mathcal{T}} \int_K |\nabla_x v|^2 \mathrm{d}x + \left(\int_{\mathbb{T}^3} r|v| \ \mathrm{d}x \right)^2, \tag{2.16}$$

for any $v \in V_h(\mathbb{T}^3)$, where the constant c depends on a and b but not on h.

Lemma 2.4. ([21, Lemma 2.1]) *There holds:*

$$\sum_{K \in \mathcal{T}} \int_K q \ \operatorname{div} \Pi_h^V[\boldsymbol{v}] \, \mathrm{d}x = \int_{\mathbb{T}^3} q \ \operatorname{div} \boldsymbol{v} \, \mathrm{d}x, \tag{2.17}$$

for all $\boldsymbol{v} \in C^1(\mathbb{T}^3, R^d)$ and all $q \in Q_h(\mathbb{T}^3)$. In particular, if $\operatorname{div}\boldsymbol{v} = 0$, then $\operatorname{div} \Pi_h^V[\boldsymbol{v}] = 0$ on each triangle $K \in \mathcal{T}$.

Lemma 2.5. (Jumps over faces in the Crouzeix-Raviart space [21, Lemma 2.2]) *For all $v \in V_h(\mathbb{T}^3)$ there holds*

$$\sum_{\sigma \in \mathcal{E}} \frac{1}{h} \int_\sigma [v]^2_{\sigma, \boldsymbol{n}_\sigma} \, \mathrm{d}S \lesssim \sum_{K \in \mathcal{T}} \int_K |\nabla_x v|^2 \mathrm{d}x, \tag{2.18}$$

where $[v]_{\sigma, \boldsymbol{n}_\sigma}$ is a jump of v with respect to a normal $\boldsymbol{n}_\sigma$ to the face σ,

$$\forall x \in \sigma = K|L \in \mathcal{E}, \quad [v]_{\sigma, \boldsymbol{n}_\sigma}(x) = \begin{cases} v|_K(x) - v|_L(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,K} \\ v|_L(x) - v|_K(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,L}. \end{cases}$$

We will frequently use the Poincaré, Sobolev and interpolation inequalities on tetrahedra reported in the following lemma. We refer to the book Boffi et al. [5] for a general introduction to this subject, see also [22, Lemma 9.1].

Lemma 2.6.

(1) We have,

$$\|v - v_K\|_{L^p(K)} \lesssim h \|\nabla v\|_{L^p(K;R^3)} \text{ for all } K \in \mathcal{T}, \tag{2.19}$$

$$\|v - v_\sigma\|_{L^p(K)} \lesssim h \|\nabla v\|_{L^p(K;R^3)} \text{ for all } \sigma \in \mathcal{E}(K), \tag{2.20}$$

$v \in W^{1,p}(K)$, where $1 \le p \le \infty$.

(2) There holds

$$\|v - v_K\|_{L^{p^*}(K)} \lesssim \|\nabla v\|_{L^p(K;R^3)} \text{ for all } K \in \mathcal{T}, \tag{2.21}$$

$$\|v - v_\sigma\|_{L^{p^*}(K)} \lesssim \|\nabla v\|_{L^p(K;R^3)} \text{ for all } \sigma \in \mathcal{E}(K), \tag{2.22}$$

$v \in W^{1,p}(K)$, $1 \leq p < 3$, where $p^* = \frac{3p}{3-p}$.

(3) We have,

$$\|v - v_K\|_{L^q(K)} \lesssim h^\beta \|\nabla v\|_{L^p(K;R^3)} \text{ for all } K \in \mathcal{T}, \tag{2.23}$$

$$\|v - v_\sigma\|_{L^q(K)} \lesssim h^\beta \|\nabla v\|_{L^p(K;R^3)} \text{ for all } \sigma \in \mathcal{E}(K), \tag{2.24}$$

$v \in W^{1,p}(K)$, $1 \leq p < 3$, where $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{p^*}$, $p \leq q \leq p^*$.

The following estimates are easy to obtain by means of scaling arguments:

$$\|v\|_{L^q(\partial K)} \lesssim \frac{1}{h^{1/q}} \left(\|v\|_{L^q(K)} + h\|\nabla_x v\|_{L^q(K;R^3)} \right), \ 1 \leq q \leq \infty \text{ for any } v \in C^1(K), \tag{2.25}$$

from which we readily deduce that

$$\|w\|_{L^q(\partial K)} \lesssim \frac{1}{h^{1/q}} \|w\|_{L^q(K)} \text{ for any } 1 \leq q \leq \infty, \ w \in P_m, \tag{2.26}$$

where P_m denotes the space of polynomials of degree m.

In a similar way, we obtain

$$\|w\|_{L^p(K)} \lesssim h^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(K)}, \ 1 \leq q < p \leq \infty, \ w \in P_m, \tag{2.27}$$

and making use of the algebraic inequality (2.30) we deduce the global version

$$\|w\|_{L^p(\mathbb{T}^3)} \lesssim h^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(\mathbb{T}^3)}, \ 1 \leq q < p \leq \infty, \text{ for any } w|_K \in P_m, \ K \in \mathcal{T}. \tag{2.28}$$

For future use, we record a version of (2.27) and (2.28) for the functions of the time variable $t \in (0, T)$,

$$\|w\|_{L^p(0,T)} \lesssim (\Delta t)^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(0,T)}, \ 1 \leq q < p \leq \infty, \tag{2.29}$$

where w is piecewise constant in time.

We finish the section of preliminaries by recalling two usefull algebraic inequalities: the 'imbedding' inequality

$$\left(\sum_{i=1}^L |a_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^L |a_i|^q \right)^{1/q}, \tag{2.30}$$

for all $a = (a_1, \ldots, a_L) \in \mathbb{R}^L$, $1 \leq q \leq p < \infty$, and the discrete Hölder inequality

$$\sum_{i=1}^L |a_i||b_i| \leq \left(\sum_{i=1}^L |a_i|^q \right)^{1/q} \left(\sum_{i=1}^L |a_i|^p \right)^{1/p}, \tag{2.31}$$

for all $a = (a_1, \ldots, a_L) \in \mathbb{R}^L$, $b = (b_1, \ldots, b_L) \in R^L$, $\frac{1}{q} + \frac{1}{p} = 1$.

3 Main results

We will systematically use the following abbreviated notation:

$$\hat{\phi} = \Pi_h^Q[\phi], \quad \phi_h = \Pi_h^V[\phi], \tag{3.1}$$

where the projections Π_h^Q, Π_h^V are defined in (2.6) and (2.8). For a function $v \in C([0,T], L^1(\mathbb{T}^3))$ we set

$$v^n(x) = v(t_n, x), \tag{3.2}$$

where $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1} < \cdots < t_N = T$ is a partition of the interval $[0,T]$. Finally, for a function $v \in V_h(\mathbb{T}^3)$ we denote

$$\nabla_h v(x) = \sum_{K\in\mathcal{T}} \nabla_x v(x) 1_K(x), \quad \mathrm{div}_h \boldsymbol{v}(x) = \sum_{K\in\mathcal{T}} \mathrm{div}_x \boldsymbol{v}(x) 1_K(x). \tag{3.3}$$

To ensure positivity of the approximate densities, we shall use an upwinding technique to discretize the convective term in the mass equation. For $q \in Q_h(\mathbb{T}^3)$ and $\boldsymbol{u} \in \boldsymbol{V}_h(\mathbb{T}^3; R^3)$, the upwinding of q with respect to $\boldsymbol{u}$ is defined, for $\sigma = K|L \in \mathcal{E}$, by

$$q_\sigma^{\mathrm{up}} = \begin{cases} q_K \text{ if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} > 0 \\ q_L \text{ if } \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} \leq 0. \end{cases} \tag{3.4}$$

3.1 Numerical scheme

Solutions $(\varrho, \boldsymbol{u})$ of the scaled system (1.1)–(1.5) will be approximated by $(\varrho^n, \boldsymbol{u}^n) = (\varrho^{n,(\Delta t,h,\varepsilon)}, \boldsymbol{u}^{n,(\Delta t,h,\varepsilon)})$ satisfying the following system of algebraic equations (numerical scheme):

For given initial data $(\varrho^0, \boldsymbol{u}^0)$ we define

$$\varrho^n \in Q_h(\mathbb{T}^3), \ \varrho^n > 0, \ \boldsymbol{u}^n \in V_h(\mathbb{T}^3; R^3), \quad n = 0, 1, \ldots, N, \tag{3.5}$$

$$\sum_{K\in\mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} \phi_K + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}) \phi_K = 0 \text{ for any } \phi \in Q_h(\mathbb{T}^3) \text{ and } n = 1, \ldots, N, \tag{3.6}$$

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t} \Big(\varrho_K^n \hat{\boldsymbol{u}}_K^n - \varrho_K^{n-1} \hat{\boldsymbol{u}}_K^{n-1} \Big) \cdot \boldsymbol{v}_K + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \cdot \boldsymbol{v}_K \\ &- \frac{1}{\varepsilon^2} \sum_{K\in\mathcal{T}} p(\varrho_K^n) \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{v}_\sigma \cdot \boldsymbol{n}_{\sigma,K} + \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v} \ \mathrm{d}x \\ &+ \Big(\frac{\mu}{3} + \eta \Big) \sum_{K\in\mathcal{T}} \int_K \mathrm{div} \boldsymbol{u}^n \mathrm{div} \boldsymbol{v} \ \mathrm{d}x = 0, \ \text{for any } \boldsymbol{v} \in V_h(\mathbb{T}^3; R^3) \text{ and } n = 1, \ldots, N. \end{aligned} \tag{3.7}$$

Although the numerical solutions depend on the size h of the space discretization, the time step Δt, and the Mach number ε, we shall use a concise notation $(\varrho^n, \boldsymbol{u}^n)$ instead of $(\varrho^{n,(\Delta t,h,\varepsilon)}, \boldsymbol{u}^{n,(\Delta t,h,\varepsilon)})$.

The numerical method (3.5)–(3.7) was proposed in [29, Definition 3.1]; it is strongly nonlinear and implicit. As shown by Karper [29, Proposition 3.3], problem (3.5–3.7) admits a solution $(\varrho_h^n, \boldsymbol{u}_h^n)$,

$$\varrho_h^n \in Q_h(\mathbb{T}^3), \ \boldsymbol{u}_h^n \in V_h(\mathbb{T}^3; R^3), \ n = 0, 1, \ldots, N,$$

for any fixed $h > 0$, $\Delta t > 0$, and, moreover, $\varrho_h^n > 0$, $n = 1, \ldots, N$, provided $\varrho_h^0 > 0$. The proof uses topological degree theory in the spirit of [20].

3.2 Error estimates

We introduce the *relative energy functional*

$$\mathcal{E}_\varepsilon(\varrho, \boldsymbol{u}|z, \boldsymbol{v}) = \int_{\mathbb{T}^3} \Big(\varrho|\boldsymbol{u} - \boldsymbol{v}|^2 + \frac{1}{\varepsilon^2} E(\varrho|z) \Big) \mathrm{d}x, \tag{3.8}$$

where

$$E(\varrho|z) = H(\varrho) - H'(z)(\varrho - z) - H(z), \ H(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s. \tag{3.9}$$

Note that under the assumption $p'(\varrho) > 0$, the function $\varrho \mapsto H(\varrho)$ is strictly convex in $(0, \infty)$; whence

$$E(\varrho|z) \geq 0 \ \text{ and } \ E(\varrho|z) = 0 \ \Leftrightarrow \ \varrho = z.$$

We are ready to state the main result of the paper.

Theorem 3.1. *Let the pressure satisfy (1.4) with* $\gamma \geq 3/2$. *Let* $\{\varrho^n, \boldsymbol{u}^n\}_{0 \leq n \leq N}$ *be a family of numerical solutions resulting from the scheme (3.5)–(3.7), where the mesh satisfies assumptions* (2.1)-(2.4) *and initial data* $(\varrho^0, \boldsymbol{u}^0)$ *obey*

$$\mathcal{E}_\varepsilon(\varrho^0_\varepsilon, \boldsymbol{u}^0_\varepsilon \Big| \overline{\varrho}, \boldsymbol{V}_0) \leq E_0 < \infty, \ \ M_0/2 \leq \int_{\mathbb{T}^3} \varrho^0_\varepsilon \mathrm{d}x \leq 2M_0, \ \ M_0 = \overline{\varrho}|\mathbb{T}^3| \tag{3.10}$$

with $M_0 > 0$, $E_0 > 0$ *and* $\overline{\varrho} > 0$ *independent of* $\varepsilon, h, \Delta t$. *Moreover, suppose that* $[\Pi, \boldsymbol{V}]$ *is a classical solution to the initial-boundary value problem (1.6), (1.7) in* $[0, T] \times \mathbb{T}^3$ *in the regularity class (1.8), emanating from the initial data* $\boldsymbol{V}_0 \in W^{k,2}(\mathbb{T}^3; R^3)$, $\mathrm{div}_x \boldsymbol{V}_0 = 0$, $k \geq 4$.

Then there exists a positive number

$$c = c\Big(M_0, E_0, \overline{\varrho}, |p'|_{C^1[\overline{\varrho}/2, 2\overline{\varrho}]}, \|\boldsymbol{V}, \Pi\|_{\mathcal{X}^k_{T,\mathbb{T}^3}}\Big), \tag{3.11}$$

such that

$$\begin{aligned} \sup_{1 \leq n \leq N} \mathcal{E}_\varepsilon\Big(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \overline{\varrho}, \boldsymbol{V}(t_n, \cdot)\Big) + \Delta t \sum_{1 \leq n \leq N} \int_{\mathbb{T}^3} |\nabla_h \boldsymbol{u}^n - \nabla_x \boldsymbol{V}(t_n, \cdot)|^2 \ \mathrm{d}x \qquad (3.12) \\ \leq c \left(\sqrt{\Delta t} + h^a + \varepsilon + \mathcal{E}_\varepsilon\Big(\varrho^0_\varepsilon, \hat{\boldsymbol{u}}^0_\varepsilon \Big| \overline{\varrho}, \boldsymbol{V}_0\Big) \right), \end{aligned}$$

where

$$a = \min\left\{ \frac{2\gamma - 3}{\gamma}, 1 \right\}. \tag{3.13}$$

Here and hereafter, we denote by c a generic positive constant that may depend on the parameters from Remark 2.1 and also on T. Its possible dependence on other parameters is always explicitly indicated in the argument of c (as in (3.11)). Moreover, c is always independent of ε, Δt h, θ and $\mathcal{T}_h$.

Remark 3.1.

- *If the initial data are ill prepared, specifically*

$$E(\varrho^0_\varepsilon|\overline{\varrho}) \stackrel{<}{\sim} \varepsilon^2, \quad \int_{\mathbb{T}^3} \varrho^0_\varepsilon |\hat{\boldsymbol{u}}^0_\varepsilon - \boldsymbol{V}_0|^2 \ \mathrm{d}x \stackrel{<}{\sim} 1,$$

we obtain just an upper bound on the error.

On the other hand, if the initial data are well prepared, meaning

$$E(\varrho_\varepsilon^0|\overline{\varrho}) \lesssim \varepsilon^{2+\xi}, \quad \int_{\mathbb{T}^3} \varrho_\varepsilon^0 |\hat{\boldsymbol{u}}_\varepsilon^0 - \boldsymbol{V}_0|^2 \ \mathrm{d}x \lesssim \varepsilon^\xi, \ \xi > 0,$$

Theorem 3.1 gives uniform convergence as $(h, \Delta t, \varepsilon) \to 0$ *of the numerical solution to the strong solution of the incompressible Navier-Stokes equations, including a rate of convergence on the time interval on which the strong solution is known to exist. The distance is measured in the "norm" induced by the relative energy functional, namely*

$$\mathcal{E}_\varepsilon\Big(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \overline{\varrho}, \boldsymbol{V}^n\Big) \approx \int_{\mathbb{T}^3} \varrho^n |\hat{\boldsymbol{u}}^n - \boldsymbol{V}(t_n)|^2 \ \mathrm{d}x + \left\| \frac{\varrho^n - \overline{\varrho}}{\varepsilon} \right\|^2_{L^q(\mathbb{T}^3)}, \ q = \min\{2, \gamma\}.$$

- *Theorem 3.1 holds also in the 2D case for any* $0 \le a < \frac{2\gamma - 2}{\gamma}$ *if* $\gamma \in (1, 2]$ *and* $a = 1$ *if* $\gamma > 2$. *Note that in this case the limit system (1.6), (1.7) admits global-in-time smooth solutions as long as the initial data are regular.*

The rest of the paper is devoted to the proof of Theorem 3.1. The main tool is a discrete version of the *relative energy inequality* derived in [22]. To simplify the presentation, we drop the index ε in the notation used for the initial data.

