UNIQUE SOLVABILITY OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS MODELING A WARM CLOUD PARCEL*

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Abstract. We analyze the solvability of a system of ordinary differential equations modeling a warm cloud. A unique feature of this model is the automatic onset of nucleation when the moist air parcel becomes supersaturated; this is made possible by a non-Lipschitz right-hand side of the differential equation, which allows for nontrivial solutions. Here we prove under mild assumptions on the external forcing that this system of equations has a unique physically consistent solution, i.e., a solution with a nonzero droplet population in the supersaturated regime.

Key words. Cloud physics, nucleation, dynamical system, Fuchsian reduction

AMS subject classifications. 37N10, 86A10, 92F05

1. Introduction.

Every software for numerical weather prediction comes with a cloud physics module for the simulation of the microphysical processes within clouds. In these systems the resolution of a typical cloud is extremely low, not much more than a handful of finite volumes. Therefore the corresponding cloud models are rather crude and can only capture limited details of the underlying physics.

In general such a model consists of a system of ordinary differential equations for a dozen or so bulk variables per volume element for the most relevant water phases (vapor, small droplets, rain, ice, snow, hail, graupel, etc.), cf., e.g., Seifert and Beheng [10]. The precipitation of liquid and solid water particles couples vertical layers of neighboring volume elements, the release of latent heat and the interaction with radiation provides the important feedback to the atmospheric dynamical system in the synoptic scale.

Recently (cf. [7]) we have developed a new cloud model, which distinguishes itself from previous ones in the way the nucleation of small water droplets is being realized. Roughly speaking, this model starts from the differential equation

$$\dot{y} \;=\; k(t)y^{\alpha}\,, \qquad y(0) = y^{\circ}\,, \tag{1.1}$$

with some parameter $0 < \alpha < 1$, where, as usual, $\dot{y}$ refers to the derivative of y with respect to time t. Problem (1.1) with $y^{\circ} = 0$ is a popular textbook example of an initial value problem for a non-Lipschitz differential equation with multiple solutions, cf., e.g., Walter [13]. In our context this differential equation may be viewed as a simplistic model for the time evolution of the bulk mass y of liquid water in the form of floating droplets, i.e., water, which has condensed on nucleation kernels (aerosols) in the atmosphere. The function k reflects the thermodynamic state of the two-phase system consisting of liquid water and vapor: For $k < 0$ the air parcel is subsaturated and liquid water is evaporating; if $k > 0$ then the air parcel is supersaturated and condensation takes place.

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If $y^\circ = 0$, i.e., if no cloud droplets are present at time $t = 0$, then $y = 0$ is a trivial solution of (1.1). However, the nonuniqueness of solutions of (1.1) allows to switch (in a differentiable manner) from the trivial solution for $t < 0$ to a positive solution for $t > 0$, if the system changes at time $t = 0$ from the subsaturated into the supersaturated regime: nucleation sets in. Such a situation is very common in nature: Consider an air parcel near surface with an amount of water vapor below saturation level, and neither cloud droplets nor rain drops are present. As the air parcel is ascending and the environmental temperature is decreasing, the saturation level decreases, and the system will become supersaturated eventually. At this point water vapor starts to condensate and a cloud develops.

Another feature of the simplistic model (1.1) is that for $y^\circ > 0$ and $k < 0$ the solution y can vanish in finite time, i.e., liquid water can evaporate completely when the system is in the subsaturated regime. Nonetheless, it goes without saying that the justification for using an exponent $\alpha < 1$ in (1.1) requires a sophisticated physical derivation, and this is provided in [7].

Of course, the full model introduced in [7] is more complicated. It is a system of six ordinary differential equations (the model focusses on warm or liquid clouds, where the ice phase can be neglected, and which commonly occur in a temperature regime between 250 and 310 K), see (2.1) below. The particular equation (2.1b) which describes the evolution of the droplet mass has additional terms that have been neglected in (1.1) for the ease of presentation. Similar to (1.1) the full model admits a trivial solution (no droplet mass) for certain initial data and a nontrivial “physically consistent” one (a notion, which will be specified below) in the supersaturated regime; however, whereas all solutions of (1.1) can be written down explicitly by using separation of variables, existence and uniqueness of the physically consistent solution of the full model is a nontrivial question. It is the purpose of this paper to settle this problem.

