Error Analysis of stochastic Stokes and Navier-Stokes

Julian Frech

Institut für Mathematik,
Johannes Gutenberg-Universität

Betreuer: Prof. Dr. Mária Lukácová-Medvidová

13.09.2018

Setting

$$
\begin{aligned}
-\mu \Delta \boldsymbol{u}(\boldsymbol{x}) + \nabla p(\boldsymbol{x}) &= f(\boldsymbol{x}) && \forall \boldsymbol{x} \in D \\
\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) &= 0 && \forall \boldsymbol{x} \in D \\
\boldsymbol{u}(\boldsymbol{x}) &= 0 && \forall \boldsymbol{x} \in \partial D
\end{aligned}
$$

Setting

$$\begin{aligned} -\mu\Delta \boldsymbol{u}(\boldsymbol{x}) + \nabla p(\boldsymbol{x}) &= f(\boldsymbol{x}) && \forall \boldsymbol{x} \in D \\ \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) &= 0 && \forall \boldsymbol{x} \in D \\ \boldsymbol{u}(\boldsymbol{x}) &= 0 && \forall \boldsymbol{x} \in \partial D \end{aligned}$$

Problem: Viscosity μ is only known to a certain degree of precision.

Setting

$$
\begin{aligned}
-\mu \Delta \boldsymbol{u}(\boldsymbol{x}) + \nabla p(\boldsymbol{x}) &= f(\boldsymbol{x}) && \forall \boldsymbol{x} \in D \\
\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) &= 0 && \forall \boldsymbol{x} \in D \\
\boldsymbol{u}(\boldsymbol{x}) &= 0 && \forall \boldsymbol{x} \in \partial D
\end{aligned}
$$

Problem: Viscosity μ is only known to a certain degree of precision.

Result: Velocity and pressure influenced by uncertainty.

Introduction of Randomness

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Introduction of Randomness

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω set of outcomes, $\mathcal{F}$ the σ-algebra of events and $\mathbb{P} : \mathcal{F} \Rightarrow [0, 1]$ probability measure.

Introduction of Randomness

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω set of outcomes, $\mathcal{F}$ the σ-algebra of events and $\mathbb{P} : \mathcal{F} \Rightarrow [0, 1]$ probability measure.
- $\mu(\boldsymbol{x}, \omega)$, $\omega \in \Omega$

Introduction of Randomness

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω set of outcomes, $\mathcal{F}$ the σ-algebra of events and $\mathbb{P} : \mathcal{F} \Rightarrow [0, 1]$ probability measure.
- $\mu(\boldsymbol{x}, \omega)$, $\omega \in \Omega$
- Velocity and pressure become random fields

Stokes Equation with Randomness

$$
\begin{aligned}
-\mu(\omega)\Delta \boldsymbol{u}(\boldsymbol{x},\omega) + \nabla p(\boldsymbol{x},\omega) &= \boldsymbol{f}(\boldsymbol{x}) && \forall \boldsymbol{x} \in D, \omega \in \Omega \\
\operatorname{div}\, \boldsymbol{u}(\boldsymbol{x},\omega) &= 0 && \forall \boldsymbol{x} \in D, \omega \in \Omega \\
\boldsymbol{u}(\boldsymbol{x},\omega) &= 0 && \forall \boldsymbol{x} \in \partial D, \omega \in \Omega
\end{aligned}
$$

Assumptions I

- $\mu(\boldsymbol{x}, \omega) : D \times \Omega \Rightarrow \mathbb{R}$ is a second-order random field.

Assumptions I

- $\mu(\boldsymbol{x}, \omega) : D \times \Omega \Rightarrow \mathbb{R}$ is a second-order random field.