4 Uniform estimates

If we take $\phi = 1$ in the formula (3.6) we get immediately conservation of mass:

$$\forall n = 1, \dots, N, \quad \int_{\mathbb{T}^3} \varrho^n \, \mathrm{d}x = \int_{\mathbb{T}^3} \varrho^0 \, \mathrm{d}x. \tag{4.1}$$

It is absolutely crucial for the subsequent discussion that the numerical scheme (3.5)–(3.7) gives rise to a discrete version of the energy inequality stated below. The reader can consult Section 4.1 in Gallouët et al. [20, Lemma 4.1] for its laborious but rather straightforward proof.

Lemma 4.1. *Let* $(\varrho^n, \boldsymbol{u}^n)$ *be a solution of the discrete problem (3.5)–(3.7) with the pressure p satisfying (1.4). Then there exist*

$$\overline{\varrho}_\sigma^n \in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)], \ \sigma = K|L \in \mathcal{E}, \ n = 1, \dots, N,$$
$$\overline{\varrho}_K^{n-1,n} \in [\min(\varrho_K^{n-1}, \varrho_K^n), \max(\varrho_K^{n-1}, \varrho_K^n)], \ K \in \mathcal{T}, \ n = 1, \dots, N,$$

such that

$$\sum_{K \in \mathcal{T}} |K| \Big(\frac{1}{2} \varrho_K^m |\boldsymbol{u}_K^m|^2 + \frac{1}{\varepsilon^2} E(\varrho_K^m | \overline{\varrho}) \Big) - \sum_{K \in \mathcal{T}} |K| \Big(\frac{1}{2} \varrho_K^0 |\boldsymbol{u}_K^0|^2 + \frac{1}{\varepsilon^2} E(\varrho_K^0 | \overline{\varrho}) \Big)$$
$$+ \Delta t \sum_{n=1}^m \sum_{K \in \mathcal{T}} \Big(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2 \, \mathrm{d}x + (\frac{\mu}{3} + \eta) \int_K |\mathrm{div} \boldsymbol{u}^n|^2 \, \mathrm{d}x \Big)$$
$$+ [D_{\mathrm{time}}^{m,|\Delta \boldsymbol{u}|}] + [D_{\mathrm{time}}^{m,|\Delta \varrho|}] + [D_{\mathrm{space}}^{m,|\Delta \boldsymbol{u}|}] + [D_{\mathrm{space}}^{m,|\Delta \varrho|}] = 0, \tag{4.2}$$

for all $m = 1, \ldots, N$, with the dissipation defect

$$[D_{\rm time}^{m,|\Delta \boldsymbol{u}|}] = \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{|\boldsymbol{u}_K^n - \boldsymbol{u}_K^{n-1}|^2}{2}, \tag{4.3a}$$

$$[D_{\rm time}^{m,|\Delta \varrho|}] = \frac{1}{\varepsilon^2} \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| H''(\overline{\varrho}_K^{n-1,n}) \frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}, \tag{4.3b}$$

$$[D_{\rm space}^{m,|\Delta \boldsymbol{u}|}] = \Delta t \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}} |\sigma| \varrho_\sigma^{n,\rm up} \frac{(\boldsymbol{u}_K^n - \boldsymbol{u}_L^n)^2}{2} \, |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|, \tag{4.3c}$$

$$[D_{\rm space}^{m,|\Delta \varrho|}] = \frac{\Delta t}{\varepsilon^2} \sum_{n=1}^{m} \sum_{\sigma=K|L\in\mathcal{E}} |\sigma| H''(\overline{\varrho}_\sigma^n) \frac{(\varrho_K^n - \varrho_L^n)^2}{2} \, |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|. \tag{4.3d}$$

Lemma 4.1 yields the following corollary.

Corollary 4.1. *Under the assumptions of Lemma 4.1, and provided (3.10) holds, there exists a constant $c = c(M_0, E_0) > 0$ (independent of n, h and Δt) such that*

$$\Delta t \sum_{n=1}^{N} \int_K |\nabla_x \boldsymbol{u}^n|^2 \, {\rm d}x \le c, \tag{4.4}$$

$$\Delta t \sum_{n=1}^{N} \|\boldsymbol{u}^n\|^2_{L^6(\mathbb{T}^3;R^3)} \le c, \tag{4.5}$$

$$\sup\nolimits_{n=0,\ldots N} \|\varrho^n |\hat{\boldsymbol{u}}^n|^2\|_{L^1(\mathbb{T}^3)} \le c. \tag{4.6}$$

$$\sup\nolimits_{n=0,\ldots N} \|\varrho^n\|_{L^\gamma(\mathbb{T}^3)} \le c, \tag{4.7}$$

$$\sup\nolimits_{n=0,\ldots N} \int_{\mathbb{T}^3} E(\varrho^n|\overline{\varrho}) \le c\varepsilon^2. \tag{4.8}$$

$$\Delta t \sum_{n=1}^{N} \mathcal{D}(\varrho^n, \boldsymbol{u}^n) \le c\varepsilon^2, \tag{4.9}$$

where

$$\mathcal{D}(\varrho^n, \boldsymbol{u}^n)$$

$$= \sum_{\sigma=K|L\in\mathcal{E}} |\sigma| \frac{(\varrho_K^n - \varrho_L^n)^2}{[\max(\varrho_K^n, \varrho_L^n)]^{(2-\gamma)^+}} 1_{\{\overline{\varrho}_\sigma^n \ge 1\}} \, |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}| + \sum_{\sigma=K|L\in\mathcal{E}} |\sigma| (\varrho_K^n - \varrho_L^n)^2 1_{\{\overline{\varrho}_\sigma^n < 1\}} \, |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|$$

Proof
If condition (3.10) is satisfied, then the right-hand side of the energy inequality (4.2) is uniformly bounded. Estimates (4.4), (4.6)-(4.8) thus follow directly from this inequality. Estimate (4.5) can be deduced from (4.1), (4.4), (4.6), (4.7), the discrete Sobolev inequality (2.16), and the discrete Poincaré inequality established in Lemma 2.3. Finally, estimate (4.9) follows from the bound (4.3d) for the upwind dissipation by standard elementary reasoning employing assumption (1.4).

5 Discrete relative energy inequality

The starting point of our error analysis is the discrete relative energy inequality for the numerical scheme (3.5)–(3.7) derived in Gallouët et al. [22, Theorem 5.1], slightly modified in order to accommodate the presence of the small scaling parameter ε.

Lemma 5.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) with the pressure p satisfying (1.4). Then there holds for all $m = 1, \dots, N$,*

$$\begin{aligned}\sum_{K\in\mathcal{T}} \frac{1}{2}|K|\Big(\varrho_K^m|\boldsymbol{u}_K^m - \boldsymbol{U}_K^m|^2 - \varrho_K^0|\boldsymbol{u}_K^0 - \boldsymbol{U}_K^0|^2\Big) + \frac{1}{\varepsilon^2}\sum_{K\in\mathcal{T}}|K|\Big(E(\varrho_K^m|r_K^m) - E(\varrho_K^0|r_K^0)\Big) \\ + \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\Big(\mu\int_K|\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2\,\mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K|\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2\,\mathrm{d}x\Big) \le \sum_{i=1}^6 T_i,\end{aligned} \tag{5.1}$$

for any $0 < r^n \in Q_h(\mathbb{T}^3)$, $\boldsymbol{U}^n \in V_h(\mathbb{T}^3; R^3)$, $n = 1, \dots, N$, where

$$\begin{aligned}
T_1 &= \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\Big(\mu\int_K\nabla_x\boldsymbol{U}^n : \nabla_x(\boldsymbol{U}^n - \boldsymbol{u}^n)\,\mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K\mathrm{div}\boldsymbol{U}^n\mathrm{div}(\boldsymbol{U}^n - \boldsymbol{u}^n)\,\mathrm{d}x\Big),\\
T_2 &= \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}\frac{\boldsymbol{U}_K^n - \boldsymbol{U}_K^{n-1}}{\Delta t}\cdot\Big(\frac{\boldsymbol{U}_K^{n-1}+\boldsymbol{U}_K^n}{2} - \boldsymbol{u}_K^{n-1}\Big),\\
T_3 &= \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\frac{\boldsymbol{U}_K^n+\boldsymbol{U}_L^n}{2} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big)\cdot\boldsymbol{U}_K^n[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}],\\
T_4 &= -\frac{\Delta t}{\varepsilon^2}\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|p(\varrho_K^n)[\boldsymbol{U}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}],\\
T_5 &= \frac{\Delta t}{\varepsilon^2}\sum_{n=1}^m\sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}(r_K^n - \varrho_K^n)\Big(H'(r_K^n) - H'(r_K^{n-1})\Big),\\
T_6 &= \frac{\Delta t}{\varepsilon^2}\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\varrho_\sigma^{n,\mathrm{up}}H'(r_K^{n-1})[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}].
\end{aligned} \tag{5.2}$$

Remark 5.1. *It is worth noting that the relative energy inequality in the form (5.1) holds for any pair of "test functions" r^n, $\boldsymbol{U}^n$ that can be chosen conveniently according to the expected asymptotic limit.*

6 Approximate discrete relative energy inequality

The next step is to take the discrete relative energy inequality (5.1) with test function $(\overline{\varrho}, \boldsymbol{V}_h^n)$, where $\boldsymbol{V}$ belongs to the same regularity class (1.8) as the velocity field of the target system, $\mathrm{div}_x\boldsymbol{V} = 0$, and $\overline{\varrho}$ is a positive constant. Since $\overline{\varrho}$ is a constant, the terms T_5 and T_6 vanish trivially. The term T_4 vanishes as well, by virtue of Lemma 2.4. We keep the term T_1 in its present form and transform the terms T_2 and T_3 conveniently. This will be done in several steps.

Transforming the term T_2

We have

$$T_2 = T_{2,1} + R_{2,1} + R_{2,2}, \text{ with } T_{2,1} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}\frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t}\cdot\Big(\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n\Big), \tag{6.1}$$

and

$$R_{2,1} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}R_{2,1}^{n,K}, \quad R_{2,2} = \Delta t\sum_{n=1}^m R_{2,2}^n,$$

where

$$R_{2,1}^{n,K} = -\frac{|K|}{2}\varrho_K^{n-1}\frac{(\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1})^2}{\Delta t} = -\frac{|K|}{2}\varrho_K^{n-1}\frac{([\boldsymbol{V}^n - \boldsymbol{V}^{n-1}]_{h,K})^2}{\Delta t},$$

and

$$R_{2,2}^n = -\sum_{K\in\mathcal{T}} |K|\varrho_K^{n-1}\frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t}\cdot\Big(\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n\Big).$$

As the function $t\mapsto \boldsymbol{V}(t,x)$ is continuously differentiable, we get

$$\Big|\frac{[\boldsymbol{V}^n - \boldsymbol{V}^{n-1}]_{h,K}}{\Delta t}\Big| = \Big|\frac{1}{|K|}\int_K\Big[\frac{1}{\Delta t}\Big[\int_{t_{n-1}}^{t_n}\partial_t\boldsymbol{V}(z,x)\mathrm{d}z\Big]_h\Big]\mathrm{d}x\Big|$$

$$= \Big|\frac{1}{|K|}\int_K\Big[\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\big[\partial_t\boldsymbol{V}(z)\big]_h(x)\mathrm{d}z\Big]\mathrm{d}x\Big| \le \|[\partial_t\boldsymbol{V}]_{h,0}\|_{L^\infty((0,T)\times\mathbb{T}^3;R^3)} \le \|\partial_t\boldsymbol{V}\Big\|_{L^\infty((0,T)\times\mathbb{T}^3;R^3)},$$

where we have used (2.12). Therefore, thanks to the mass conservation (4.1), we get

$$|R_{2,1}^{n,K}| \le \frac{M_0}{2}|K|\Delta t\|\partial_t\boldsymbol{V}\|^2_{L^\infty((0,T)\times\mathbb{T}^3;R^3)}. \tag{6.2}$$

To treat the term $R_{2,2}^n$ we use the discrete Hölder inequality and identity (4.1) in order to get

$$|R_{2,2}^n| \le \Delta t\; cM_0\|\partial_t\boldsymbol{V}\|^2_{L^\infty(0,T;W^{1,\infty}(\mathbb{T}^3;R^3))}$$

$$+cM_0^{1/2}\Big(\sum_{K\in\mathcal{T}}|K|\varrho_K^{n-1}|\boldsymbol{u}_K^{n-1} - \boldsymbol{u}_K^n|^2\Big)^{1/2}\|\partial_t\boldsymbol{V}\|_{L^\infty((0,T)\times\mathbb{T}^3;R^3)};$$

whence, by virtue of estimate (4.2) for the upwind dissipation term (4.3a), one obtains

$$|R_{2,2}| \le \sqrt{\Delta t}\, c(M_0, E_0, \|\partial_t\boldsymbol{V}\|_{L^\infty((0,T)\times\mathbb{T}^3;R^3)}). \tag{6.3}$$

Transforming the term T_3.

Employing the definition (3.4) of upwind quantities, we easily establish that

$T_3 = T_{3,1} + R_{3,1}$,

with $T_{3,1} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}}\Big)\cdot\boldsymbol{V}_{h,K}^n\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}$, $\quad R_{3,1} = \Delta t\sum_{n=1}^m\sum_{\sigma\in\mathcal{E}}R_{3,1}^{n,\sigma}$,

and $R_{3,1}^{n,\sigma} = |\sigma|\varrho_K^n\frac{|\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,L}^n|^2}{2}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,K}]^+ + |\sigma|\varrho_L^n\frac{|\boldsymbol{V}_{h,L}^n - \boldsymbol{V}_{h,K}^n|^2}{2}[\boldsymbol{u}_\sigma^n\cdot\boldsymbol{n}_{\sigma,L}]^+$, $\forall\sigma = K|L\in\mathcal{E}$.

Writing $\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,L}^n = (\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_h^n) + (\boldsymbol{V}_h^n - \boldsymbol{V}_{h,\sigma}^n) + (\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_h^n) + (\boldsymbol{V}_h^n - \boldsymbol{V}_{h,L}^n)$, $\sigma = K|L\in\mathcal{E}$, and employing estimates (2.19) then $(2.14)_{s=1}$ and (2.20) after $(2.14)_{s=1}$ to evaluate the L^∞-norm of the first and second terms, and performing the same for the last two terms, we get

$$\|\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,L}^n\|_{L^\infty(K\cup L;R^3)} \le ch\|\nabla\boldsymbol{V}\|_{L^\infty(K\cup L;R^9)}; \tag{6.4}$$

consequently

$$|R_{3,1}^{n,\sigma}| \le h^2\; c\|\nabla\boldsymbol{V}\|^2_{L^\infty((0,T)\times\mathbb{T}^3;R^9)}|\sigma|(\varrho_K^n + \varrho_L^n)|\boldsymbol{u}_\sigma^n|,\ \forall\sigma = K|L\in\mathcal{E},$$

whence

$$\begin{aligned}|R_{3,1}| \le h\; c\|\nabla\boldsymbol{V}\|^2_{L^\infty((0,T)\times\mathbb{T}^3;R^9)}\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}(K)}h|\sigma|(\varrho_K^n + \varrho_L^n)^{6/5}\Big)^{5/6}\times\\ \Big[\Delta t\sum_{n=1}^m\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}h|\sigma||\boldsymbol{u}_\sigma^n|^6\Big)^{1/3}\Big]^{1/2} \le h\; c(M_0, E_0, \|\nabla\boldsymbol{V}\|_{L^\infty(Q_T;R^9)}),\end{aligned} \tag{6.5}$$

provided $\gamma \geq 6/5$, thanks to the discrete Hölder inequality, the equivalence of norms (2.15) and the energy bounds listed in Corollary 4.1.