The discussion of nonunique solutions, respectively the existence of unique positive solutions of initial value problems for non-Lipschitz ordinary differential equations is a classical topic of applied mathematics. Extensions of the example (1.1) have been investigated, e.g., in [2, 3, 4, 15]; an extensive survey of this field is provided by the monograph by Agarwal and Lakshmikantham [1]. However, the known results do not seem to be applicable in our case. Instead we employ a technique known as Fuchsian reduction (Kichenassamy [5], see also Rendall and Schmidt [8]), but this comes for the price that we need to assume that the external forcing of our system – the updraft velocity of the air parcel – is sufficiently regular, e.g., analytic. We emphasize, though, that the theory developed in [5] is based on an assumption which is not fulfilled in our situation. To overcome this hurdle we resort to a sophisticated application of Weissinger’s fixed point theorem [14, 12] by exploiting the particular structure of our system.

The outline of this paper is as follows. In the following section we present the cloud model from [7], explaining briefly the origin of its individual terms, but leaving away all physical details which are irrelevant for the present purpose; for those we refer to the original paper. At the end of this section we also fix the notion of physically consistent solutions. Section 3 is devoted to the proof of existence and uniqueness of a physically consistent solution of the full model. In Section 4 we show that the model is self-consistent in the sense that the two different variables of the model which monitor the amount of rain drops within the cloud provide consistent information about the presence of these drops. Finally, some concluding remarks and acknowledgements are presented in Section 5.

2. The model equations. In the following we recall the system of ordinary differential equations from [7] which constitutes a microphysical model of an adiabatic warm cloud parcel in the atmosphere moving in vertical direction. Besides water vapor this model distinguishes between cloud droplets and rain drops, the latter being sufficiently heavy to allow for sedimentation due to gravity, whereas the former ones are assumed to be floating within the air parcel. Accordingly, the model simulates the time evolution of three variables q_{v}, q_{c}, and q_{r}, respectively, for the cumulated water mass of vapor, cloud droplets, and rain drops in the air parcel, together with another variable n_{r} for the total number of rain drops*:

$$\begin{aligned}
\dot q_{\mathrm{v}} &= -d_1(q_{\mathrm{v}}-q_{\mathrm{vs}})q_{\mathrm{c}}^{1/3} - (q_{\mathrm{v}}-q_{\mathrm{vs}})_-E\,, && (2.1a)\\
\dot q_{\mathrm{c}} &= d_1(q_{\mathrm{v}}-q_{\mathrm{vs}})q_{\mathrm{c}}^{1/3} - d_2q_{\mathrm{c}}^2 - d_3vq_{\mathrm{c}}q_{\mathrm{r}}^{2/3}n_{\mathrm{r}}^{1/3}, && (2.1b)\\
\dot q_{\mathrm{r}} &= (q_{\mathrm{v}}-q_{\mathrm{vs}})_-E + d_2q_{\mathrm{c}}^2 + d_3vq_{\mathrm{c}}q_{\mathrm{r}}^{2/3}n_{\mathrm{r}}^{1/3} - vq_{\mathrm{r}}\,, && (2.1c)\\
\dot n_{\mathrm{r}} &= \frac{n_{\mathrm{r}}}{q_{\mathrm{r}}}(q_{\mathrm{v}}-q_{\mathrm{vs}})_-E + d_4q_{\mathrm{c}} - \lambda vn_{\mathrm{r}}\,. && (2.1d)
\end{aligned}$$

Here, $q_{\mathrm{vs}} = q_{\mathrm{vs}}(p,T)$ denotes the vapor mass at saturation for given values of the pressure p and temperature T, cf. (3.1) below, and

$$(q_{\mathrm{v}}-q_{\mathrm{vs}})_- = \min\{q_{\mathrm{v}}-q_{\mathrm{vs}},0\}$$

is short hand notation for the negative part of $q_{\mathrm{v}}-q_{\mathrm{vs}}$, i.e., $(q_{\mathrm{v}}-q_{\mathrm{vs}})_-$ is negative when the air parcel is subsaturated, and zero else. The term

$$E = d_5q_{\mathrm{r}}^{1/3}n_{\mathrm{r}}^{2/3} + d_6v^{1/2}q_{\mathrm{r}}^{1/2}n_{\mathrm{r}}^{1/2} \tag{2.2}$$

is a simplified approximation of the evaporation rate of rain drops in a subsaturated regime, and in (2.2) and (2.1),

$$v = d_7\Big(\frac{mq_{\mathrm{r}}}{q_{\mathrm{r}}+mn_{\mathrm{r}}}\Big)^{\mu} \tag{2.3}$$

is proportional to an estimate for the terminal fall velocity of spherical rain drops, with $m,\mu>0$ being suitable constants. For later use we note that

$$c_v\min\big\{\,1,(q_{\mathrm{r}}/n_{\mathrm{r}})^{\mu}\,\big\} \leqslant v \leqslant C_v\min\big\{\,1,(q_{\mathrm{r}}/n_{\mathrm{r}})^{\mu}\,\big\} \tag{2.4}$$

for some uniform constants $c_v, C_v > 0$.