$\Rightarrow$ $\mathbb{E}[\mu](\boldsymbol{x}) = \int_{\Omega} (\boldsymbol{x}, \omega) d\mathbb{P}(\omega) \in L^2(D)$

Assumptions I

- $\mu(\boldsymbol{x}, \omega) : D \times \Omega \Rightarrow \mathbb{R}$ is a second-order random field.

$$\Rightarrow \mathbb{E}[\mu](\boldsymbol{x}) = \int_{\Omega} (\boldsymbol{x}, \omega) d\mathbb{P}(\omega) \in L^2(D)$$

$$\Rightarrow C[\mu](\boldsymbol{x}, \boldsymbol{x}') := Cov\Big(\mu(\boldsymbol{x}, \cdot), \mu(\boldsymbol{x}', \cdot)\Big) \in L^2(D \times D)$$

Assumptions I

- $\mu(\boldsymbol{x},\omega) : D \times \Omega \Rightarrow \mathbb{R}$ is a second-order random field.

$\Rightarrow$ $\mathbb{E}[\mu](\boldsymbol{x}) = \int_\Omega (\boldsymbol{x},\omega) d\mathbb{P}(\omega) \in L^2(D)$

$\Rightarrow$ $C[\mu](\boldsymbol{x},\boldsymbol{x}') := Cov\Big(\mu(\boldsymbol{x},\cdot),\mu(\boldsymbol{x}',\cdot)\Big) \in L^2(D \times D)$

- $\mu(\boldsymbol{x},\omega) \in L^\infty(D \times \Omega)$ uniformly bounded away from zero.

$\Leftrightarrow$ $0 < C_{\min} \leqslant \mu(\boldsymbol{x},\omega) \leqslant C_{\max} < \infty$

Solution Spaces

- $\boldsymbol{u}(\boldsymbol{x},\omega) \in \boldsymbol{\mathcal{V}} := L^2_{\mathbb{P}}(\Omega, \boldsymbol{H}^1_0(\mathrm{div}, D))$
- $p(\boldsymbol{x},\omega) \in \mathcal{W} := L^2_{\mathbb{P}}(\Omega, L^2(D))$
- $||\boldsymbol{v}||_{\boldsymbol{\mathcal{V}}} := \left(\mathbb{E}[||v||_{\boldsymbol{H}^1(\mathrm{div},D)}]\right)^{\frac{1}{2}}$
- $||q||_{\mathcal{W}} := \left(\mathbb{E}[||q||_{L^2(D)}]\right)^{\frac{1}{2}}$
- $||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\mathrm{div},D)} := ||\boldsymbol{v}||^2_{L^2(D)} + ||\mathrm{div}\ \boldsymbol{v}||^2_{L^2(D)}$

Weak Formulation

Find $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}$ and $p \in \mathcal{W}$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \ell(\boldsymbol{v}) \qquad && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}} \\ b(\boldsymbol{u}, q) &= 0 && \forall q \in \mathcal{W}. \end{aligned}$$

With bilineare forms,

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= \mathbb{E}\Big[\int_D \mu(\cdot) \nabla \boldsymbol{u}(\boldsymbol{x}, \cdot) \nabla \boldsymbol{v}(\boldsymbol{x}, \cdot) \; d\boldsymbol{x} \Big] \\ b(\boldsymbol{v}, q) &:= -\mathbb{E}\Big[\int_D q(\boldsymbol{x}, \cdot) \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, \cdot) \; d\boldsymbol{x} \Big] \\ \ell(\boldsymbol{v}) &:= \mathbb{E}\Big[\int_D \boldsymbol{v}(\boldsymbol{x}, \cdot) \; \boldsymbol{f}(\boldsymbol{x}) \; d\boldsymbol{x} \Big]. \end{aligned}$$

Truncation I

- μ depending on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

Truncation I

- μ depending on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$\Rightarrow$ Karhunen-Loève Expansion to separate out the dependence on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

Truncation I

- μ depending on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$\Rightarrow$ Karhunen-Loève Expansion to separate out the dependence on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$$\mu(\boldsymbol{x}, \omega) = \mathbb{E}[\mu](\boldsymbol{x}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \phi_j(\boldsymbol{x}) \xi_j(\boldsymbol{x})$$