Clearly, for each face $\sigma = K|L \in \mathcal{E}$, $\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} + \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,L} = 0$; whence, finally

$$T_{3,1} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}}\Big) \cdot \Big(\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,\sigma}^n\Big) \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}. \tag{6.6}$$

Before the next transformation of the term $T_{3,1}$, we write

$$\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,\sigma}^n = (\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_h^n) + (\boldsymbol{V}_h^n - \boldsymbol{V}_{h,\sigma}^n);$$

whence by virtue of (2.19), (2.20) and $(2.14)_{s=1}$, similarly to (6.4),

$$\|\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,\sigma}^n\|_{L^\infty(K;R^3)} \leq ch \|\nabla_x \boldsymbol{V}\|_{L^\infty((0,T)\times\mathbb{T}^3;R^{3\times 3}))}, \ \ \sigma \subset K. \tag{6.7}$$

Let us now decompose the term $T_{3,1}$ as

$$T_{3,1} = T_{3,2} + R_{3,2}, \text{ with } R_{3,2} = \Delta t \sum_{n=1}^{m} R_{3,2}^n,$$

$$T_{3,2} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big) \cdot \Big(\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_{h,K}^n\Big) \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}, \text{ and}$$

$$R_{3,2}^n = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big) \cdot \Big(\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_{h,K}^n\Big) \Big(\boldsymbol{u}_\sigma^n - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big) \cdot \boldsymbol{n}_{\sigma,K}.$$

By virtue of the discrete Hölder's inequality and estimate (6.7), we get

$$|R_{3,2}^n| \leq c \|\nabla_x \boldsymbol{V}\|_{L^\infty(Q_T;R^9)} \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big|\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}}\Big|^2\Big)^{1/2}$$
$$\times \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| |\varrho_\sigma^{n,\mathrm{up}}|^{\gamma_0}\Big)^{1/(2\gamma_0)} \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \Big|\boldsymbol{u}_\sigma^n - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big|^q\Big)^{1/q},$$

where $\frac{1}{2} + \frac{1}{2\gamma_0} + \frac{1}{q} = 1$, $\gamma_0 = \min\{\gamma, 3\}$ and $\gamma \geq 3/2$. For the sum in the last term of the above product, we have

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| \Big|\boldsymbol{u}_\sigma^n - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big|^q \leq c \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma| |\boldsymbol{u}_\sigma^n - \boldsymbol{u}_K^n|^q$$

$$\leq c\Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \Big(\|\boldsymbol{u}_\sigma^n - \boldsymbol{u}^n\|_{L^q(K;R^3)}^q + \sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^q(K;R^3)}^q \Big) \leq c h^{\frac{2\gamma_0 - 3}{2\gamma_0} q} \Big(\sum_{K\in\mathcal{T}} \|\nabla_x \boldsymbol{u}^n\|_{L^2(K;R^9)}^2 \Big)^{q/2},$$

where we have used the definition (3.4), the discrete Minkowski inequality, the interpolation inequalities (2.23), (2.24) and the discrete "imbedding" inequality (2.30). Now we can go back to the estimate of $R_{3,2}^n$, and, taking into account the upper bounds (4.4), (4.7), (4.8), we may use Young's inequality to get

$$|R_{3,2}| \leq h^a \, c(M_0, E_0, \|\nabla_x \boldsymbol{V}\|_{L^\infty(Q_T;R^9)}) + \Delta t \sum_{n=1}^{m} \mathcal{E}_\varepsilon(\varrho^n, \hat{\boldsymbol{u}}^n | \overline{\varrho}, \hat{\boldsymbol{V}}_h^n), \tag{6.8}$$

provided $\gamma \geq 3/2$, where a is given in (6.13).

Finally, we rewrite the term $T_{3,2}$ as

$$
\begin{aligned}
&T_{3,2} = T_{3,3} + R_{3,3}, \text{ with } R_{3,3} = \Delta t \sum_{n=1}^{m} R_{3,3}^n, \\
&T_{3,3} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \Big) \cdot \Big(\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_{h,K}^n \Big) \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}, \text{ and} \qquad (6.9)\\
&R_{3,3}^n = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \Big) \cdot \Big(\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_{h,K}^n \Big) \Big(\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} - \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} \Big) \cdot \boldsymbol{n}_{\sigma,K};
\end{aligned}
$$

whence

$$
|R_{3,3}| \le c(\|\nabla_x \boldsymbol{V}\|_{L^\infty(Q_T,R^9)})\, \Delta t \sum_{n=1}^{m} \mathcal{E}_\varepsilon(\varrho^n, \hat{\boldsymbol{u}}^n \,|\, \hat{r}^n, \hat{\boldsymbol{V}}_{h,0}^n). \qquad (6.10)
$$

We have proved the following lemma:

Lemma 6.1 (Approximate relative energy inequality)**.** *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) emanating from the initial data (3.10), where the pressure satisfies (1.4) with $\gamma \ge 3/2$. Then there exists a positive constant*

$$
c = c\Big(M_0, E_0, \|\boldsymbol{V}\|_{W^{1,\infty}((0,T)\times\mathbb{T}^3;R^3)}\Big),
$$

such that for all $m = 1, \dots, N$, we have:

$$
\begin{aligned}
&\int_{\mathbb{T}^3} \Big(\varrho^m |\hat{\boldsymbol{u}}^m - \hat{\boldsymbol{V}}_h^m|^2 + \frac{1}{\varepsilon^2} E(\varrho^m|\overline{\varrho}) \Big) \mathrm{d}x - \int_{\mathbb{T}^3} \Big(\varrho^0 |\hat{\boldsymbol{u}}^0 - \hat{\boldsymbol{V}}_{h,0}^0|^2 + \frac{1}{\varepsilon^2} E(\varrho^0|\overline{\varrho}) \Big) \mathrm{d}x \\
&+ \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{V}_h^n)|^2 \,\mathrm{d}x + (\frac{\mu}{3} + \eta) \int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{V}_h^n)|^2 \,\mathrm{d}x \Big) \le \sum_{i=1}^{3} S_i + R_{h,\Delta t}^m + G^m,
\end{aligned} \qquad (6.11)
$$

for any pair of functions $(\overline{\varrho}, \boldsymbol{V})$, where $\overline{\varrho} = const. > 0$ and

$$
\boldsymbol{V} \in C^1([0,T]\times\mathbb{T}^3; R^3), \quad \mathrm{div}_x \boldsymbol{V} = 0.
$$

Here

$$
\begin{aligned}
&S_1 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \mu \int_K \nabla_x \boldsymbol{V}_h^n : \nabla_x(\boldsymbol{V}_h^n - \boldsymbol{u}^n) \,\mathrm{d}x \\
&S_2 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} |K| \varrho_K^{n-1} \frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t} \cdot \Big(\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n \Big), \qquad (6.12)\\
&S_3 = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_\varepsilon(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}} \Big) \cdot \Big(\boldsymbol{V}_{h,\sigma}^n - \boldsymbol{V}_{h,K}^n \Big) \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K},
\end{aligned}
$$

and

$$
|G^m| \le c\, \Delta t \sum_{n=1}^{m} \mathcal{E}_\varepsilon\Big(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \overline{\varrho}, \hat{\boldsymbol{V}}^n\Big), \quad |R_{h,\Delta t}^m| \le c(\sqrt{\Delta t} + h^a), \qquad (6.13)
$$

with the power a given by (3.13) and with the relative energy functional $\mathcal{E}_\varepsilon$ introduced in (3.8). Here, in accordance with our notational convention (2.9), (3.1)–(3.3), we have set $\boldsymbol{V}_h^n = \Pi_h^V[\boldsymbol{V}(t_n)]$, $\boldsymbol{V}_{h,K}^n = [\boldsymbol{V}_h^n]_K$, $\boldsymbol{V}_{h,\sigma}^n = [\boldsymbol{V}_h^n]_\sigma$, $\hat{r}^n = \Pi_h^Q[r(t_n)]$, where the projections Π^Q, Π^V are defined in (2.6) and (2.8).

7 Consistency error

This section is devoted to the proof of a discrete identity satisfied by any strong solution of problem (1.6), (1.7) in the class (1.8).

Lemma 7.1 (A discrete identity for strong solutions)**.** *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) emanating from the initial data (3.10), with the pressure satisfying (1.4), where $\gamma \geq 3/2$. Let $(\Pi, \boldsymbol{V})$ belonging to the regularity class (1.8) with $k \geq 4$ be a (strong) solution of the incompressible Navier-Stokes system (1.6), (1.7).*

Then there exists a constant

$$c = c\Big(M_0, E_0, \|\boldsymbol{V}, \Pi\|_{\mathcal{X}^k_{T,\mathbb{T}^3}}\Big) > 0,$$

such that

$$\sum_{i=1}^{3} \mathcal{S}_i + \mathcal{R}^m_{h,\Delta t} = 0, \text{ for all } m = 1, \dots, N, \tag{7.1}$$

where

$$\mathcal{S}_1 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \mu \int_K \nabla_x \boldsymbol{V}^n_h : \nabla_x (\boldsymbol{V}^n_h - \boldsymbol{u}^n) \,\mathrm{d}x$$

$$\mathcal{S}_2 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} |K| \overline{\varrho} \frac{\boldsymbol{V}^n_{h,K} - \boldsymbol{V}^{n-1}_{h,K}}{\Delta t} \cdot \Big(\boldsymbol{V}^n_{h,K} - \boldsymbol{u}^n_K\Big),$$

$$\mathcal{S}_3 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \overline{\varrho} \Big(\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_{\sigma}\Big) \cdot \Big(\boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_{h,K}\Big) \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} \cdot \boldsymbol{n}_{\sigma,K}$$

and

$$|\mathcal{R}^m_{h,\Delta t}| \leq c\Big(h^b + \Delta t + \varepsilon\Big), \; b = \min\left\{\frac{5\gamma - 6}{2\gamma}, 1\right\}.$$

The remaining part of this section is devoted to the proof of Lemma 7.1.

7.1 Essential and residual sets

We start with an auxiliary algebraic inequality whose straightforward proof is left to the reader.

Lemma 7.2. *Let p satisfy assumptions (1.4). Let $\overline{\varrho} > 0$. Then there exists a constant $c = c(\overline{\varrho}) > 0$ such that for all $\varrho \in [0, \infty)$ we have*

$$\varrho \mapsto E(\varrho|\overline{\varrho}) \geq c\Big(1_{R_+ \setminus [\overline{\varrho}/2, 2\overline{\varrho}]} + \varrho^\gamma 1_{R_+ \setminus [\overline{\varrho}/2, 2\overline{\varrho}]} + (\varrho - \overline{\varrho})^2 1_{[\overline{\varrho}/2, 2\overline{\varrho}]}\Big), \tag{7.2}$$

where $E(\cdot|\cdot)$ is defined in (3.9).

Now, for a fixed number $\overline{\varrho}$ and fixed functions ϱ^n, $n = 0, \dots, N$, we introduce the residual and essential subsets of $\mathbb{T}^3$ (relative to ϱ^n) as follows:

$$N^n_{\mathrm{ess}} = \{x \in \mathbb{T}^3 \,\Big|\, \frac{1}{2}\overline{\varrho} \leq \varrho^n(x) \leq 2\overline{\varrho}\}, \; N^n_{\mathrm{res}} = \mathbb{T}^3 \setminus N^n_{\mathrm{ess}}, \tag{7.3}$$

and we set

$$[g]_{\mathrm{ess}}(x) = g(x) 1_{N^n_{\mathrm{ess}}}(x), \; [g]_{\mathrm{res}}(x) = g(x) 1_{N^n_{\mathrm{res}}}(x), \;\; x \in \mathbb{T}^3, \;\; g \in L^1(\mathbb{T}^3).$$

Integrating inequality (7.2) we deduce that

$$\frac{c(\overline{\varrho})}{\varepsilon^2} \sum_K \int_K \Big(\Big[1\Big]_{\mathrm{res}} + \Big[(\varrho^n)^\gamma\Big]_{\mathrm{res}} + \Big[\varrho^n - \overline{\varrho}\Big]^2_{\mathrm{ess}}\Big) \,\mathrm{d}x \leq \mathcal{E}_\varepsilon\Big(\varrho^n, \boldsymbol{u}^n \Big| \overline{\varrho}, \boldsymbol{V}^n\Big), \tag{7.4}$$

for any pair $(\overline{\varrho}, \boldsymbol{V})$ with $\boldsymbol{V}$ belonging to the class (1.8) and any $\varrho^n \in Q_h(\mathbb{T}^3)$, $\varrho^n \geq 0$, $\boldsymbol{u}_n \in V_h(\mathbb{T}^3; R^3)$.

We are now ready to prove Lemma 7.1.

Since $(\Pi, \boldsymbol{V})$ satisfies (1.6), (1.7) on $(0,T) \times \mathbb{T}^3$ and belongs to the class (1.8), equation (1.6) can be rewritten in the form

$$\overline{\varrho}\partial_t \boldsymbol{V} + \overline{\varrho}\boldsymbol{V} \cdot \nabla \boldsymbol{V} + \nabla \Pi - \mu \Delta \boldsymbol{V} = 0 \text{ in } (0,T) \times \mathbb{T}^3.$$

From this fact, we deduce the identity

$$\sum_{i=1}^{4} \mathcal{T}_i = 0, \tag{7.5}$$

where

$$\mathcal{T}_1 = -\Delta t \sum_{n=1}^{m} \int_{\mathbb{T}^3} \Big(\mu \Delta \boldsymbol{V}^n\Big) \cdot (\boldsymbol{V}^n - \boldsymbol{u}^n) \, \mathrm{d}x, \quad \mathcal{T}_2 = \Delta t \sum_{n=1}^{m} \int_{\mathbb{T}^3} \overline{\varrho} [\partial_t \boldsymbol{V}]^n \cdot (\boldsymbol{V}^n - \boldsymbol{u}^n) \, \mathrm{d}x,$$

$$\mathcal{T}_3 = \Delta t \sum_{n=1}^{m} \int_{\mathbb{T}^3} \overline{\varrho} \boldsymbol{V}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}^n - \boldsymbol{u}^n) \, \mathrm{d}x, \quad \mathcal{T}_4 = -\Delta t \sum_{n=1}^{m} \int_{\mathbb{T}^3} \nabla \Pi^n \cdot \boldsymbol{u}^n \, \mathrm{d}x.$$

In the steps below, we handle each term $\mathcal{T}_i$.

7.2 Viscous term

Integrating by parts, we get:

$$\begin{aligned}
&\mathcal{T}_1 = \mathcal{T}_{1,1} + \mathcal{R}_{1,1}, \\
&\text{with } \mathcal{T}_{1,1} = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \mu \nabla \boldsymbol{V}_h^n : \nabla (\boldsymbol{V}_h^n - \boldsymbol{u}^n) \, \mathrm{d}x, \text{ and } \mathcal{R}_{1,1} = I_1 + I_2, \text{ with} \\
&I_1 = \Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K \mu \nabla (\boldsymbol{V}^n - \boldsymbol{V}_h^n) : \nabla (\boldsymbol{V}_h^n - \boldsymbol{u}^n) \, \mathrm{d}x, \\
&I_2 = -\Delta t \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \mu \boldsymbol{n}_{\sigma,K} \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_h^n - \boldsymbol{u}^n) \, \mathrm{d}S \\
&\;\; = -\Delta t \sum_{n=1}^{m} \sum_{\sigma \in \mathcal{E}} \int_\sigma \mu \boldsymbol{n}_\sigma \cdot \nabla \boldsymbol{V}^n \cdot \Big[\boldsymbol{V}_h^n - \boldsymbol{u}^n\Big]_{\sigma, \boldsymbol{n}_\sigma} \mathrm{d}S,
\end{aligned} \tag{7.6}$$

where in the last line $\boldsymbol{n}_\sigma$ is the unit normal to the face σ and $[\cdot]_{\sigma,\boldsymbol{n}_\sigma}$ is the jump over sigma (with respect to $\boldsymbol{n}_\sigma$) defined in Lemma 2.5.