Concerning the cloud droplet mass evolution (2.1b), the first term on the right-hand side corresponds to the model (1.1) from the introduction and represents the condensation/vaporization process, i.e., the phase transition between vapor and cloud droplets. It therefore reappears as a sink in (2.1a). Likewise, the source of vapor mass due to evaporating rain drops, as described by the second term on the right-hand side of (2.1a), constitutes a sink of rain drop mass, respectively. The final two terms in (2.1b) represent collision processes by which cloud droplets turn into rain drops. Precipitation is accounted for by the last term in (2.1c). Finally, the

*In cloud physics these quantities are commonly normalized per unit mass of dry air, and this model adjusts to this standard. Nonetheless, we will simply refer to mass and number throughout the text of this paper.

differential equation (2.1d) for the rain drop number is a suitable adaptation of the previous equation (2.1c) for their mass. A common assumption in cloud physics is to distinguish between the precipitation fluxes of mass and number of rain drops, and this is taken care of by introducing a (constant) parameter λ. The fraction $n_{\mathrm{r}}/q_{\mathrm{r}}$ in front of the evaporation term in (2.1d) is used to put the evaporated drop mass in perspective to the average drop mass to provide an estimate for the number of evaporating rain drops; if n_{r} and q_{r} both happen to be identically zero, then no evaporation can take place, and the corresponding term in (2.1d) is taken to be zero.

The coefficients

$$d_i \;=\; d_i(p,T,q_{\mathrm{c}})\,, \qquad i=1,\dots,7\,, \tag{2.5}$$

in (2.1)-(2.3) depend on p, T, and q_{c}, and we assume this dependency to be Lipschitz continuous; moreover, these coefficients are positive, bounded away from zero and from above as long as the same is true for p and T (see the appendix for more details).

The system (2.1) is closed by the following two differential equations for the time evolution of pressure and temperature, namely

$$\dot p \;=\; -\frac{g}{R_{\mathrm{a}}}\frac{p}{T}\,w \qquad \text{and} \qquad \dot T \;=\; -\frac{g}{c_{\mathrm{p}}}w \;-\; \frac{L}{c_{\mathrm{p}}}\dot q_{\mathrm{v}}\,, \tag{2.6}$$

where w is the driving updraft velocity of the air parcel, i.e., the external forcing of our system, g is the gravitational acceleration[†], $L=L(T)$ is the latent heat of vaporization, and R_{a} and c_{p} are the specific gas constant and specific heat capacity of dry air, respectively; in the temperature regime of warm clouds the latter can be assumed to be independent of T. (These ingredients of (2.6) are listed in more detail because they become relevant in (3.5) later on.)

By an *admissible* solution of (2.1), (2.6) we understand nonnegative functions q_{v}, q_{c}, q_{r}, and n_{r}, and strictly positive functions p and T. We therefore request that the corresponding initial data $q_{\mathrm{v}}^\circ, q_{\mathrm{c}}^\circ, q_{\mathrm{r}}^\circ, n_{\mathrm{r}}^\circ, p^\circ$, and T° at time $t=0$ have the same properties. Concerning the updraft velocity w we only assume for the moment that it is continuous in time, and that it drives the air parcel through reasonable elevations

$$z(t) \;=\; z^\circ \;+\; \int_0^t w(\tau)\,\mathrm{d}\tau\,, \qquad t>0\,, \tag{2.7}$$

above surface; for the existence and uniqueness result in the following section we make the stronger assumption that w be analytic (see also Remark 3.3 in Section 3).