Truncation I

- μ depending on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$\Rightarrow$ Karhunen-Loève Expansion to separate out the dependence on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$$\mu(\boldsymbol{x}, \omega) = \mathbb{E}[\mu](\boldsymbol{x}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \phi_j(\boldsymbol{x}) \xi_j(\boldsymbol{x})$$

- Integral operator $C_\mu : L^2(D) \Rightarrow L^2(D)$ defined by
- $(C_\mu w)(\boldsymbol{x}) = \int_D C[\mu](\boldsymbol{x}, \boldsymbol{x}') w(x') dx'$

Truncation I

- μ depending on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$\Rightarrow$ Karhunen-Loève Expansion to separate out the dependence on $\boldsymbol{x} \in D$ and $\omega \in \Omega$

$$\mu(\boldsymbol{x}, \omega) = \mathbb{E}[\mu](\boldsymbol{x}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \phi_j(\boldsymbol{x}) \xi_j(\boldsymbol{x})$$

- Integral operator $C_\mu : L^2(D) \Rightarrow L^2(D)$ defined by
- $(C_\mu w)(\boldsymbol{x}) = \int_D C[\mu](\boldsymbol{x}, \boldsymbol{x}') w(x') dx'$
- Ordered eigenpairs: $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$, $\lambda_1 \geqslant \lambda \geqslant \dots$

Assumptions on the Randomness II

- $\{\xi_j\}_{j=1}^{\infty}$ family of uncorrelated random variables with

Assumptions on the Randomness II

- $\{\xi_j\}_{j=1}^{\infty}$ family of uncorrelated random variables with
- $\mathbb{E}[\xi] = 0, \qquad Var(\xi) = 1$

Assumptions on the Randomness II

- $\{\xi_j\}_{j=1}^{\infty}$ family of uncorrelated random variables with
- $\mathbb{E}[\xi] = 0, \qquad Var(\xi) = 1$
- $\exists C_\xi > 0$ such that $||\xi_j||_{L^\infty_{\mathbb{P}(\Omega)}} \leqslant C_\xi$ for all $j \geqslant 1$.

Truncation II

- μ depending only on ω

Truncation II

- μ depending only on ω
- $\mu(\omega) := \mu_0 + \sum_{j=1}^{M} \mu_j \xi_j(\omega)$

Truncation II

- μ depending only on ω
- $\mu(\omega) := \mu_0 + \sum_{j=1}^{M} \mu_j \xi_j(\omega)$
- μ already has an exact finite-dimesional representation

Truncation II

- μ depending only on ω
- $\mu(\omega) := \mu_0 + \sum_{j=1}^{M} \mu_j \xi_j(\omega)$
- μ already has an exact finite-dimesional representation

$\Rightarrow$ No Error

Truncated Weak Formulation

Find $\boldsymbol{u}^{(M)} \in \boldsymbol{\mathcal{V}}$ and $p^{(M)} \in \mathcal{W}$ such that

$$\begin{aligned} a(\boldsymbol{u}^{(M)}, \boldsymbol{v}) + b(\boldsymbol{v}, p^{(M)}) &= \ell(\boldsymbol{v}) && \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}} \\ b(\boldsymbol{u}^{(M)}, q) &= 0 && \forall q \in \mathcal{W}. \end{aligned}$$

Assumptions on the Randomness III

- $\xi_j : \Omega \Rightarrow \mathbb{R}$ independent random variables

Assumptions on the Randomness III

- $\xi_j : \Omega \Rightarrow \mathbb{R}$ independent random variables
- $\Gamma_j := \xi_j(\Omega)$ is a bounded intervall in $\mathbb{R}$ for all $j = 1, 2, \ldots$.

Assumptions on the Randomness III

- $\xi_j : \Omega \Rightarrow \mathbb{R}$ independent random variables
- $\Gamma_j := \xi_j(\Omega)$ is a bounded intervall in $\mathbb{R}$ for all $j = 1, 2, \ldots$.
- Probability density functions $\rho_j : \Gamma_j \Rightarrow \mathbb{R}^+$ of each ξ_j are given.