To estimate I_1 we use the Cauchy-Schwarz inequality, employ the estimates $(2.14)_{s=2}$ to evaluate the norms involving $\nabla(\boldsymbol{V}^n - \boldsymbol{V}_h^n)$, and use $(2.13)_{s=1}$, (4.4), the Minkowski inequality to estimate the norms involving $\nabla(\boldsymbol{V}_h^n - \boldsymbol{u}^n)$. We get

$$|I_1| \leq h \, c(M_0, E_0, \|\nabla \boldsymbol{V}, \nabla^2 \boldsymbol{V}\|_{L^\infty(0,T;L^\infty(\mathbb{T}^3;R^{36}))}).$$

Since the integral over any face $\sigma \in \mathcal{E}$ of the jump of a function from $V_h(\mathbb{T}^3)$ is zero, we may write

$$I_2 = \Delta t \sum_{n=1}^{m} \sum_{\sigma \in \mathcal{E}} \int_\sigma \mu \boldsymbol{n}_\sigma \cdot \Big(\nabla_x \boldsymbol{V}^n - (\nabla_x \boldsymbol{V}^n)_\sigma\Big) \cdot \Big[\boldsymbol{u}^n - \boldsymbol{V}_h^n\Big]_{\sigma, \boldsymbol{n}_\sigma} \mathrm{d}S;$$

whence by means of the mean value formula applied to $x \mapsto \nabla \boldsymbol{V}^n(x)$ to evaluate the differences $\nabla_x \boldsymbol{V}^n - (\nabla_x \boldsymbol{V}^n)_\sigma$ and Hölder's inequality,

$$
\begin{aligned}
|I_2| &\le \Delta t\, h\, c\, \|\nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T;R^{27})} \sum_{n=1}^m \sum_{\sigma\in\mathcal{E}} \sqrt{|\sigma|}\sqrt{h}\, \Big(\frac{1}{\sqrt{h}} \Big\| \big[\boldsymbol{u}^n - \boldsymbol{V}_h^n\big]_{\sigma,\boldsymbol{n}_\sigma}\Big\|_{L^2(\sigma;R^3)}\Big) \\
&\le \Delta t\, h\, c\, \|\nabla^2 \boldsymbol{V}\|_{L^\infty(Q_T;R^{27})} \sum_{n=1}^m \sum_{\sigma\in\mathcal{E}} \Big(|\sigma| h + \frac{1}{h} \Big\| \big[\boldsymbol{u}^n - \boldsymbol{V}_h^n\big]_{\sigma,\boldsymbol{n}_\sigma}\Big\|^2_{L^2(\sigma;R^3)}\Big).
\end{aligned}
$$

Therefore,

$$
|\mathcal{R}_{1,1}| \le h\, c(M_0, E_0, \|\boldsymbol{V}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{T}^3;R^3))}), \tag{7.7}
$$

where we have employed Lemma 2.5, together with (4.4) and (2.13).

7.3 Term with the time derivative

Let us now decompose the term $\mathcal{T}_2$.

Step 1:

$$
\begin{aligned}
&\mathcal{T}_2 = \mathcal{T}_{2,1} + \mathcal{R}_{2,1}, \\
&\text{with } \mathcal{T}_{2,1} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K \overline{\varrho} \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x, \quad \mathcal{R}_{2,1} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \mathcal{R}_{2,1}^{n,K}, \\
&\text{and } \mathcal{R}_{2,1}^{n,K} = \int_K \overline{\varrho} [\partial_t \boldsymbol{V}]^n \cdot (\boldsymbol{V}^n - \boldsymbol{V}_h^n)\,\mathrm{d}x + \int_K \overline{\varrho}\Big([\partial_t \boldsymbol{V}]^n - \frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big) \cdot (\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x.
\end{aligned}
$$

The remainder $\mathcal{R}_{2,1}^{n,K}$ can be rewritten as follows

$$
\mathcal{R}_{2,1}^{n,K} = \int_K \overline{\varrho} [\partial_t \boldsymbol{V}]^n \cdot (\boldsymbol{V}^n - \boldsymbol{V}_h^n)\,\mathrm{d}x + \frac{1}{\Delta t}\int_K \overline{\varrho}\Big[\int_{t_{n-1}}^{t_n}\int_s^{t_n} \partial_t^2 \boldsymbol{V}(z,\cdot)\mathrm{d}z\mathrm{d}s\Big] \cdot (\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x;
$$

whence, by the Hölder inequality,

$$
\begin{aligned}
|\mathcal{R}_{2,1}^{n,K}| \le (\Delta t + h)\overline{\varrho}\Big[&\|\partial_t \boldsymbol{V}\|_{L^\infty(Q_T;R^3)} |K|^{5/6} (\|\boldsymbol{V}^n\|_{L^6(K)} + \|\boldsymbol{V}_{h,0}^n\|_{L^6(K)}) \\
&+ \|\partial_t^2 \boldsymbol{V}^n\|_{L^{6/5}(\mathbb{T}^3;R^3))} (\|\boldsymbol{u}^n\|_{L^6(K)} + \|\boldsymbol{V}_{h,0}^n\|_{L^6(K)})\Big].
\end{aligned}
$$

Consequently,

$$
|\mathcal{R}_{2,1}| \le (\Delta t + h)\, c\Big(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \partial_t \boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;R^{15})}, \|\partial_t^2 \boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\mathbb{T}^3;R^3))}\Big), \tag{7.8}
$$

where we have used the discrete Hölder and Young inequalities, the estimates (2.12) and the energy bound (4.4) from Corollary 4.1.

Step 2: *Term* $\mathcal{T}_{2,1}$. We decompose the term $\mathcal{T}_{2,1}$ as

$$
\begin{aligned}
&\mathcal{T}_{2,1} = \mathcal{T}_{2,2} + \mathcal{R}_{2,2}, \\
&\text{with } \mathcal{T}_{2,2} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \int_K \overline{\varrho} \frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t} \cdot (\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x,\ \mathcal{R}_{2,2} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \mathcal{R}_{2,2}^{n,K}, \\
&\text{and } \mathcal{R}_{2,2}^{n,K} = \int_K \overline{\varrho}\Big(\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t} - \Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_h\Big) \cdot (\boldsymbol{V}_{h,0}^n - \boldsymbol{u}^n)\,\mathrm{d}x + \\
&\int_K \overline{\varrho}\Big(\Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_h - \Big[\frac{\boldsymbol{V}^n - \boldsymbol{V}^{n-1}}{\Delta t}\Big]_{h,K}\Big) \cdot (\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x = I_1^K + I_2^K.
\end{aligned}
$$

Next, we calculate

$$
\begin{aligned}
|I_2^K| &= \frac{1}{\Delta t}\overline{\varrho}\Big|\int_K \Big(\Big[\int_{t_{n-1}}^{t_n}\partial_t \boldsymbol{V}(z)\mathrm{d}z\Big]_h - \Big[\int_{t_{n-1}}^{t_n}\partial_t \boldsymbol{V}(z)\mathrm{d}z\Big]_{h,K}\Big)\cdot(\boldsymbol{u}^n - \boldsymbol{V}_h^n)\,\mathrm{d}x\Big| \\
&\le \overline{\varrho}\frac{h}{\Delta t}\int_{t_{n-1}}^{t_n}\Big\|\nabla_x\Big[\partial_t \boldsymbol{V}(z)\Big]_h\Big\|_{L^{6/5}(K;R^3)}\|\boldsymbol{u}^n - \boldsymbol{V}_{h,0}^n\|_{L^6(K;R^3)},
\end{aligned}
$$

where we have used the Fubini theorem, Hölder's inequality and (2.19), $(2.14)_{s=1}$. Further, employing the Sobolev inequality on the Crouzeix-Raviart space $V_h(\mathbb{T}^3)$, (2.16), the Hölder inequality and estimate $(2.14)_{s=1}$, we get

$$
\sum_{K\in\mathcal{T}}|I_2^K| \le \frac{h}{\Delta t}\overline{\varrho}\|\boldsymbol{u}^n - \boldsymbol{V}_h^n\|_{L^6(\mathbb{T}^3;R^3)}\int_{t_{n-1}}^{t_n}\Big\|\nabla_x\partial_t\boldsymbol{V}(z)\Big\|_{L^{6/5}(\mathbb{T}^3;R^3)}\mathrm{d}z.
$$

We apply a similar argument to the term I_1^K. Resuming these calculations, and summing over n from 1 to m we get by using Corollary 4.1 , (2.12),

$$
|\mathcal{R}_{2,2}| \le h\,c(M_0, E_0, \overline{\varrho}, \|(\boldsymbol{V}, \nabla_x\boldsymbol{V}, \partial_t\boldsymbol{V})\|_{L^\infty(Q_T;R^{16})}, \|\partial_t\nabla\boldsymbol{V}\|_{L^2(0,T;L^{6/5}(\mathbb{T}^3;R^9))}). \tag{7.9}
$$

Step 3: *Term* $\mathcal{T}_{2,2}$. We rewrite this term in the form

$$
\begin{aligned}
&\mathcal{T}_{2,2} = \mathcal{T}_{2,3} + \mathcal{R}_{2,3},\ \mathcal{R}_{2,3} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\mathcal{R}_{2,3}^{n,K}, \\
&\text{with } \mathcal{T}_{2,3} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K\bar{\rho}\frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t}\cdot(\boldsymbol{u}_K^n - \boldsymbol{V}_{h,K}^n)\,\mathrm{d}x, \\
&\text{and } \mathcal{R}_{2,3}^{n,K} = \int_K\bar{\rho}\frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t}\cdot\Big((\boldsymbol{u}^n - \boldsymbol{u}_K^n) - (\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n)\Big)\,\mathrm{d}x.
\end{aligned} \tag{7.10}
$$

We estimate the L^∞ norm of $\frac{\boldsymbol{V}_{h,K}^n - \boldsymbol{V}_{h,K}^{n-1}}{\Delta t}$ as in (6.2) and get, similarly to the above,

$$
|\mathcal{R}_{2,3}| \le h\ c\Big(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}\|_{W^{1,\infty}((0,T)\times\mathbb{T}^3;R^3)}\Big)\,. \tag{7.11}
$$

7.4 Convective term

Step 1: *Term* $\mathcal{T}_3$.
Let us first decompose $\mathcal{T}_3$ as

$$
\begin{aligned}
&\mathcal{T}_3 = \mathcal{T}_{3,1} + \mathcal{R}_{3,1}, \\
&\text{with } \mathcal{T}_{3,1} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K\overline{\varrho}\boldsymbol{V}_{h,K}^n\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x, \quad \mathcal{R}_{3,1} = \Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\mathcal{R}_{3,1}^{n,K}, \\
&\text{and } \mathcal{R}_{3,1}^{n,K} = \int_K\overline{\varrho}(\boldsymbol{V}^n - \boldsymbol{V}_h^n)\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x \\
&\qquad + \int_K\overline{\varrho}(\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n)\cdot\nabla\boldsymbol{V}^n\cdot(\boldsymbol{V}_h^n - \boldsymbol{u}^n)\,\mathrm{d}x \\
&\qquad + \int_K\overline{\varrho}\boldsymbol{V}_{h,K}^n\cdot\nabla\boldsymbol{V}^n\cdot\Big(\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n - (\boldsymbol{u}^n - \boldsymbol{u}_K^n)\Big)\,\mathrm{d}x.
\end{aligned}
$$

We have, by virtue of $(2.13)_{s=1}$,

$$
\|\boldsymbol{V}^n - \boldsymbol{V}_h^n\|_{L^\infty(K;R^3)} \lesssim h\|\nabla\boldsymbol{V}^n\|_{L^\infty(K;R^9)},
$$

and, by virtue of (2.19), $(2.13)_{s=1}$ $(2.14)_{s=1}$,

$$\|\boldsymbol{V}_h^n - \boldsymbol{V}_{h,K}^n\|_{L^\infty(K;R^3)} \stackrel{<}{\sim} h\|\nabla \boldsymbol{V}^n\|_{L^\infty(K;R^9)}$$

and, finally, by virtue of (2.19),

$$\|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|_{L^2(K;R^3)} \stackrel{<}{\sim} h\|\nabla_x \boldsymbol{u}^n\|_{L^2(K;R^9)}.$$

Consequently, employing Hölder's inequality (for the integrals over K) and the discrete Hölder's inequality (for the sums over $K \in \mathcal{T}$), and using estimate (4.4), we arrive at

$$|\mathcal{R}_{3,1}| \le h\, c(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla_x \boldsymbol{V})\|_{L^\infty(Q_T;R^{12})}). \tag{7.12}$$

Now we shall deal with the term $\mathcal{T}_{3,1}$. Integrating by parts, we get:

$$\int_K \overline{\varrho}\boldsymbol{V}_{h,K}^n \cdot \nabla \boldsymbol{V}^n \cdot (\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n)\,\mathrm{d}x = \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\overline{\varrho}[\boldsymbol{V}_{h,K}^n \cdot \boldsymbol{n}_{\sigma,K}](\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n)\cdot(\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n),$$

due to the fact that $\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \boldsymbol{V}_{h,K}^n \cdot \boldsymbol{n}_{\sigma,K}\mathrm{d}S = 0$.