For any admissible solution of (2.1) we observe by taking the sum of the three differential equations (2.1a)-(2.1c) that the total mass of water is a nonincreasing function of time; hence, for such a solution the three functions q_{v}, q_{c}, and q_{r} are uniformly bounded from above by the total water mass $q_{\mathrm{v}}^\circ + q_{\mathrm{c}}^\circ + q_{\mathrm{r}}^\circ$ in the cloud parcel at time $t=0$. From (2.6) and the mean-value theorem it further follows that

$$T(t) \;=\; T^\circ \;-\; \frac{g}{c_{\mathrm{p}}}\bigl(z(t)-z^\circ\bigr) \;-\; \frac{\tilde L}{c_{\mathrm{p}}}\bigl(q_{\mathrm{v}}(t)-q_{\mathrm{v}}^\circ\bigr)\,, \tag{2.8}$$

where $\tilde L = L(\tilde T)$ is the latent heat for some intermediate value of the temperature. Concerning the pressure we conclude from (2.6) that

$$p(t) \;=\; p^\circ \exp\Bigl(-\frac{g}{R_{\mathrm{a}}}\int_0^t \frac{w(\tau)}{T(\tau)}\,\mathrm{d}\tau\Bigr)\,,$$

[†]Please note the corresponding misprint in [7].

and hence, by the mean-value theorem,

$$p(t) \;=\; p^\circ\Big(\frac{T^\circ}{T(t)}\Big)^{g\tilde w/R_\mathrm{a}} \tag{2.9}$$

for some intermediate updraft velocity $\tilde w$. Given our assumptions on w it therefore follows from (2.8) and (2.9) that temperature and pressure of any admissible solution of (2.1), (2.6) are uniformly bounded from above and away from zero in every compact time interval, and hence, so are the coefficients $d_1, d_2, \ldots, d_7$. Finally, concerning n_r we readily conclude from (2.1d) that

$$\dot n_\mathrm{r} \;\leqslant\; d_4 q_\mathrm{c}\,, \tag{2.10}$$

showing that n_r is also bounded from above in every finite time interval.

According to the Picard-Lindelöf Theorem (cf., e.g., Teschl [12]) a unique solution of (2.1), (2.6) exists for every strictly positive tuple of initial data $q_\mathrm{v}^\circ, q_\mathrm{c}^\circ, q_\mathrm{r}^\circ, n_\mathrm{r}^\circ, p^\circ$, and T°, and such a solution is obviously admissible locally. However, if one (or some) of the input data $q_\mathrm{c}^\circ, q_\mathrm{r}^\circ$, and n_r° happens to be zero, then the right-hand side of (2.1) is no longer Lipschitz near the initial data, and the corresponding initial value problem need not have a unique solution; we thus refer to *critical initial data*, if at least one of these three numbers $q_\mathrm{c}^\circ, q_\mathrm{r}^\circ, n_\mathrm{r}^\circ$ is zero. For example – like in the simple model discussed in the introduction – if $q_\mathrm{c}^\circ = q_\mathrm{r}^\circ = n_\mathrm{r}^\circ = 0$, then constant values for $q_\mathrm{v}, q_\mathrm{c}, q_\mathrm{r}$, and n_r provide a trivial solution of (2.1), which is physically consistent as long as the air parcel is in the subsaturated regime, but this solution is nonphysical when the vapor mass exceeds the saturation level. Rather, if $q_\mathrm{v} - q_\mathrm{vs}$ turns positive, then we expect to see the emergence of a positive cloud droplet mass, and we will refer to a *physically consistent* solution of (2.1), (2.6), if

$$q_\mathrm{c}(t) > 0\,, \qquad \text{whenever} \;\; q_\mathrm{v}(t) > q_\mathrm{vs}\big(p(t), T(t)\big)\,. \tag{2.11}$$

We further require the solution to be *self-consistent* in the sense that the two variables q_r and n_r, which keep track of the amount of rain water, are either both zero, or both nonzero.

In the following section we investigate initial value problems with critical initial data in more detail to settle existence and uniqueness of physically consistent solutions. Subsequently, in Section 4, we turn to the self-consistency of this solution.

3. Critical initial value problems. For given values of p and T the vapor mass q_vs at saturation level is given by

$$q_\mathrm{vs}(p,T) \;=\; \varepsilon\,\frac{p_\mathrm{s}(T)}{p}\,, \tag{3.1}$$

where $\varepsilon = R_\mathrm{a}/R_\mathrm{v}$ is the ratio of the specific gas constants of dry air and water vapor, respectively, and p_s is the saturation vapor pressure. The latter satisfies the Clausius-Clapeyron differential equation

$$p_\mathrm{s}'(T) \;=\; L(T)\,\frac{p_\mathrm{s}(T)}{R_\mathrm{v}T^2}\,, \qquad p_\mathrm{s}(T_0) = p_0\,, \tag{3.2}$$

where L, again, denotes the latent heat of vaporization, and (p_0, T_0) is the triple point of water; cf., e.g., Rogers and Yau [9].