Change of Variable

- Doob-Dynkin Lemma

$\Rightarrow$ $\boldsymbol{u}^{(M)}(\boldsymbol{x},\omega) = \boldsymbol{u}(\boldsymbol{x},\xi_1(\omega),..,\xi_M(\omega))$

Change of Variable

- Doob-Dynkin Lemma

$\Rightarrow$ $\boldsymbol{u}^{(M)}(\boldsymbol{x},\omega) = \boldsymbol{u}(\boldsymbol{x},\xi_1(\omega),..,\xi_M(\omega))$

- Define $(y_1,..,y_M) \in \Gamma := \Gamma_1 \times .. \times \Gamma_M$ with $y_j := \xi_j(\omega)$.

Change of Variable

- Doob-Dynkin Lemma

$\Rightarrow$ $\boldsymbol{u}^{(M)}(\boldsymbol{x}, \omega) = \boldsymbol{u}(\boldsymbol{x}, \xi_1(\omega), .., \xi_M(\omega))$

- Define $(y_1, .., y_M) \in \Gamma := \Gamma_1 \times .. \times \Gamma_M$ with $y_j := \xi_j(\omega)$.
- Independency of the ξ_j results in $\rho(\boldsymbol{y}) = \prod_{j=1}^{M} \rho_j(y_j)$ for all $\boldsymbol{y}_j \in \Gamma_j$.

Truncated Weak Formulation with Change of Variable

Find $\boldsymbol{u}^{(M)} \in \boldsymbol{V}$ and $p^{(M)} \in W$ such that

$$\begin{aligned} a(\boldsymbol{u}^{(M)}, \boldsymbol{v}) + b(\boldsymbol{v}, p^{(M)}) &= \ell(\boldsymbol{v}) \qquad && \forall \boldsymbol{v} \in \boldsymbol{V} \\ b(\boldsymbol{u}^{(M)}, q) &= 0 \qquad && \forall q \in W. \end{aligned}$$

Truncated Weak Formulation with Change of Variable

Find $\boldsymbol{u}^{(M)} \in \boldsymbol{V}$ and $p^{(M)} \in W$ such that

$$\begin{aligned} a(\boldsymbol{u}^{(M)}, \boldsymbol{v}) + b(\boldsymbol{v}, p^{(M)}) &= \ell(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \\ b(\boldsymbol{u}^{(M)}, q) &= 0 \qquad \forall q \in W. \end{aligned}$$

- $\boldsymbol{V} := L^2_\rho(\Gamma; \boldsymbol{H}^1_0(\mathrm{div}, D))$
- $W := L^2_\rho(\Gamma; L^2(D))$
- $||\boldsymbol{v}||^2_{\boldsymbol{V}} := \left(\int\limits_\Gamma ||\boldsymbol{v}(\cdot, \boldsymbol{y})||^2_{\boldsymbol{H}^1(\mathrm{div}, D)} \rho(\boldsymbol{y}) \; d\boldsymbol{y} \right)$
- $||q||^2_W := \left(\int\limits_\Gamma ||q(\cdot, \boldsymbol{y})||^2_{L^2(D)} \rho(\boldsymbol{y}) \; d\boldsymbol{y} \right)$
- Viscosity: $\mu(\boldsymbol{y}) := \mu_o + \sum_{j=1}^{M} \mu_j y_j$.