Step 2: *Term* $\mathcal{T}_{3,1}$
Next we write

$$\mathcal{T}_{3,1} = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}, \quad \mathcal{R}_{3,2} = \Delta t \sum_{n=1}^m \mathcal{R}_{3,2}^n,$$

$$\mathcal{T}_{3,2} = \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\overline{\varrho}[\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}](\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n)\cdot(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}), \tag{7.13}$$

$$\text{and } \mathcal{R}_{3,2}^n \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\overline{\varrho}\Big[\Big(\boldsymbol{V}_{h,K}^n - \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}}\Big)\cdot \boldsymbol{n}_{\sigma,K}\Big](\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n)\cdot(\boldsymbol{V}_{h,K}^n - \boldsymbol{u}_K^n)$$
$$+\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\overline{\varrho}[\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}} \cdot \boldsymbol{n}_{\sigma,K}](\boldsymbol{V}_\sigma^n - \boldsymbol{V}_{h,K}^n)\cdot\Big((\boldsymbol{V}_{h,K}^n - \hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}}) - (\boldsymbol{u}_K^n - \hat{\boldsymbol{u}}_{h,\sigma}^{n,\mathrm{up}})\Big).$$

We use repeatedly the Taylor formula along with $(2.13)_{s=1}$, (2.19), $(2.14)_{s=1}$, to get the bound

$$|\mathcal{R}_{3,2}^n| \le h\, c\overline{\varrho}\Big(1 + \|\boldsymbol{V}\|_{W^{1,\infty}(Q_T;R^3)}\Big)^3 \sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n|$$
$$+c\overline{\varrho}\Big(1 + \|\boldsymbol{V}\|_{W^{1,\infty}(Q_T;R^3)}\Big)^2 \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_K^n - \boldsymbol{u}_\sigma^n|.$$

We obtain, by Hölder's inequality,

$$\sum_{K\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n| \le c\Big(\sum_{\sigma\in\mathcal{T}} h|\sigma||\boldsymbol{u}_K^n|^6\Big)^{1/6} \le c\Big(\sum_{K\in\mathcal{T}} \|\nabla_x \boldsymbol{u}_n\|^2_{L^2(K;R^9)}\Big)^{1/2},$$

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} h|\sigma||\boldsymbol{u}_K^n - \boldsymbol{u}_\sigma^n| \le c\Big[\Big(\sum_{K\in\mathcal{T}} \|\boldsymbol{u}^n - \boldsymbol{u}_K^n\|^2_{L^2(K;R^3)}\Big)^{1/2} \le h\, c\Big(\sum_{K\in\mathcal{T}} \|\nabla_x \boldsymbol{u}_n\|^2_{L^2(K;R^9)}\Big)^{1/2},$$

where we have used $(2.21)_{p=2}$, $(2.19–2.20)_{p=2}$. Consequently, we may use (4.4) to conclude that

$$|\mathcal{R}_{3,2}| \leq h\, c\Big(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla_x \boldsymbol{V}\|_{L^\infty(Q_T;R^{12})}\Big). \tag{7.14}$$

Step 3: *Term $\mathcal{T}_{3,2}$*
Finally, we replace in $\mathcal{T}_{3,2}$ the term $\boldsymbol{V}^n_\sigma - \boldsymbol{V}^n_{h,K}$ by $\boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_{h,K}$. We get

$$\mathcal{T}_{3,2} = \mathcal{T}_{3,3} + \mathcal{R}_{3,3}, \quad \mathcal{R}_{3,3} = \Delta t \sum_{n=1}^{m} \mathcal{R}^n_{3,3},$$

$$\mathcal{T}_{3,3} = \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \overline{\varrho} [\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} \cdot \boldsymbol{n}_{\sigma,K}] (\boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_{h,K}) \cdot (\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma), \tag{7.15}$$

and

$$\mathcal{R}^n_{3,3} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \overline{\varrho} \boldsymbol{V}^n_{h,K} \cdot \boldsymbol{n}_{\sigma,K} \Big([\boldsymbol{V}^n - \boldsymbol{V}_h]^n_\sigma - [\boldsymbol{V}^n_h - \boldsymbol{V}^n_h]_K \Big) \cdot (\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma),$$

resulting in the bound

$$|\mathcal{R}^n_{3,3}| \leq h\, c\Big(M_0, E_0, \overline{\varrho}, \|\boldsymbol{V}, \nabla_x \boldsymbol{V}\|_{L^\infty(Q_T;R^{12})}\Big). \tag{7.16}$$

7.5 Pressure term

Step 1: *A useful formula for upwind discretization*
We will need the following formula proved in [12, Formula (2.16), Lemma 8.1].

$$\int_{\mathbb{T}^3} r\boldsymbol{u} \cdot \nabla_x \phi \mathrm{dx} = - \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r^{\mathrm{up}}_\sigma \boldsymbol{u} \cdot \boldsymbol{n}_{\sigma,K} F \tag{7.17}$$

$$+ \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma (F - \phi) \Big[r\Big]_{\sigma,\boldsymbol{n}_{\sigma,K}} [\boldsymbol{u}\cdot\boldsymbol{n}_{\sigma,K}]^- \mathrm{d}S + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \phi r (\boldsymbol{u} - \boldsymbol{u}_\sigma) \cdot \boldsymbol{n}_{\sigma,K} + \sum_{K\in\mathcal{T}} \int_K (F - \phi) r \mathrm{div}\boldsymbol{u} \mathrm{dx}$$

for any $r, F \in Q_h(\mathbb{T}^3)$, $\boldsymbol{u} \in V_h(\mathbb{T}^3)$, $\phi \in C^1(R^3)$.

We also report the straightforward formula:

$$- \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r^{\mathrm{up}}_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\sigma,K} F = \sum_{\sigma\in\mathcal{E}} |\sigma| r^{\mathrm{up}}_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_\sigma \Big[F\Big]_{\sigma,\boldsymbol{n}_\sigma} \tag{7.18}$$

for any $r, F \in Q_h(\mathbb{T}^3)$, $\boldsymbol{u} \in V_h(\mathbb{T}^3)$.

The strategy is to use formula (7.17) with $r = \varrho^n$, $\boldsymbol{u} = \boldsymbol{u}^n$, $\phi = \Pi^n$, $F = \hat{\Pi}^n$ in combination with the discrete continuity equation (3.6) to show that the term $\mathcal{T}_4$ is proportional to ε.

Step 2: *Discrete time derivative in the continuity equation (3.6)*
We get by direct calculation

$$\mathcal{J} \equiv \Delta t \sum_{n=1}^{N} \int_{\mathbb{T}^3} \frac{\varrho^n - \varrho^{n-1}}{\Delta t} \hat{\Pi}^n \mathrm{dx} = \sum_{n=1}^{N} \int_{\mathbb{T}^3} \Big((\varrho^n - \overline{\varrho}) - (\varrho^{n-1} - \overline{\varrho}) \Big) \hat{\Pi}^n \mathrm{dx}$$

$$= \sum_{n=1}^{N} \int_{\mathbb{T}^3} \Big((\varrho^n - \overline{\varrho}) \Pi^n - (\varrho^{n-1} - \overline{\varrho}) \Pi^{n-1} \Big) \mathrm{dx} + \sum_{n=1}^{N} \int_{\mathbb{T}^3} (\varrho^{n-1} - \overline{\varrho}) (\hat{\Pi}^{n-1} - \hat{\Pi}^n) \mathrm{dx}$$

$$= \int_{\mathbb{T}^3} (\varrho^n - \overline{\varrho})\Pi^N - \int_{\mathbb{T}^3} (\varrho^0 - \overline{\varrho})\Pi^0 + \Delta t \sum_{n=1}^{N} \int_{\mathbb{T}^3} (\varrho^{n-1} - \overline{\varrho}) \frac{\hat{\Pi}^{n-1} - \hat{\Pi}^n}{\Delta t} \mathrm{dx}.$$

Therefore, as in (6.2),

$$|\mathcal{J}| = \Delta t \Big| \sum_{n=1}^{N} \sum_{K\in\mathcal{T}} \int_K \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} \Pi_K^n \mathrm{dx} \Big| \le \varepsilon(1+\Delta t)\, c(M_0, E_0, \|\Pi\|_{L^\infty(Q_T)}, \|\partial_t \Pi\|_{L^1(0,T;L^p(\mathbb{T}^3))}), \quad (7.19)$$

$p = \max\{2, \gamma'\}$, where we have used (4.8) and Lemma 7.2.

Step 3: *Error of upwind discretization*
We write

$$-\mathcal{T}_4 = \frac{\Delta t}{\overline{\varrho}} \sum_{n=1}^{N} \int_{\mathbb{T}^3} \varrho^n \boldsymbol{u}^n \cdot \nabla_x \Pi^n \mathrm{d}x + \frac{\Delta t}{\overline{\varrho}} \sum_{n=1}^{N} \int_{\mathbb{T}^3} (\overline{\varrho} - \varrho^n) \boldsymbol{u}^n \cdot \nabla_x \Pi^n \mathrm{d}x = -\mathcal{T}_{4,1} - \mathcal{R}_{4,1}, \quad (7.20)$$

where

$$|\mathcal{R}_{4,1}| = \Big| \frac{\Delta t}{\overline{\varrho}} \sum_{n=1}^{N} \int_{\mathbb{T}^3} (\overline{\varrho} - \varrho^n) \boldsymbol{u}^n \cdot \nabla_x \Pi^n \Big| \le \varepsilon\, c(M_0, E_0, \overline{\varrho}, \|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)}).$$

Using the formula (7.17) with $r = \varrho^n$, $\boldsymbol{u} = \boldsymbol{u}^n$, $F = \Pi_h^Q[\Pi^n]$ we check without difficulty that

$$\mathcal{R}_{4,2} \equiv -\int_{\mathbb{T}^3} \varrho^n \boldsymbol{u}^n \cdot \nabla_x \Pi^n \mathrm{d}x - \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K} \Pi_K^n$$

$$= \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \left(\Pi^n - \widehat{\Pi^n}\right) \left[\varrho^n\right]_{\sigma,\boldsymbol{n}_{\sigma,K}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}]^- \mathrm{d}S_x$$

$$- \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \Pi^n \varrho^n (\boldsymbol{u}^n - \boldsymbol{u}_\sigma^n) \cdot \boldsymbol{n}_{\sigma,K}\ \mathrm{dS}_x + \sum_{K\in\mathcal{T}} \left(\Pi^n - \widehat{\Pi^n}\right) \varrho^n \mathrm{div}_x \boldsymbol{u}^n \mathrm{d}x.$$

$$= I_1^n + I_2^n + I_3^n. \quad (7.21)$$

We shall now estimate conveniently the terms I_1^n, I_2^n, I_3^n in three steps.

Step 3a: *Term* I_1.
First, we have by using Hölder's inequality,

$$\Big| \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} 1_E \int_\sigma \left(\Pi^n - \widehat{\Pi^n}\right) \left[\varrho^n\right]_{\sigma,\boldsymbol{n}_{\sigma,K}} [\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}]^- \mathrm{d}S_x \Big|$$

$$\lesssim h \|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)} \Big(\sum_{\sigma=K|L\in\mathcal{E}} \int_\sigma 1_E \frac{\left[\varrho^n\right]^2_{\sigma,\boldsymbol{n}_{\sigma,K}}}{\max(\varrho_K^n, \varrho_L^n)^\delta} \left| [\boldsymbol{u}_{\sigma,K}^n \cdot \boldsymbol{n}]^- \right| \mathrm{d}S_x \Big)^{1/2}$$

$$\times \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \max(\varrho_K^n, \varrho_L^n)^\gamma \mathrm{d}S_x \Big)^{\frac{\delta}{2\gamma}} \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}|^{\frac{\gamma}{\gamma-\delta}} \mathrm{d}S_x \Big)^{\frac{\gamma-\delta}{2\gamma}}$$

with any $0 \le \delta < \gamma$ and any $E \subset \mathbb{T}^3$, where we have used estimate (2.19). Therefore employing (2.25), (2.26) and the estimates (4.5), (4.7), (4.9), we obtain

$$\Delta t \sum_{n=1}^{N} |I_1^n| \lesssim \varepsilon h^{1/2} c(M_0, E_0, \|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)}). \quad (7.22)$$

Step 3b: *Term I_2.*
We observe that

$$|I_2^n| = \Big| \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma (\Pi^n - \Pi_\sigma^n) \varrho^n (\boldsymbol{u}^n - \boldsymbol{u}_\sigma^n) \cdot \boldsymbol{n}_{\sigma,K} \mathrm{d}S_x \Big|$$

$$\lesssim \sum_{\sigma\in\mathcal{E}} \|\Pi^n - \Pi_\sigma^n\|_{L^\infty(\sigma)} \|\varrho^n\|_{L^{\gamma_0}(\sigma)} \Big\| \boldsymbol{u}_h^n - \boldsymbol{u}_{h,\sigma}^n \Big\|_{L^{\frac{\gamma_0}{\gamma_0-1}}(\sigma;R^3)}$$

$$\lesssim \|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)} \sum_{K\in\mathcal{T}} \|\varrho^n\|_{L^{\gamma_0}(K)} \|\boldsymbol{u}^n - \boldsymbol{u}_\sigma^n\|_{L^{\frac{\gamma_0}{\gamma_0-1}}(K;R^3)}$$

$$\lesssim h^{\frac{5\gamma_0-6}{2\gamma_0}} \|\varrho^n\|_{L^{\gamma_0}(\mathbb{T}^3)} \Big(\sum_{K\in\mathcal{T}} \|\nabla_x \boldsymbol{u}^n\|^2_{L^2(K;R^{3\times3})} \Big)^{1/2} \|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)},$$

with $\gamma_0 = \min\{2,\gamma\}$, where we have used Hölder's inequality, the trace estimates (2.25), (2.26), and the interpolation estimates (2.24) together with the estimate (4.4).

Consequently,

$$\Delta t \sum_{n=1}^N |I_2^n(t)| \le h^{\frac{5\gamma_0-6}{2\gamma_0}} c(M_0, E_0, \|\nabla_x \Pi\|_{L^\infty(\mathbb{T}^3)}) \tag{7.23}$$

Step 3c: *Term I_3.*
We have

$$|I_3^n| = \Big| \sum_{K\in\mathcal{T}} \int_K (\widehat{\Pi^n} - \Pi^n) \varrho^n \mathrm{div}_x \boldsymbol{u}^n \mathrm{d}x \Big| \lesssim h \sum_{K\in\mathcal{T}} \|\varrho^n\|_{L^\gamma(K)} \|\mathrm{div}_x \boldsymbol{u}^n\|_{L^2(K)} \|\|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)}$$

whence

$$\Delta t \sum_{n=1}^N |I_2^n(t)| \le hc(M_0, E_0, \|\nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3)}). \tag{7.24}$$

Step 4: *Conclusion*
According to (7.20), (7.21) and (3.6),

$$\mathcal{T}_4 = \frac{\Delta t}{\overline{\varrho}} \sum_{n=1}^N \mathcal{R}_{4,2}^n + \mathcal{R}_{4,1} - \frac{1}{\overline{\varrho}} \mathcal{J};$$

whence, taking into account (7.19), (7.22), (7.23), (7.24), we infer that

$$|\mathcal{T}_4| \le (\varepsilon + h^{\frac{5\gamma_0-6}{2\gamma_0}}) c(M_0, E_0, \overline{\varrho}, \|\Pi, \nabla_x \Pi\|_{L^\infty((0,T)\times\mathbb{T}^3;R^4)} \|\partial_t \Pi\|_{L^1(0,T;L^3(\mathbb{T}^3))}), \ \gamma_0 = \min\{2,\gamma\}. \tag{7.25}$$

Now it suffices to put together formulas (7.6), (7.10), (7.15) together with (7.7), (7.8), (7.9), (7.11), (7.12), (7.14), (7.16) and (7.25) in order to deduce the statement of Lemma 7.1.

8 A Gronwall type inequality

In this section we put together the relative energy inequality (6.11) and the identity (7.1) derived in the previous section. The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 8.1. *Let $(\varrho^n, \boldsymbol{u}^n)$ be a solution of the discrete problem (3.5)–(3.7) emanating from the initial data (3.10) with the pressure satisfying (1.4), where $\gamma \ge 3/2$. Then there exists a positive number*

$$c = c\Big(M_0, E_0, \overline{\varrho}, |p'|_{C^1[\overline{\varrho}/2, 2\overline{\varrho}]}, \|\boldsymbol{V}, \Pi\|_{\mathcal{X}^k_{T,\mathbb{T}^3}}\Big)$$

such that for all $m=1,\dots,N$, there holds:

$$\mathcal{E}_\varepsilon(\varrho^m,\boldsymbol{u}^m|\overline{\varrho},\hat{\boldsymbol{V}}_h^m)+\Delta t\frac{\mu}{2}\sum_{n=1}^m\sum_{K\in\mathcal{T}}\int_K|\nabla_x(\boldsymbol{u}^n-\boldsymbol{V}_h^n)|^2\mathrm{d}x$$

$$\le c\Big[h^a+\sqrt{\Delta t}+\varepsilon+\mathcal{E}_\varepsilon(\varrho^0,\boldsymbol{u}^0|\overline{\varrho},\hat{\boldsymbol{V}}_h(0))\Big]+c\,\Delta t\sum_{n=1}^m\mathcal{E}_\varepsilon(\varrho^n,\boldsymbol{u}^n|\overline{\varrho},\hat{\boldsymbol{V}}_h^n),$$

with any couple $(\Pi,\boldsymbol{V})$ belonging to the class (1.8)$_{k\ge4}$ satisfying (1.6), (1.7) on $[0,T)\times\mathbb{T}^3$, where a is defined in (3.13) and $\mathcal{E}_\varepsilon$ is given in (3.8).