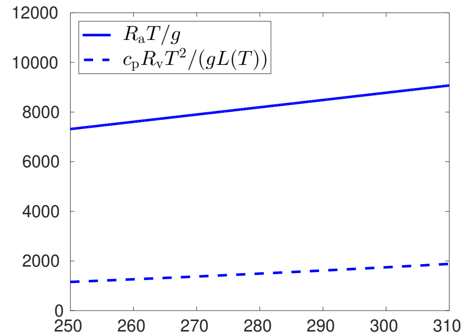

FIG. 3.1. *The graphs of the two competing functions in* (3.5)*, in units of* m *versus* K.

Using (3.1) and (3.2) we can substitute $y = q_{\rm v} - q_{\rm vs}$ in (2.1), (2.6), to obtain the system

$$\begin{aligned}
\dot{y} \;&=\; -d_8 y q_{\rm c}^{1/3} \,-\, d_9 y_- E \,+\, cw\,, && \text{(3.3a)}\\
\dot{q}_{\rm c} \;&=\; d_1 y q_{\rm c}^{1/3} \,-\, d_2 q_{\rm c}^2 \,-\, d_3 v q_{\rm c} q_{\rm r}^{2/3} n_{\rm r}^{1/3}\,, && \text{(3.3b)}\\
\dot{q}_{\rm r} \;&=\; y_- E \,+\, d_2 q_{\rm c}^2 \,+\, d_3 v q_{\rm c} q_{\rm r}^{2/3} n_{\rm r}^{1/3} \,-\, v q_{\rm r}\,, && \text{(3.3c)}\\
\dot{n}_{\rm r} \;&=\; \frac{n_{\rm r}}{q_{\rm r}} y_- E \,+\, d_4 q_{\rm c} \,-\, \lambda v n_{\rm r}\,, && \text{(3.3d)}\\
\dot{p} \;&=\; -d_{10} w\,, && \text{(3.3e)}\\
\dot{T} \;&=\; -d_{11} w \,+\, d_{12}\bigl(d_1 y q_{\rm c}^{1/3} \,+\, y_- E\bigr)\,, && \text{(3.3f)}
\end{aligned}$$

where, again, y_- is short hand notation for $\min\{y, 0\}$,

$$d_8 \;=\; d_1\Bigl(1 + \varepsilon \frac{L(T)p_{\rm s}'(T)}{c_{\rm p}p}\Bigr)\,, \qquad d_9 \;=\; 1 + \varepsilon \frac{L(T)p_{\rm s}'(T)}{c_{\rm p}p}\,, \tag{3.4}$$

d_{10}, d_{11}, and d_{12} are abbreviations for the corresponding coefficients in (2.6), and

$$\begin{aligned}
c \;=\; c(p,T) \;&=\; \frac{\varepsilon}{p}\Bigl(\frac{g}{c_{\rm p}} p_{\rm s}'(T) \,-\, \frac{g}{R_{\rm a}} \frac{p_{\rm s}(T)}{T}\Bigr)\\
&=\; \frac{g^2}{c_{\rm p}R_{\rm v}^2} \frac{p_{\rm s}(T)L(T)}{pT^3} \Bigl(\frac{R_{\rm a}T}{g} \,-\, \frac{c_{\rm p}R_{\rm v}T^2}{gL(T)}\Bigr)\,.
\end{aligned} \tag{3.5}$$

Take note that $d_8, \dots, d_{12}$ are also Lipschitz continuous functions of pressure, temperature, and droplet mass $q_{\rm c}$, and due to (2.8) and (2.9) they are uniformly bounded from above and away from zero, because $p_{\rm s}'$ is always positive by virtue of (3.2).

Finally, c of (3.5) is also Lipschitz continuous, and in the relevant temperature regime c can be considered to be strictly positive: Figure 3.1 provides the graphs of $R_{\rm a}T/g$ and $c_{\rm p}R_{\rm v}T^2/(gL(T))$ for a temperature range from 250 up to 310 K, based on a common approximation of the latent heat of vaporization taken from [9][‡]; this plot clearly demonstrates that the term in parantheses in (3.5) is strictly positive, and hence, so is c.

[‡]The approximation of the latent heat from [6] yields the same plot.