Bilinear Forms

- $a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Gamma \rho(\boldsymbol{y})\mu(\boldsymbol{y}) \int_D \nabla \boldsymbol{u}^{(M)}(\boldsymbol{x}, \boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \; dx \; dy$
- $b(\boldsymbol{v}, q) := \int_\Gamma \rho(\boldsymbol{y}) \int_D q(\boldsymbol{x}, \boldsymbol{y}) \; \mathrm{div} \; \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \; dx \; dy$
- $\ell(\boldsymbol{v}) := \int_\Gamma \rho(\boldsymbol{y}) \int_D \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \; \boldsymbol{f}(\boldsymbol{x}) \; dx \; dy$

  for $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in W$

Unique Solvability

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \ell(\boldsymbol{v})$$

Unique Solvability

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \ell(\boldsymbol{v})$$

Lemma

Assumptions:

- *Bounded bilinear forms* $a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \Rightarrow \mathbb{R}$ *and* $b(\cdot,\cdot) : \boldsymbol{V} \times W \Rightarrow \mathbb{R}$
- *Norms* $||\cdot||_{\boldsymbol{V}}$ *and* $||\cdot||_W$
- *Bounded Right-hand side*
- $a(\cdot,\cdot)$ *is coercive on* $\boldsymbol{V}^0 := \{\boldsymbol{v} \in \boldsymbol{V};\ b(\boldsymbol{v}, q) = 0 \quad \forall q \in W\}$

$\Leftrightarrow$ *There exists a constant* c *such that* $a(\boldsymbol{v}, \boldsymbol{v}) \geqslant c||\boldsymbol{v}||^2_{\boldsymbol{V}}\ \forall \boldsymbol{v} \in \boldsymbol{V}^0$.

- *The inf-sup condition holds.*

Then the saddle point problem admits unique solutions and the solutions are bounded.

Properties

- Inf-sup condition: $\inf\limits_{q \in W_{hp,k}} \sup\limits_{\boldsymbol{v} \in \boldsymbol{V}_{hp,k}} \frac{|b(\boldsymbol{v},q)|}{||\boldsymbol{v}||_{\boldsymbol{V}}||q||_W} \geqslant \beta$

Properties

- Inf-sup condition: $\inf\limits_{q \in W_{hp,k}} \sup\limits_{\boldsymbol{v} \in \boldsymbol{V}_{hp,k}} \frac{|b(\boldsymbol{v},q)|}{||\boldsymbol{v}||_{\boldsymbol{V}}||q||_W} \geqslant \beta$
- Coercivity: $a(\boldsymbol{u},\boldsymbol{u}) \geqslant \mu_{\min}||\boldsymbol{u}||_{\boldsymbol{V}}^2; \quad \forall v \in \boldsymbol{V}_{hp,k}^0$

Properties

- Inf-sup condition: $\inf\limits_{q \in W_{hp,k}} \sup\limits_{\boldsymbol{v} \in \boldsymbol{V}_{hp,k}} \frac{|b(\boldsymbol{v},q)|}{||\boldsymbol{v}||_{\boldsymbol{V}}||q||_W} \geqslant \beta$
- Coercivity: $a(\boldsymbol{u},\boldsymbol{u}) \geqslant \mu_{\min}||\boldsymbol{u}||^2_{\boldsymbol{V}}; \quad \forall v \in \boldsymbol{V}^0_{hp,k}$
- $\boldsymbol{V}^0_{ph,k} := \left\{\boldsymbol{u} \in \boldsymbol{V}_{hp,k};\ b(\boldsymbol{u},q) = 0, \quad \forall q \in W_{hp,k}\right\}$

Properties

- Inf-sup condition: $\inf\limits_{q \in W_{hp,k}} \sup\limits_{\boldsymbol{v} \in \boldsymbol{V}_{hp,k}} \frac{|b(\boldsymbol{v},q)|}{||\boldsymbol{v}||_{\boldsymbol{V}}||q||_W} \geqslant \beta$
- Coercivity: $a(\boldsymbol{u}, \boldsymbol{u}) \geqslant \mu_{\min}||\boldsymbol{u}||^2_{\boldsymbol{V}}; \quad \forall v \in \boldsymbol{V}^0_{hp,k}$
- $\boldsymbol{V}^0_{ph,k} := \left\{ \boldsymbol{u} \in \boldsymbol{V}_{hp,k};\ b(\boldsymbol{u}, q) = 0, \quad \forall q \in W_{hp,k} \right\}$
- Continuity: $\left| a(\boldsymbol{u}, \boldsymbol{v}) \right| \leqslant \mu_{\max}||\boldsymbol{u}||_{\boldsymbol{V}}||\boldsymbol{v}||_{\boldsymbol{V}}$