Proof. Gathering the formulas (6.11) and (7.1), one gets

$$\mathcal{E}_\varepsilon(\varrho^m,\boldsymbol{u}^m\Big|\overline{\varrho},\hat{\boldsymbol{V}}_h^m)-\mathcal{E}_\varepsilon(\varrho^0,\boldsymbol{u}^0\Big|\overline{\varrho},\hat{\boldsymbol{V}}_h(0))+\mu\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\Big|\nabla(\boldsymbol{u}^n-\boldsymbol{V}_h^n)\Big|^2_{L^2(K;R^3)}\le\mathcal{P}_1+\mathcal{P}_2+\mathcal{Q},\quad(8.1)$$

where

$$\mathcal{P}_1=\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}|K|(\varrho_K^{n-1}-\overline{\varrho})\frac{\boldsymbol{V}_{h,K}^n-\boldsymbol{V}_{h,K}^{n-1}}{\Delta t}\cdot\Big(\boldsymbol{V}_{h,K}^n-\boldsymbol{u}_K^n\Big),$$

$$\mathcal{P}_2=\Delta t\sum_{n=1}^m\sum_{K\in\mathcal{T}}\sum_{\sigma=K|L\in\mathcal{E}_K}|\sigma|\Big(\varrho_\sigma^{n,\mathrm{up}}-\overline{\varrho}\Big)\Big(\hat{\boldsymbol{V}}_{h,\sigma}^{n,\mathrm{up}}-\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}\Big)\cdot\Big(\boldsymbol{V}_{h,\sigma}^n-\boldsymbol{V}_{h,K}^n\Big)\boldsymbol{V}_{h,\sigma}^{n,\mathrm{up}}\cdot\boldsymbol{n}_{\sigma,K},$$

$$\mathcal{Q}=\mathcal{R}_{h,\Delta t}^m+R_{h,\Delta t}^m+G^m.$$

Now, we estimate conveniently the terms $\mathcal{P}_1$ and $\mathcal{P}_2$ in two steps.

Step 1: *Term $\mathcal{P}_1$.* We estimate the L^∞ norm of $\frac{\boldsymbol{V}_{h,K}^n-\boldsymbol{V}_{h,K}^{n-1}}{\Delta t}$ by the L^∞ norm of $\partial_t\boldsymbol{V}$ in the same manner as in (6.2). According to Lemma 7.2, $|\varrho-r|^\gamma 1_{R_+\setminus[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho)\le c(p)E^p(\varrho|r)$, with any $p\ge1$; in particular,

$$|\varrho-r|^{6/5}1_{R_+\setminus[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho)\le cE(\varrho|r).\quad(8.2)$$

We get by using the Hölder inequality,

$$\Big|\sum_{K\in\mathcal{T}}|K|(\varrho_K^{n-1}-\overline{\varrho})\frac{\boldsymbol{V}_{h,K}^n-\boldsymbol{V}_{h,K}^{n-1}}{\Delta t}\cdot\Big(\boldsymbol{V}_{h,K}^n-\boldsymbol{u}_K^n\Big)\Big|\le c\|\partial_t\boldsymbol{V}\|_{L^\infty(Q_T;R^3)}\times$$

$$\Big[\Big(\sum_{K\in\mathcal{T}}|K||\varrho_K^{n-1}-\overline{\varrho}|^2 1_{[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho_K)\Big)^{1/2}+\Big(\sum_{K\in\mathcal{T}}|K||\varrho_K^{n-1}-\overline{\varrho}|^{6/5}1_{R_+\setminus[\overline{\varrho}/2,2\overline{\varrho}]}(\varrho_K)\Big)^{5/6}\Big]\times$$

$$\Big(\sum_{K\in\mathcal{T}}|K|\Big|\boldsymbol{V}_{h,K}^n-\boldsymbol{u}_K^n\Big|^6\Big)^{1/6}\le c(\|(\partial_t\boldsymbol{V})\|_{L^\infty(Q_T;R^3)})\varepsilon\mathcal{E}_\varepsilon^{1/2}(\varrho^{n-1},\hat{\boldsymbol{u}}^{n-1}|\overline{\varrho},\hat{\boldsymbol{V}}_h^{n-1})$$

$$+\varepsilon^{5/3}\mathcal{E}_\varepsilon^{5/6}(\varrho^{n-1},\hat{\boldsymbol{u}}^{n-1}|\overline{\varrho},\hat{\boldsymbol{V}}_h^{n-1})\Big)\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{V}_{h,K}^n-\boldsymbol{u}_K^n\|^6_{L^6(K;R^3)}\Big)^{1/6},$$

where we have used (8.2) and estimate (4.8) to obtain the last line. Now, we write $\boldsymbol{V}_{h,K}^n-\boldsymbol{u}_K^n=([\boldsymbol{V}_h^n-\boldsymbol{u}^n]_K-(\boldsymbol{V}_h^n-\boldsymbol{u}^n))+(\boldsymbol{V}_h^n-\boldsymbol{u}^n)$ and use the Minkowski inequality together with formulas (2.21), (2.16) to get

$$\Big(\sum_{K\in\mathcal{T}}\|\boldsymbol{V}_{h,K}^n-\boldsymbol{u}_K^n\|^6_{L^6(K;R^3)}\Big)^{1/6}\le\Big(\sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{V}_h^n-\boldsymbol{u}^n)\|^2_{L^2(K;R^3)}\Big)^{1/2}.$$

Finally, employing Young's inequality, and estimate (4.8), we arrive at

$$|\mathcal{P}_1| \leq \varepsilon\, c(\delta, M_0, E_0, \overline{\varrho}, \|(\boldsymbol{V}, \nabla_x \boldsymbol{V}, \partial_t \boldsymbol{V})\|_{L^\infty(Q_T,R^{15})}) \times \Big(\Delta t \mathcal{E}_\varepsilon(\varrho^0, \hat{\boldsymbol{u}}^0 | \hat{r}^0, \hat{\boldsymbol{V}}_h^0) + \Delta t \sum_{n=1}^m \mathcal{E}_\varepsilon(\varrho^n, \hat{\boldsymbol{u}}^n | \hat{r}^n, \hat{\boldsymbol{V}}_h^n)\Big) + \delta \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \|\nabla(\boldsymbol{V}_h^n - \boldsymbol{u}^n)\|^2_{L^2(K;R^3)}, \quad (8.3)$$

with any $\delta > 0$, provided $\varepsilon \in (0,1)$.

Step 2: *Term* $\mathcal{P}_2$. We rewrite $\boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_{h,K} = \boldsymbol{V}^n_{h,\sigma} - \boldsymbol{V}^n_{h,K} + [\boldsymbol{V}^n_h - \boldsymbol{V}^n_h]_\sigma + [\boldsymbol{V}^n_h - \boldsymbol{V}^n_h]_K$ and estimate the L^∞ norm of this expression by $h\|\nabla_x \boldsymbol{V}\|_{L^\infty(Q_T;R^9)}$ by virtue of (2.19), (2.20), $(2.14)_{s=1}$. Now we write $\mathcal{P}_2 = \Delta t \sum_{n=1}^m \mathcal{P}_2^n$ where Lemma 7.2 and the Hölder inequality yield, similarly as in the previous step,

$$\begin{aligned}
|\mathcal{P}_2^n| &\leq c(\overline{\varrho}, \|\nabla \boldsymbol{V}\|_{L^\infty(Q_T;R^9)}) \times \\
&\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \Big(E^{1/2}(\varrho_\sigma^{n,\mathrm{up}} | \overline{\varrho}) + E^{2/3}(\varrho_\sigma^{n,\mathrm{up}} | \overline{\varrho}) \Big) |\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma}| \, |\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,0,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma| \\
&\leq c(\overline{\varrho}, \|(\boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;R^{12})}) \Big[\Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \Big(E(\varrho_\sigma^{n,\mathrm{up}} | \overline{\varrho} \Big)^{1/2} \\
&+ \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h E(\varrho_\sigma^{n,\mathrm{up}} | \overline{\varrho}) \Big)^{2/3} \Big] \times \Big(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h \Big| \hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma \Big|^6 \Big)^{1/6},
\end{aligned}$$

provided $\gamma \geq 3/2$. Next, we observe that the contribution of the face $\sigma = K|L$ to the sums $\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h E(\varrho_\sigma^{n,\mathrm{up}} | \overline{\varrho})$ and $\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| h |\hat{\boldsymbol{V}}^{n,\mathrm{up}}_{h,\sigma} - \hat{\boldsymbol{u}}^{n,\mathrm{up}}_\sigma|^6$ is less than or equal to $2|\sigma| h (E(\varrho_K^n | \overline{\varrho}) + E(\varrho_L^n | \overline{\varrho}))$, and less than or equal to $2|\sigma| h (|\boldsymbol{V}^n_{h,K} - \boldsymbol{u}^n_K|^6 + |\boldsymbol{V}^n_{h,L} - \boldsymbol{u}^n_L|^6)$, respectively. Consequently, we get by the same reasoning as in the previous step, under the assumption $\gamma \geq 3/2$,

$$|\mathcal{P}_2| \leq c(\delta, M_0, E_0, \overline{\varrho}, \|(\boldsymbol{V}, \nabla \boldsymbol{V})\|_{L^\infty(Q_T;R^{12})})\, \Delta t \sum_{n=1}^m \mathcal{E}_\varepsilon(\varrho^n, \hat{\boldsymbol{u}}^n | \overline{\varrho}, \hat{\boldsymbol{V}}_h^n) + \delta \Delta t \sum_{n=1}^m \sum_{K\in\mathcal{T}} \|\nabla(\boldsymbol{V}_h^n - \boldsymbol{u}^n)\|^2_{L^2(K;R^3)}. \quad (8.4)$$

Gathering the formulas (8.1) and (8.3)-(8.4) with δ sufficiently small (with respect to μ), we conclude the proof of Lemma 8.1 and a fortiori also the proof of Theorem 3.1 by applying Gronwall's inequality to Lemma 8.1. □

9 Numerical experiments

To illustrate the behavior of the present scheme, we perform two numerical tests in the simplified 2D geometry. We use the equation of state $p(\varrho) = \varrho^\gamma$ with the Poisson adiabatic constant of dry air $\gamma = 1.4$. Note that due to the implicit time discretization there is no stability condition between the time step and the spatial mesh parameter. On the other hand, we need to control inner time substeps used in the fixed point iteration to solve numerically the nonlinear system at each time step. Alternatively, one can use Newton's method or a quasi-Newton method for solving this nonlinear system, which will further relax this time step restriction.

9.1 Example 1

Following Haack et al. [25] we consider the flow of a vortex in the box $\Omega = [-1,1]^2$, supplemented with the no-slip boundary conditions for the velocity. The initial data of the compressible Navier-Stokes system are

$$\begin{aligned} u_1(x,y,0) &= \sin^2(\pi x)\sin(2\pi y) \\ u_2(x,y,0) &= -\sin(2\pi x)\sin^2(\pi y) \\ \rho(x,y,0) &= 1 - \tfrac{\varepsilon^2}{2}\tanh(y-0.5). \end{aligned} \tag{9.1}$$

The time evolution of the flow field at $t = 0,\ 0.1,\ 0.2$ is shown in Figure 1. As time evolves the amplitude of the solution decreases due to physical diffusion. Table 1 presents the experimental order of convergence for the scheme (3.5)–(3.7) for different values of the Mach number ε ranging from 0.8 to 0.001. The error is evaluated by means of the reference solution computed on a fine mesh $h = 1/256$. The simulations are performed on consecutively refined meshes with $h = 1/2^N$, $N = 3, \ldots, 7$, $T = 0.01$. We can clearly recognize the expected first order convergence rate in the momentum (q_1, q_2), which is uniform with respect to ε. The numerical convergence rate for the velocity tends to second order, as will be expected. The velocity gradients as well as the density converge linearly. The first order convergence rate for the density for the low Mach number $\varepsilon = 0.001$ would be recovered using finer grid resolution.

Relative energy error between the compressible and the incompressible Navier-Stokes equations in the low Mach number limit

In order to highlight the convergence result of Theorem 3.1, we evaluate the difference between the compressible and incompressible numerical solutions. The solution of the compressible Navier-Stokes system is computed by means of the scheme (3.5)–(3.7) for different meshes. The solution of the limit incompressible Navier-Stokes equations is obtained by the pressure stabilized Lagrange-Galerkin method of Notsu and Tabata [38] on a fine mesh $h = 1/512$.

Let $\varrho, p = \varrho^\gamma, \boldsymbol{u}$ be the numerical solution of the compressible Navier-Stokes system, and let $\pi, \boldsymbol{V}$ be the solution of incompressible Navier-Stokes equations; we set

$$z = (1 + \varepsilon^2 \Pi)^{1/\gamma}.$$

The error is measured by

$$e_{\mathcal{E}} = \sup_{1\le n\le N} \mathcal{E}_\varepsilon(\rho^n, \hat{\boldsymbol{u}}^n | z(t^n,\cdot), \boldsymbol{V}(t^n,\cdot)),\ e_{\nabla_x \boldsymbol{u}} = \|\nabla_x(\boldsymbol{u} - \boldsymbol{V})\|_{L^2(0,T;L^2(\Omega))},$$
$$e_{\boldsymbol{u}} = \|\boldsymbol{u} - \boldsymbol{V}\|_{L^2(0,T;L^2(\Omega))},\ e_\rho = \|\rho - z\|_{L^2(0,T;L^2(\Omega))},\ e_p = \|p - z^\gamma\|_{L^2(0,T;L^2(\Omega))}.$$

Table 2 presents the results for two different viscosity constants, $\mu = 0.01$ and $\mu = 1$. We set $\varepsilon = h$. Numerical simulations show second order convergence rate for the relative entropy with respect to h and ε and indicate that our theoretical convergence rates of first order (for $\gamma \ge 3$) are suboptimal. Furthermore, numerical experiments show that the pressure and the density of the compressible Navier-Stokes system converge to their incompressible limit with second order rate.

9.2 Example 2

In this numerical experiment we choose a particular solution of the incompressible Navier-Stokes equations, the so-called unsteady Taylor vortex flow with periodic boundary conditions in the x and y directions:

$$\begin{aligned} V_1(x,y,t) &= \sin(2\pi x)\cos(2\pi y)\, e^{-8\pi^2\mu t}, \\ V_2(x,y,t) &= -\cos(2\pi x)\sin(2\pi y)\, e^{-8\pi^2\mu t}, \\ \Pi(x,y,t) &= \tfrac{1}{4}\left(\cos(4\pi x) + \cos(4\pi y)\right) e^{-16\pi^2\mu t}. \end{aligned} \tag{9.2}$$

For the compressible Navier-Stokes system the corresponding initial data are

$$\begin{aligned} \rho(x,y,0) &= 1 + \varepsilon^2 \Pi(x,y,0), \\ u_1(x,y,0) &= V_1(x,y,0), \\ u_2(x,y,0) &= V_2(x,y,0). \end{aligned} \tag{9.3}$$

The time evolution of the flow field at $t = 0, 0.1, 0.2$ is shown in Figure 2. Analogously as in the previous experiment we present in Table 3 experimental orders of convergence for different Mach numbers varying from 0.8 to 0.001. We obtain first order convergence rates for the momentum, velocity gradients and density uniformly with respect to the Mach number. The L^2-norm of the velocity converges with second order.

To study the relative energy error between the compressible solution and its incompressible limit we use the exact solution of the incompressible Taylor vortex flow (9.2) and apply the numerical scheme (3.5)–(3.7) to solve numerically the compressible Navier-Stokes system (1.1), (1.2). In Figure 3 the error of the velocity component u_1 for different Mach numbers and mesh sizes is presented. As expected, the error between the compressible and incompressible solutions decreases, when either the mesh size h or the Mach number ε tend to 0. Tables 4 and 5 demonstrate the same results. Table 4 presents the convergence rates for the physically relevant adiabatic exponent $\gamma = 1.4$, which is not covered by our theoretical analysis. On the other hand, Table 5 shows the convergence rates, for the case, when Theorem 3.1 gives first order convergence of the relative energy. In both cases we obtained for different viscosities $\mu = 0.01$ and $\mu = 1$ second order convergence of the relative energy, density and pressure and first order rates for the velocity and its gradient.