Obviously, the two systems of differential equations are equivalent, i.e., any physically consistent solution of (2.1), (2.6) yields a physically consistent solution of (3.3) with $y^\circ \geqslant -q_{\mathrm{vs}}(p^\circ, T^\circ)$, and vice versa. To see the latter, determine

$$q_{\mathrm{v}} \;=\; y \,+\, q_{\mathrm{vs}}$$

from the corresponding component y of the solution of (3.3) by using q_{vs} of (3.1), (3.2); it follows that q_{v} satisfies (2.1a) with initial data $q_{\mathrm{v}}(0) \geqslant 0$, and hence, q_{v} will be nonnegative throughout, because the right-hand side of (2.1a) is positive whenever q_{v} is below the positive value of q_{vs}.

Now we consider critical initial value problems in the subsaturated regime; by $y^\circ = q_{\mathrm{v}}^\circ - q_{\mathrm{vs}}(p^\circ, T^\circ)$ we denote the initial data to be used for $y(0)$.

THEOREM 3.1. *Let the coefficient* (3.5) *satisfy* $c(p^\circ, T^\circ) > 0$ *for given positive values* p° *and* T°*, and assume that* y° *is either negative with* $y^\circ \geqslant -q_{\mathrm{vs}}(p^\circ, T^\circ)$*, or* $y^\circ = 0$ *and the external forcing* w *satisfies* $w(t) < 0$ *for* t *within some open time interval* $(0, t_0)$*. Then the system* (3.3) *with initial data*

$$q_{\mathrm{c}}^\circ = 0\,, \qquad q_{\mathrm{r}}^\circ \geqslant 0\,, \qquad n_{\mathrm{r}}^\circ \geqslant 0\,,$$

and p°*,* T°*, and* y° *as above admits a unique physically consistent solution in some open time interval* $(0, t_0')$*, provided that* q_{r}° *and* n_{r}° *are either both zero, or both nonzero. The component* q_{c} *of this solution is identically zero, and if* $q_{\mathrm{r}}^\circ = n_{\mathrm{r}}^\circ = 0$ *then* q_{r} *and* n_{r} *are also identically zero in* $(0, t_0')$*.*

Proof. For any admissible solution of (3.3), $y(t)$ will be negative for positive time t near $t = 0$ due to (3.3a) and the premises of this theorem. On the other hand, as long as y is nonpositive, the right-hand side of (3.3b) is nonpositive either. This proves that q_{c} must vanish identically in some interval $(0, t_0')$, and in this interval the other components of the solution must satisfy the reduced system

$$\begin{aligned}
\dot{y} &= cw - d_9 y_- E\,, && \text{(3.6a)}\\
\dot{q}_{\mathrm{r}} &= y_- E - v q_{\mathrm{r}}\,, && \text{(3.6b)}\\
\dot{n}_{\mathrm{r}} &= \frac{n_{\mathrm{r}}}{q_{\mathrm{r}}} y_- E - \lambda v n_{\mathrm{r}}\,, && \text{(3.6c)}\\
\dot{p} &= -d_{10} w\,, && \text{(3.6d)}\\
\dot{T} &= -d_{11} w + d_{12} y_- E\,. && \text{(3.6e)}
\end{aligned}$$

When q_{r}° and n_{r}° are both nonzero, we can thus use the Picard-Lindelöf Theorem to obtain a unique solution of the corresponding initial value problem for (3.6), and together with $q_{\mathrm{c}} = 0$ we have found a physically consistent solution of (3.3), which satisfies the given initial data.

Consider next the case that $q_{\mathrm{r}}^\circ = n_{\mathrm{r}}^\circ = 0$. Since the right-hand sides of (3.6b) and (3.6c) are nonpositive, each, in $(0, t_0')$, the components q_{r} and n_{r} of any admissible solution of (3.3) must stay zero in $(0, t_0')$ in this case. In fact, this zero solution, completed with $q_{\mathrm{c}} = 0$ and the (unique) solution of the system

$$\dot{y} = cw\,, \qquad \dot{p} = -d_{10} w\,, \qquad \dot{T} = -d_{11} w\,,$$

for the corresponding initial data y°, p°, and T°, does indeed provide a physically consistent solution of (3.3) in $[0, t_0')$, and this is the only physically consistent one for these initial data; recall that the first term on the right-hand side of (3.6c) is considered to be zero for this solution. □

Next we turn to the more difficult supersaturated regime.