Properties

- Inf-sup condition: $\inf_{q\in W_{hp,k}} \sup_{\boldsymbol{v}\in \boldsymbol{V}_{hp,k}} \frac{|b(\boldsymbol{v},q)|}{||\boldsymbol{v}||_{\boldsymbol{V}}||q||_W} \geqslant \beta$
- Coercivity: $a(\boldsymbol{u},\boldsymbol{u}) \geqslant \mu_{\min}||\boldsymbol{u}||^2_{\boldsymbol{V}}; \quad \forall v \in \boldsymbol{V}^0_{hp,k}$
- $\boldsymbol{V}^0_{ph,k} := \left\{\boldsymbol{u} \in \boldsymbol{V}_{hp,k};\ b(\boldsymbol{u},q) = 0, \quad \forall q \in W_{hp,k}\right\}$
- Continuity: $\left|a(\boldsymbol{u},\boldsymbol{v})\right| \leqslant \mu_{\max}||\boldsymbol{u}||_{\boldsymbol{V}}||\boldsymbol{v}||_{\boldsymbol{V}}$
- $\Rightarrow$ Unique Solvability

Discretization Idea

- Finite Element Method in space

Discretization Idea

- Finite Element Method in space
- Generalized Polynomial Chaos in random variable

Discretization Idea

- Finite Element Method in space
- Generalized Polynomial Chaos in random variable
- Askey Chaos: Mapping of probability distribution and orthonormal polynomials

Solution Spaces

- $\boldsymbol{X}_{hp}^{\mathrm{div}} := \left\{ v \in \boldsymbol{H}_0^1(\mathrm{div}, D);\ \boldsymbol{v}\Big|_K \in \mathcal{P}_p(K) \quad \forall K \in \Delta_h \right\}$

Solution Spaces

- $\boldsymbol{X}_{hp}^{\mathrm{div}} := \left\{ v \in \boldsymbol{H}_0^1(\mathrm{div}, D);\ \boldsymbol{v}\Big|_K \in \mathcal{P}_p(K) \quad \forall K \in \Delta_h \right\}$
- $X_{hp}^0 := \left\{ q \in L^2(D);\ q\Big|_K \in \mathcal{P}_{p-1}(K) \quad \forall K \in \Delta_h \right\}$

Solution Spaces

- $\boldsymbol{X}_{hp}^{\mathrm{div}} := \left\{ v \in \boldsymbol{H}_0^1(\mathrm{div}, D);\ \boldsymbol{v}\Big|_K \in \mathcal{P}_p(K) \quad \forall K \in \Delta_h \right\}$
- $X_{hp}^0 := \left\{ q \in L^2(D);\ q\Big|_K \in \mathcal{P}_{p-1}(K) \quad \forall K \in \Delta_h \right\}$
- Multi-index $\boldsymbol{k} = (k_1, .., k_M) \in \mathbb{N}_0^{(M)}$ of polynomial degrees k_j

Solution Spaces

- $\boldsymbol{X}_{hp}^{\mathrm{div}} := \left\{ v \in \boldsymbol{H}_0^1(\mathrm{div}, D);\ \boldsymbol{v}\Big|_K \in \mathcal{P}_p(K) \quad \forall K \in \Delta_h \right\}$
- $X_{hp}^0 := \left\{ q \in L^2(D);\ q\Big|_K \in \mathcal{P}_{p-1}(K) \quad \forall K \in \Delta_h \right\}$
- Multi-index $\boldsymbol{k} = (k_1, .., k_M) \in \mathbb{N}_0^{(M)}$ of polynomial degrees k_j
- $S^M = S^M(\Gamma) := S_{k_1}(\Gamma_1) \otimes .. \otimes S_{k_M}(\Gamma_M) \subset L_\rho^2(\Gamma)$