10 Conclusion

In the present paper we have studied both theoretically and numerically convergence of the solution of the compressible Navier-Stokes esystem (1.1), (1.2) to its incompressible limit (1.6), (1.7). The distance between a discrete solution of a (convenient) numerical approximation of the compressible problem and the exact solution of the incompressible Navier-Stokes equations is measured by means of the relative energy functional $\mathcal{E}$, cf. (3.8). For the numerical scheme we use a combined finite element-finite volume method based on the linear Crouzeix-Raviart finite element for the velocity and piecewise constant approximation for the density. The convective terms are approximated by means of upwinding.

Our main result is presented in Theorem 3.1. We have shown that in the case that the data are well-prepared, cf. Remark 3.1, we obtain uniform convergence of order $\mathcal{O}(\sqrt{\Delta t}, h^a, \varepsilon)$, $a = \min\left\{\frac{2\gamma-3}{\gamma}, 1\right\}$, $\gamma \geq 3/2$. First of all we have shown uniform a priori estimates in suitable norms, see Section 4. The crucial step of the proof is to derive a discrete relative energy inequality. The consistency analysis presented in Section 7 shows suitable discrete identities and estimates for the discrete relative energy, that are obtained in the case when the strong solution of the incompressible Navier-Stokes equations is a test function in the discrete relative energy. A Gronwall-type argument concludes the proof and yields the corresponding convergence rates.

As far as we know this is the first result concerning error estimates between the numerical solution of the compressible Navier-Stokes equations and the solution of the limiting incompressible Navier-Stokes equations in the zero Mach number limit. We would also like to point out that these estimates are uniform with respect to all three parameters $\Delta t, h$ and the Mach number ε. Thus, our results demonstrate the asymptotic preserving property of the proposed numerical scheme. Our extensive numerical experiments show second order convergence rate of the relative energy. Consequently, the theoretical results presented in Theorem 3.1 seem to be suboptimal.

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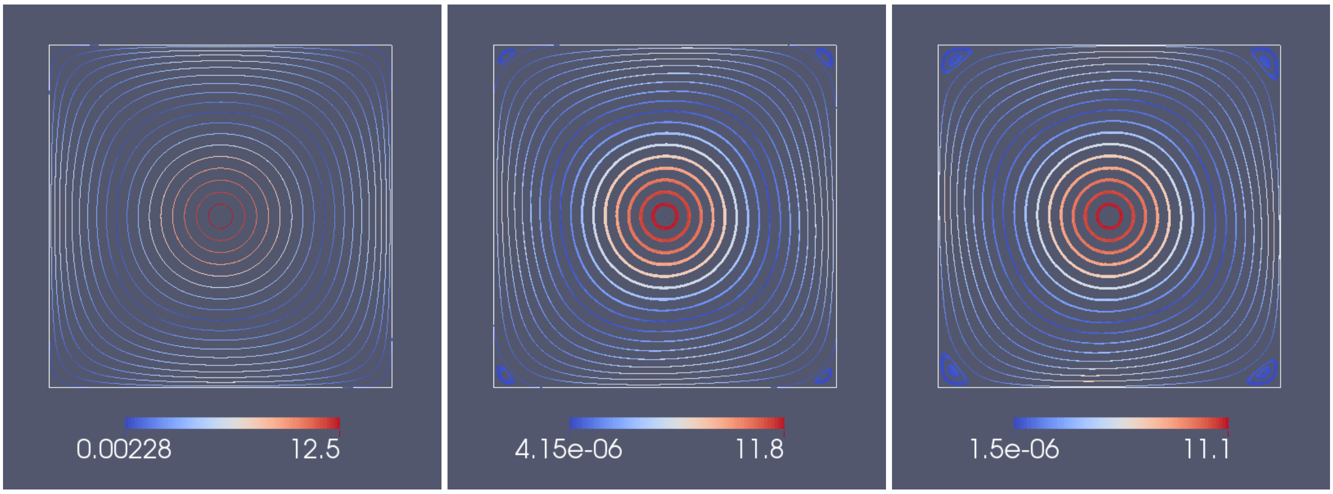
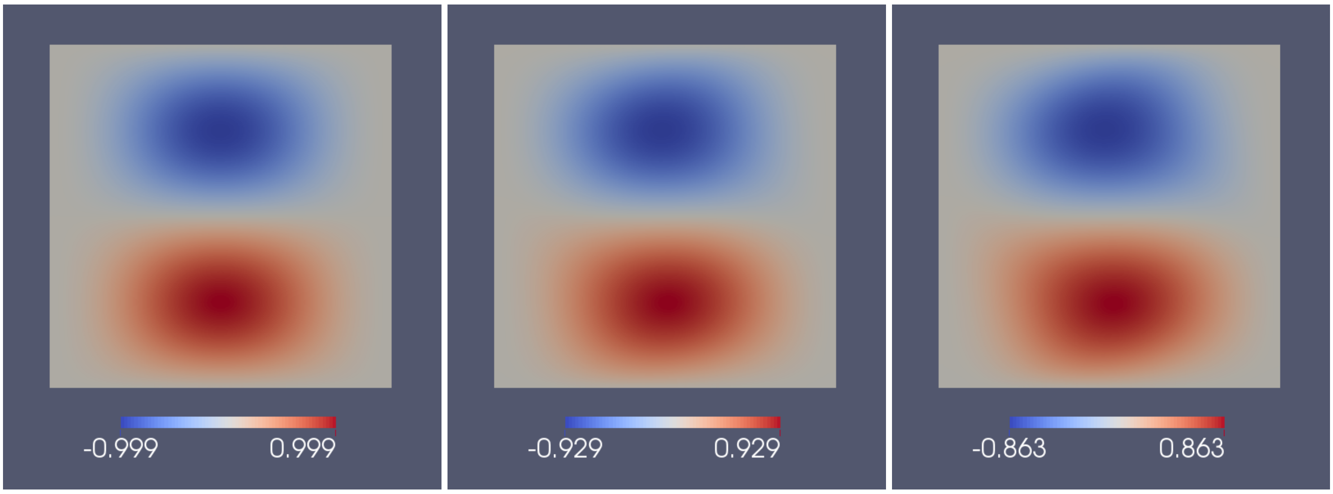
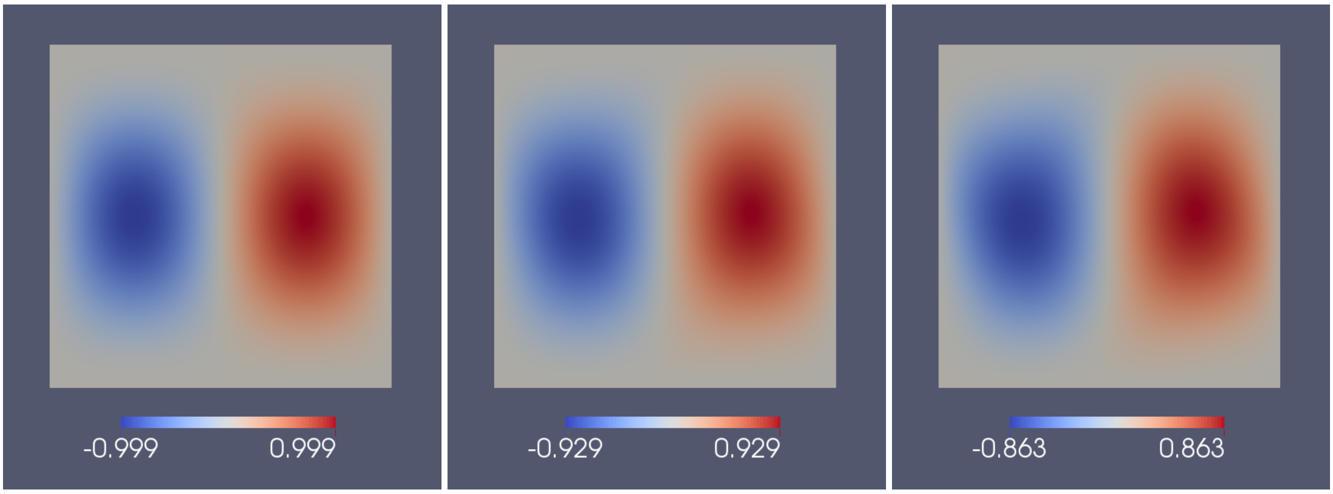


(a) streamline



(b) u_1



(c) u_2

Figure 1: Example 1: Time evolution of the streamline and velocity components with $\mu = 0.01$, $h = 1/64$, $\varepsilon = 0.001$ at $t = 0, 0.1, 0.2$ from left to right.

Table 1: Example 1: Experimental order of convergence, $\mu = 0.01$.

(a) $\varepsilon = 8e-1$

h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	6.61e-03	–	6.73e-02	–	6.70e-02	–	1.63e-02	–	9.55e-01	–
1/16	3.34e-03	0.98	3.38e-02	0.99	3.37e-02	0.99	6.56e-03	1.31	4.95e-01	0.95
1/32	1.70e-03	0.97	1.68e-02	1.01	1.67e-02	1.01	2.37e-03	1.47	2.36e-01	1.07
1/64	8.37e-04	1.02	8.30e-03	1.02	8.27e-03	1.01	7.55e-04	1.65	9.23e-02	1.35
1/128	3.99e-04	1.07	4.03e-03	1.04	3.98e-03	1.06	2.10e-04	1.85	3.32e-02	1.48

(b) $\varepsilon = 1e-1$

h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	3.49e-04	–	6.70e-02	–	6.69e-02	–	1.63e-02	–	9.33e-01	–
1/16	1.94e-04	0.85	3.37e-02	0.99	3.36e-02	0.99	7.49e-03	1.12	5.61e-01	0.73
1/32	8.50e-05	1.19	1.67e-02	1.01	1.67e-02	1.01	2.76e-03	1.44	2.79e-01	1.01
1/64	3.35e-05	1.34	8.27e-03	1.01	8.25e-03	1.02	9.02e-04	1.61	1.18e-01	1.24
1/128	1.36e-05	1.30	4.01e-03	1.04	3.97e-03	1.06	2.67e-04	1.76	4.95e-02	1.25

(c) $\varepsilon = 1e-2$

h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	2.09e-05	–	6.73e-02	–	6.74e-02	–	2.06e-02	–	1.02e+00	–
1/16	1.80e-05	0.22	3.42e-02	0.98	3.42e-02	0.98	1.36e-02	0.60	8.11e-01	0.33
1/32	1.36e-05	0.40	1.70e-02	1.01	1.70e-02	1.01	6.87e-03	0.99	5.79e-01	0.49
1/64	1.04e-05	0.39	8.32e-03	1.03	8.33e-03	1.03	2.12e-03	1.70	2.36e-01	1.29
1/128	4.33e-06	1.26	3.99e-03	1.06	4.01e-03	1.05	5.52e-04	1.94	8.35e-02	1.50

(d) $\varepsilon = 1e-3$

h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	1.17e-07	–	6.73e-02	–	6.73e-02	–	1.87e-02	–	9.40e-01	–
1/16	7.96e-08	0.56	3.37e-02	1.00	3.37e-02	1.00	7.05e-03	1.41	4.40e-01	1.10
1/32	5.69e-08	0.48	1.68e-02	1.00	1.68e-02	1.00	3.31e-03	1.09	2.70e-01	0.70
1/64	2.47e-08	1.20	8.30e-03	1.02	8.30e-03	1.02	1.74e-03	0.93	1.94e-01	0.48
1/128	3.16e-08	-0.36	4.00e-03	1.05	4.00e-03	1.05	5.10e-04	1.77	8.72e-02	1.15

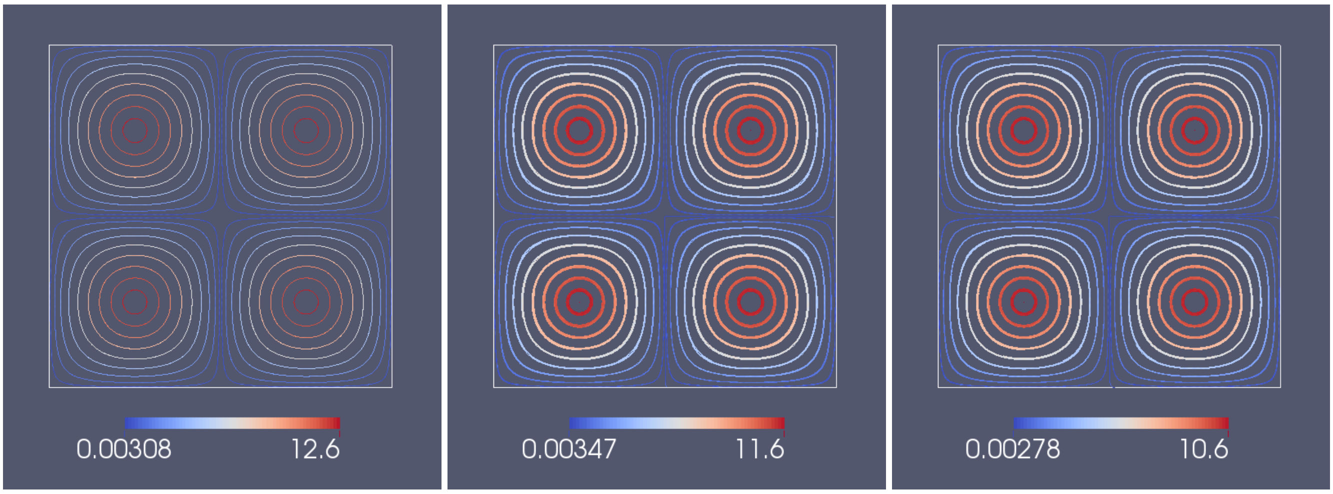
Table 2: Example 1: Convergence of the compressible solution to the incompressible solution for different viscosities μ, $t = 0.01$.