THEOREM 3.2. *Let the coefficient* (3.5) *satisfy* $c(p^\circ, T^\circ) > 0$ *for given values* $p^\circ, T^\circ > 0$. *If either* $y^\circ > 0$, *or* $y^\circ = 0$ *and the external forcing* w *is analytic near* $t = 0$ *with* $w(t) > 0$ *for* t *within some open time interval* $(0, t_0)$, *then the system* (3.3) *with initial data*

$$q_{\rm c}^\circ = 0\,, \qquad q_{\rm r}^\circ \geqslant 0\,, \qquad n_{\rm r}^\circ \geqslant 0\,,$$

and p°, T°, *and* y° *as above admits a unique physically consistent solution in some open time interval* $(0, t_0')$, *provided that* $q_{\rm r}^\circ$ *and* $n_{\rm r}^\circ$ *are either both zero, or both nonzero. In either case* t_0' *can be chosen so small that* $q_{\rm c}(t)$, $q_{\rm r}(t)$, *and* $n_{\rm r}(t)$ *are positive for* $t \in (0, t_0')$.

Proof. By assumption, if $y^\circ = 0$ then the external forcing w admits a Taylor expansion

$$w(t) \;=\; a_\nu t^\nu + O(t^{\nu+1})\,, \qquad t \to 0\,, \tag{3.7}$$

for some $\nu \in \mathbb{N}_0$ and $a_\nu > 0$.

Next we observe that for any solution of (3.3) with $q_{\rm c} > 0$ the function

$$x_{\rm c} \;=\; q_{\rm c}^{2/3} \tag{3.8}$$

solves the differential equation

$$\dot x_{\rm c} \;=\; \frac{2}{3} d_1 y \;-\; \frac{2}{3} d_2 x_{\rm c}^{5/2} \;-\; \frac{2}{3} d_3 v q_{\rm r}^{2/3} n_{\rm r}^{1/3} x_{\rm c} \tag{3.9}$$

by virtue of (3.3b).

The main body of this proof is divided in two steps. In the first step we show that (i) $q_{\rm c} > 0$ near $t = 0$ for any physically consistent solution of (3.3) with initial data as stipulated in the statement of this theorem, and (ii) that y and $x_{\rm c}$ of (3.8) can be written as

$$y(t) = t^n u_{\rm s}(t)\,, \qquad\qquad x_{\rm c}(t) = t^{n+1} u_{\rm c}(t)\,, \tag{3.10a}$$

while

$$q_{\rm r}(t) = t^{3n+4} u_{\rm q}(t)\,, \qquad\qquad n_{\rm r}(t) = t^{(3n+5)/2} u_{\rm n}(t)\,, \tag{3.10b}$$

if $q_{\rm r}^\circ = n_{\rm r}^\circ = 0$. In (3.10) the time variable t varies in $[0, t_0')$ for some $t_0' > 0$, and $u_{\rm s}, u_{\rm c}, u_{\rm q}, u_{\rm n}$ belong to $C^1(0, t_0') \cap C([0, t_0'))$ with strictly positive initial data at time $t = 0$; finally, the parameter $n \in \mathbb{N}_0$ is given by

$$n = \begin{cases} 0\,, & \text{if } y^\circ > 0\,, \\ \nu + 1\,, & \text{if } y^\circ = 0\,, \end{cases} \tag{3.11}$$

where ν is the leading order exponent in (3.7). In the second step we prove that there exists exactly one solution of (3.3) satisfying (3.10) with n as in (3.11).

Throughout this proof we will simply write $d_i(t)$ for $d_i(p(t), T(t), q_{\rm c}(t))$, $i = 1, 2, \ldots, 12$, and d_i° for $d_i(0) = d_i(p^\circ, T^\circ, 0)$. The same simplified notation is adapted for c of (3.5) and v of (2.3).

To begin with the first part of the proof, let $(y, q_{\rm c}, q_{\rm r}, n_{\rm r}, p, T)$ be a physically consistent solution of (3.3) in some time interval $(0, t_0')$ with the stipulated initial values at $t = 0$. Note that the boundedness of the right-hand sides of (3.3b), (3.3e), and (3.3f) implies that q_{c}, p, and T are Lipschitz continuous near $t = 0$, and hence, the Lipschitz continuity of d_i and c with respect to p, T, and q_{c} implies that

$$d_i(t) \;=\; d_i^\circ + O(t)\,, \quad c(t) \;=\; c^\circ + O(t)\,, \qquad t \to 0\,.$$

This will be used later on.