Solution Spaces

- $\boldsymbol{X}_{hp}^{\mathrm{div}} := \left\{ v \in \boldsymbol{H}_0^1(\mathrm{div}, D);\ \boldsymbol{v}\Big|_K \in \mathcal{P}_p(K) \quad \forall K \in \Delta_h \right\}$
- $X_{hp}^0 := \left\{ q \in L^2(D);\ q\Big|_K \in \mathcal{P}_{p-1}(K) \quad \forall K \in \Delta_h \right\}$
- Multi-index $\boldsymbol{k} = (k_1, .., k_M) \in \mathbb{N}_0^{(M)}$ of polynomial degrees k_j
- $S^M = S^M(\Gamma) := S_{k_1}(\Gamma_1) \otimes .. \otimes S_{k_M}(\Gamma_M) \subset L^2_\rho(\Gamma)$
- $S_{k_j}(\Gamma_j) = \mathrm{span}\{y_j^{\alpha_j}; 0 \leqslant \alpha_j \leqslant k_j\} \subset L^2_{\rho_j}(\Gamma_j),\ j = 1, .., M.$

- $\boldsymbol{V}_{hp,k} := \boldsymbol{X}_{hp}^{\mathrm{div}} \otimes S^M$ and $W_{hp,k} := X_{hp}^0 \otimes S^M$

Example in one Dimension

- $\boldsymbol{X}_{hp}^{\mathrm{div}} = span\{\phi_1, .., \phi_J\} \subset H_0^1(D)$
- $S^M = \{\psi_1, .., \psi_Q\}$
- $\boldsymbol{X}_{hp}^{\mathrm{div}} \otimes S^M := span\{\phi_i \psi_j : \; i = 1, .., J, j = 1, .., Q\}$

Fully discrete Problem

Find $\boldsymbol{u}_{hp,k} \in \boldsymbol{V}_{hp,k} \subset \boldsymbol{V}$ and $p_{hp,k} \in W_{hp,k} \subset W$ satisfying

$$\begin{aligned} a(\boldsymbol{u}_{hp,k}^{(M)}, \boldsymbol{v}) + b(\boldsymbol{v}, p_{hp,k}^{(M)}) &= \ell(\boldsymbol{v}) \qquad && \forall \boldsymbol{v} \in \boldsymbol{V}_{hp,k} \\ b(\boldsymbol{u}_{hp,k}^{(M)}, q) &= 0 \qquad && \forall q \in W_{hp,k}. \end{aligned}$$

Error Sources

I. Error due to Finite Element Method in Space

II. Error due to Generalized Polynomial Chaos Decomposition

Error Splitting

$$
\begin{aligned}
&||\boldsymbol{u}^{(M)} - \boldsymbol{u}_{hp,k}^{(M)}||_{\boldsymbol{V}} + ||p^{(M)} - p_{hp,k}^{(M)}||_W \\
&\leqslant C\Big[\inf_{\boldsymbol{v}\in X_{hp}^{\mathrm{div}}\otimes L^2_\rho(\Gamma)} ||\boldsymbol{u}^{(M)} - \boldsymbol{v}||_{\boldsymbol{V}} + \inf_{q\in X_{hp}^{0}\otimes L^2_\rho(\Gamma)} ||p^{(M)} - q||_W \\
&+ \sum_{j=1}^{M} \Big(\inf_{v_j\in H^1(div,D)\otimes S_j(\Gamma_j)} ||\boldsymbol{u}^{(M)} - v_j||_{\boldsymbol{V}} + \inf_{q_j\in L^2(D)\otimes S_j(\Gamma_j)} ||p^{(M)} - q_j||_W \Big) \Big]
\end{aligned}
$$

Error due to FEM

Lemma

- $D \subset \mathbb{R}^2$
- $\boldsymbol{u}^{(M)} \in L^2_\rho(\Gamma; H^s(\mathrm{div}, D)),\ p^{(M)} \in L^2_\rho(\Gamma; H^s(\mathrm{div}, D))$