(a) $\varepsilon = h, \mu = 0.01$

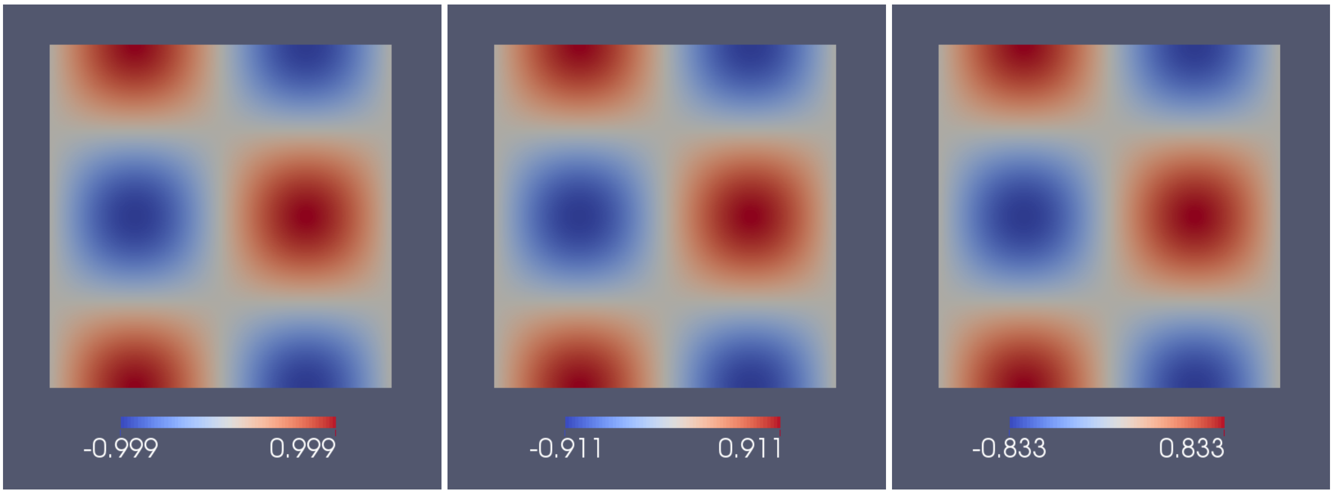
h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	1.12e-03	–	4.91e-01	–	1.82e-03	–	3.65e-04	–	5.11e-04	–
1/16	3.74e-04	1.58	2.55e-01	0.95	1.18e-03	0.63	7.90e-05	2.21	1.11e-04	2.20
1/32	1.09e-04	1.78	2.29e-01	0.16	7.87e-04	0.58	1.50e-05	2.40	2.09e-05	2.41
1/64	1.91e-05	2.51	1.26e-01	0.86	3.24e-04	1.28	3.31e-06	2.18	4.64e-06	2.17
1/128	4.50e-06	2.09	4.41e-02	1.51	1.41e-04	1.20	8.73e-07	1.92	1.22e-06	1.93
1/256	1.14e-06	1.98	1.54e-02	1.52	6.32e-05	1.16	2.06e-07	2.08	2.88e-07	2.08

(b) $\varepsilon = h, \mu = 1$

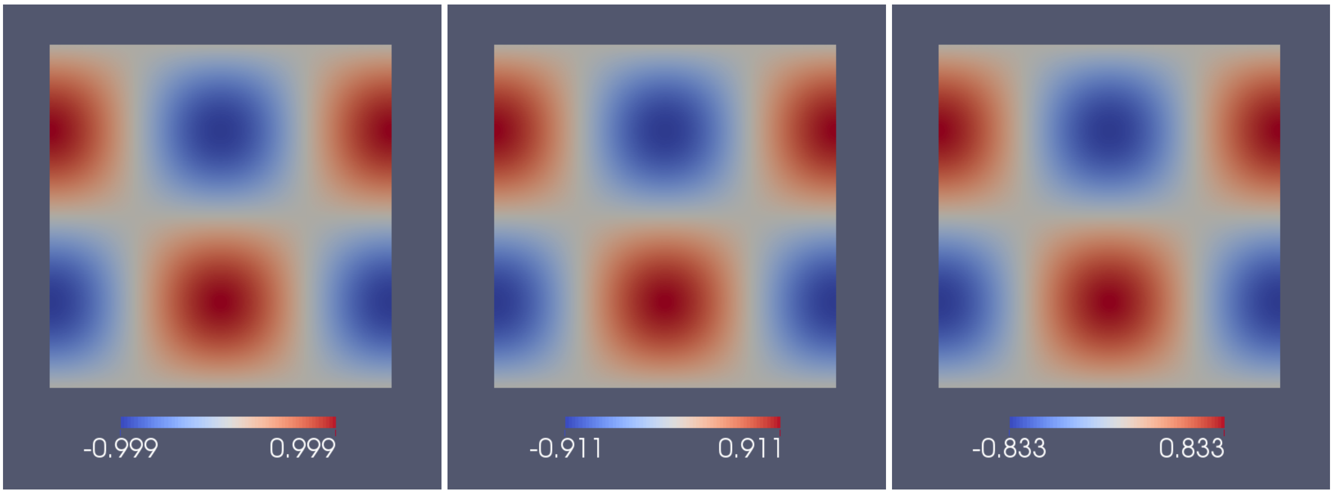
h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	5.42e-03	–	4.66e-01	–	2.99e-03	–	8.81e-04	–	1.23e-03	–
1/16	1.34e-03	2.02	1.73e-01	1.43	2.17e-03	0.46	1.79e-04	2.30	2.51e-04	2.29
1/32	3.66e-04	1.87	6.58e-02	1.39	1.17e-03	0.89	3.78e-05	2.24	5.30e-05	2.24
1/64	1.06e-04	1.79	2.37e-02	1.47	6.13e-04	0.93	8.21e-06	2.20	1.15e-05	2.20
1/128	2.96e-05	1.84	8.26e-03	1.52	2.78e-04	1.14	1.81e-06	2.18	2.54e-06	2.18
1/256	7.96e-06	1.89	2.97e-03	1.48	1.31e-04	1.09	4.09e-07	2.15	5.73e-07	2.15



(a) streamline



(b) u_1



(c) u_2

Figure 2: Example 2: Time evolution of the streamline and velocity components with $\mu = 0.01$, $h = 1/64$, $\varepsilon = 0.001$ at $t = 0, 0.1, 0.2$ from left to right.

Table 3: Example 2: Experimental order of convergence, $\mu = 0.01$.

(a) $\varepsilon = 8e-1$

h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	3.06e-02	–	9.58e-02	–	9.57e-02	–	4.04e-02	–	8.65e+00	–
1/16	1.56e-02	0.97	4.83e-02	0.99	4.83e-02	0.99	2.51e-02	0.69	7.39e+00	0.23
1/32	7.81e-03	1.00	2.40e-02	1.01	2.40e-02	1.01	9.23e-03	1.44	3.97e+00	0.90
1/64	3.87e-03	1.01	1.19e-02	1.01	1.19e-02	1.01	2.57e-03	1.84	1.50e+00	1.40
1/128	1.92e-03	1.01	5.73e-03	1.05	5.72e-03	1.06	6.64e-04	1.95	5.45e-01	1.46

(b) $\varepsilon = 1e-1$

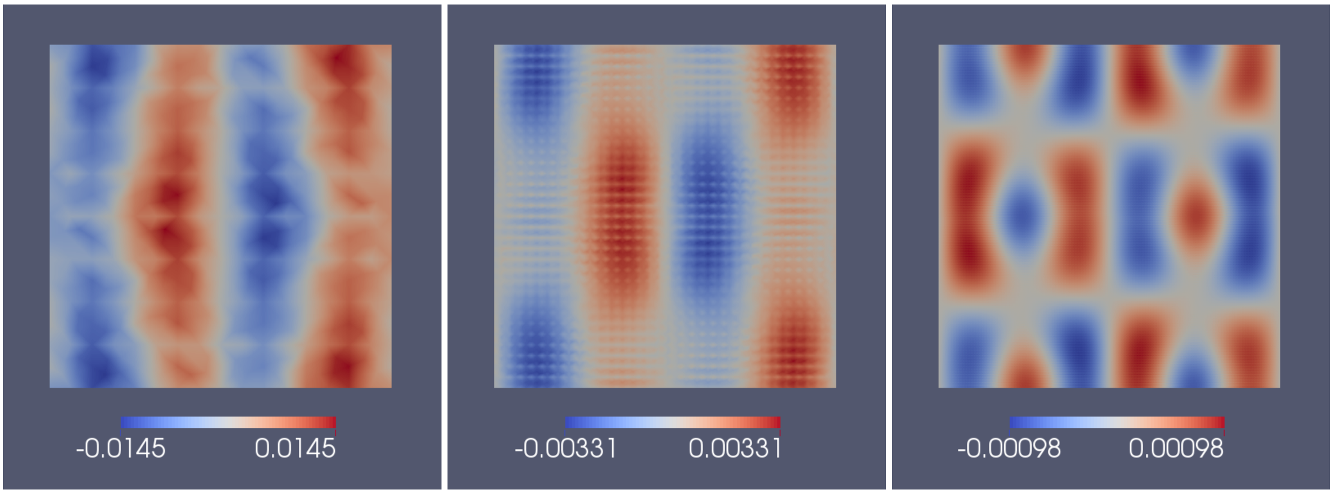
h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	1.53e-03	–	9.46e-02	–	9.46e-02	–	3.21e-02	–	6.77e+00	–
1/16	1.14e-03	0.42	4.78e-02	0.98	4.78e-02	0.98	1.93e-02	0.73	5.19e+00	0.38
1/32	5.63e-04	1.02	2.38e-02	1.01	2.38e-02	1.01	8.89e-03	1.12	3.22e+00	0.69
1/64	1.77e-04	1.67	1.17e-02	1.02	1.17e-02	1.02	2.83e-03	1.65	1.35e+00	1.25
1/128	4.66e-05	1.93	5.64e-03	1.05	5.64e-03	1.05	7.12e-04	1.99	4.99e-01	1.44

(c) $\varepsilon = 1e-2$

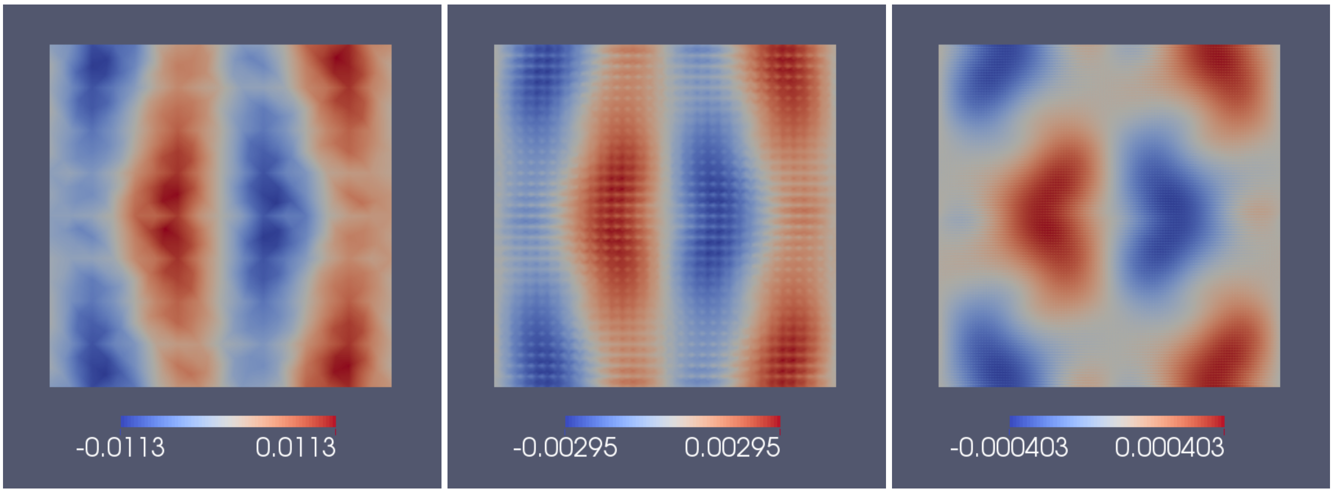
h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	1.02e-05	–	9.49e-02	–	9.49e-02	–	3.11e-02	–	6.08e+00	–
1/16	6.84e-06	0.58	4.84e-02	0.97	4.84e-02	0.97	2.05e-02	0.60	4.65e+00	0.39
1/32	1.37e-06	2.32	2.41e-02	1.01	2.41e-02	1.01	1.03e-02	0.99	3.28e+00	0.50
1/64	7.75e-07	0.82	1.18e-02	1.03	1.18e-02	1.03	3.22e-03	1.68	1.35e+00	1.28
1/128	4.38e-07	0.82	5.64e-03	1.07	5.64e-03	1.07	7.73e-04	2.06	4.94e-01	1.45

(d) $\varepsilon = 1e-3$

h	$\|\rho\|_{L^2}$	EOC	$\|q_1\|_{L^2}$	EOC	$\|q_2\|_{L^2}$	EOC	$\|\mathbf{u}\|_{L^2}$	EOC	$\|\nabla\mathbf{u}\|_{L^2}$	EOC
1/8	1.00e-07	–	9.50e-02	–	9.49e-02	–	3.13e-02	–	6.09e+00	–
1/16	4.07e-08	1.30	4.76e-02	1.00	4.76e-02	1.00	1.09e-02	1.52	2.70e+00	1.17
1/32	1.95e-08	1.06	2.37e-02	1.01	2.37e-02	1.01	4.55e-03	1.26	1.49e+00	0.86
1/64	8.34e-09	1.23	1.17e-02	1.02	1.17e-02	1.02	2.55e-03	0.84	1.13e+00	0.40
1/128	3.01e-09	1.47	5.64e-03	1.05	5.64e-03	1.05	7.49e-04	1.77	5.05e-01	1.16



(a) $\varepsilon = 1e-1$



(b) $\varepsilon = 1e-3$

Figure 3: Example 2: Error of velocity component u_1 for different mesh sizes and Mach numbers; $h = \frac{1}{16}, \frac{1}{64}, \frac{1}{256}$ (left to right), $\varepsilon = 0.1, 0.001$ (top to bottom).

Table 4: Example 2: Convergence of the compressible solution to the incompressible solution for different viscosities μ, $t = 0.01$, $\gamma = 1.4$.

(a) $\varepsilon = h, \mu = 0.01$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	1.22e-02	–	2.57e-01	–	5.54e-03	–	3.39e-04	–	3.34e-04	–
1/16	1.09e-03	3.48	1.47e-01	0.81	1.92e-03	1.53	7.40e-05	2.20	6.82e-05	2.29
1/32	2.02e-04	2.43	1.16e-01	0.34	9.09e-04	1.08	1.31e-05	2.50	1.05e-05	2.70
1/64	2.63e-05	2.94	7.90e-02	0.55	3.20e-04	1.51	2.51e-06	2.38	1.91e-06	2.46
1/128	4.45e-06	2.56	3.86e-02	1.03	8.79e-05	1.86	5.21e-07	2.27	3.91e-07	2.29
1/256	9.86e-07	2.17	2.09e-02	0.89	2.84e-05	1.63	1.22e-07	2.09	7.63e-08	2.36

(b) $\varepsilon = h, \mu = 1$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	6.68e-04	–	3.40e-02	–	1.24e-03	–	2.16e-04	–	3.54e-04	–
1/16	1.69e-04	1.98	1.80e-02	0.92	6.27e-04	0.98	3.49e-05	2.63	6.25e-05	2.50
1/32	4.19e-05	2.01	8.70e-03	1.05	2.33e-04	1.43	5.78e-06	2.59	9.09e-06	2.78
1/64	1.06e-05	1.98	4.25e-03	1.03	8.11e-05	1.52	1.25e-06	2.21	1.48e-06	2.62
1/128	2.68e-06	1.98	2.11e-03	1.01	3.18e-05	1.35	3.07e-07	2.03	2.93e-07	2.34
1/256	7.52e-07	1.83	1.05e-03	1.01	1.48e-05	1.10	7.65e-08	2.00	6.81e-08	2.11

Table 5: Example 2: Convergence of the compressible solution to the incompressible solution for different viscosities μ, $t = 0.01$, $\gamma = 3$.

(a) $\varepsilon = h, \mu = 0.01$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	3.60e-02	–	3.57e-01	–	7.56e-03	–	3.66e-04	–	4.22e-04	–
1/16	3.04e-03	3.57	1.94e-01	0.88	2.35e-03	1.69	8.67e-05	2.08	9.08e-05	2.22
1/32	2.98e-04	3.35	1.30e-01	0.58	9.44e-04	1.32	1.92e-05	2.17	1.76e-05	2.37
1/64	5.26e-05	2.50	8.35e-02	0.64	3.25e-04	1.54	4.36e-06	2.14	3.45e-06	2.35
1/128	1.46e-05	1.85	4.46e-02	0.90	1.11e-04	1.55	1.08e-06	2.01	1.06e-06	1.70
1/256	3.88e-06	1.91	2.22e-02	1.01	4.05e-05	1.45	2.65e-07	2.03	2.34e-07	2.18

(b) $\varepsilon = h, \mu = 1$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_ρ	EOC	e_p	EOC
1/8	1.54e-03	–	4.38e-02	–	2.20e-03	–	1.26e-04	–	7.06e-04	–
1/16	3.63e-04	2.08	2.14e-02	1.03	1.04e-03	1.08	3.02e-05	2.06	1.20e-04	2.56
1/32	1.18e-04	1.62	9.99e-03	1.10	4.29e-04	1.28	1.03e-05	1.55	2.41e-05	2.32
1/64	3.95e-05	1.58	4.84e-03	1.05	1.92e-04	1.16	2.70e-06	1.93	5.22e-06	2.21
1/128	1.22e-05	1.69	2.40e-03	1.01	9.26e-05	1.05	6.89e-07	1.97	1.25e-06	2.06
1/256	3.45e-06	1.82	1.20e-03	1.00	4.59e-05	1.01	1.74e-07	1.99	3.10e-07	2.01