Then, there holds

$$\inf_{v \in X^{\mathrm{div}}_{hp} \otimes L^2_\rho(\Gamma)} ||\boldsymbol{u}^{(M)} - \boldsymbol{v}||_V + \inf_{q \in X^0_{hp} \otimes L^2_\rho(\Gamma)} ||p^{(M)} - q||_W$$

$$\leqslant Ch^{\min\{s,p\}} p^{-s} \left(||\boldsymbol{u}^{(M)}||_{L^2_\rho(\Gamma; H^s(div, D))} + ||p^{(M)}||_{L^2_\rho(\Gamma; H^s(div, D))} \right).$$

Error due to GPC for the velocity

- For $0 < \tau < 1$

$$\inf_{v_j \in \boldsymbol{H}_0^1(\mathrm{div}, D) \otimes S_j(\Gamma_j)} ||\boldsymbol{u}^{(M)} - v_j||_{\boldsymbol{V}}$$

$$\leqslant \frac{c(\mu_{\max}, \beta)||\boldsymbol{f}||_{L^2(D)}}{\tau \tilde{\mu}} \sqrt{2\pi} \left(1 + \frac{1}{\sqrt{1 - \zeta_j^2}} \, \mathcal{O}(k_j^{-\frac{1}{3}})\right) \zeta_j^{k_j+1}$$

GPC Error Derivation

- Velocity and pressure are analytic with respect to $\boldsymbol{y} \in \Gamma$.

GPC Error Derivation

- Velocity and pressure are analytic with respect to $\boldsymbol{y} \in \Gamma$.
- Power series representation for a single $y_j \in \Gamma_j$.

GPC Error Derivation

- Velocity and pressure are analytic with respect to $\boldsymbol{y} \in \Gamma$.
- Power series representation for a single $y_j \in \Gamma_j$.
- Boundary of the coefficients and radius of convergence.

GPC Error Derivation

- Velocity and pressure are analytic with respect to $\boldsymbol{y} \in \Gamma$.
- Power series representation for a single $y_j \in \Gamma_j$.
- Boundary of the coefficients and radius of convergence.
- Exploit unique solvability to show equality.

Error Result

Theorem

- $D \subset \mathbb{R}^2$
- $(\boldsymbol{u}^{(M)}, p^{(M)}) \in L^2_\rho(\Gamma; \boldsymbol{H}^s(\mathrm{div}, D)) \times L^2_\rho(\Gamma; \boldsymbol{H}^s(D))$
- $(\boldsymbol{u}^{(M)}_{hp,k}, p^{(M)}_{hp,k}) \in \boldsymbol{V}_{hp,k} \times W_{hp,k}$

$$||\boldsymbol{u}^{(M)} - \boldsymbol{u}^{(M)}_{hp,k}||_{\boldsymbol{V}} + ||p^{(M)} - p^{(M)}_{hp,k}||_W \leqslant C\left(h^{\min\{s,p\}}p^{-s} + \frac{1}{\tau}\sum_{j=1}^{M} \zeta_j^{k_j+1}\right)$$

The constant $C > 0$ is only indepent of the discretization parameters h, g and $\boldsymbol{k}$. Further, $\zeta_j = (\Xi_j + \sqrt{1 - \Xi_j^2}\,)^{-1} \in (0,1)$ with $\Xi_j = 1 + \frac{2(1-\tau)\tilde{\mu}}{|\mu_j|}$ for $j = 1, .., M$ and $\tau \in (0,1)$.

Outlook

- Navier-Stokes and Oseen equation with random viscosity

Outlook

- Navier-Stokes and Oseen equation with random viscosity
- Why Navier-Stokes is not that easily treated

Outlook

- Navier-Stokes and Oseen equation with random viscosity
- Why Navier-Stokes is not that easily treated
- Error analysis for time-dependent Oseen equation