Since the right-hand side of (3.3a) is Lipschitz continuous with respect to y, this differential equation has a unique solution y with $y(0) = y^\circ$ for the given components $q_{\mathrm{c}}, q_{\mathrm{r}}, n_{\mathrm{r}}, p$, and T. It is easily checked that this solution is given near $t = 0$ by

$$y(t) \;=\; f(t)\Big(y^\circ \;+\; \int_0^t \frac{c(\tau)w(\tau)}{f(\tau)}\,\mathrm{d}\tau\Big) \tag{3.12}$$

with

$$f(t) \;=\; \exp\Big(-\int_0^t d_8(s)q_{\mathrm{c}}^{1/3}(s)\,\mathrm{d}s\Big)\,,$$

because, obviously, the right-hand side of (3.12) is positive for sufficiently small $t > 0$ by virtue of our premises on y° and w. It follows that $f(t) = 1 + O(t)$, $t \to 0$, and

$$y(t) \;=\; y^\circ + O(t)\,, \quad t \to 0\,, \qquad \text{if } y^\circ > 0\,,$$

whereas if $y^\circ = 0$, then a Taylor expansion based on (3.7) yields

$$\begin{aligned} y(t) \;&=\; \big(1 + O(t)\big)\int_0^t \frac{c^\circ a_{n-1}\tau^{n-1} + O(\tau^n)}{1 + O(\tau)}\,\mathrm{d}\tau \\ &=\; \frac{c^\circ a_{n-1}}{n}\,t^n + O(t^{n+1})\,, \quad t \to 0\,. \end{aligned}$$

We have thus established the first equation of (3.10a) for the exponent $n \in \mathbb{N}_0$ given in (3.11) and some function u_{s}, which is differentiable in $(0, t_0')$ because y is, and which extends continuously to $t = 0$ with corresponding value

$$u_{\mathrm{s}}(0) \;=\; u_{\mathrm{s}}^\circ \;=\; \begin{cases} y^\circ\,, & \text{if } y^\circ > 0\,, \\ c^\circ a_{n-1}/n\,, & \text{if } y^\circ = 0\,. \end{cases} \tag{3.13}$$

From this and (2.11) we conclude that $q_{\mathrm{c}} > 0$ for $t \in (0, t_0')$, and hence, x_{c} of (3.8) is well-defined and solves the differential equation (3.9) in this time interval. Given the representation (3.10a) of y we further conclude from (3.9) that

$$x_{\mathrm{c}}(t) \;\leqslant\; \frac{2}{3}\int_0^t d_1(\tau)y(\tau)\,\mathrm{d}\tau \;=\; \frac{2\,u_{\mathrm{s}}^\circ d_1^\circ}{3(n+1)}\,t^{n+1} + O(t^{n+2})\,, \qquad t \to 0\,,$$

and inserting this back into (3.9) we get the corresponding lower bound

$$\begin{aligned} x_{\mathrm{c}}(t) \;&\geqslant\; \frac{2}{3}\int_0^t d_1(\tau)y(\tau)\,\mathrm{d}\tau \;-\; \int_0^t \Big(C_2\tau^{5(n+1)/2} + C_3\tau^{n+1}\Big)\,\mathrm{d}\tau \\ &=\; \frac{2\,u_{\mathrm{s}}^\circ d_1^\circ}{3(n+1)}\,t^{n+1} - O(t^{n+2})\,, \qquad t \to 0\,, \end{aligned}$$

where C_2 and C_3 are suitable positive constants due to (2.4) and the boundedness of all other functions involved in (3.9). This establishes the corresponding equation (3.10a) for x_c with

$$u_\mathrm{c}(0) \;=\; u_\mathrm{c}^\circ \;=\; \frac{2\,u_\mathrm{s}^\circ d_1^\circ}{3(n+1)}\,. \tag{3.14}$$

Again, $u_\mathrm{c} \in C^1(0,t_0') \cap C([0,t_0'))$, for the same reason as above.

Finally, assume that $q_\mathrm{r}^\circ = n_\mathrm{r}^\circ = 0$. Since $y(t) > 0$ for $0 < t < t_0'$, n_r satisfies the differential equation

$$\dot n_\mathrm{r} \;=\; d_4 x_\mathrm{c}^{3/2} \;-\; \lambda v n_\mathrm{r}$$

by virtue of (3.3d) and (3.8), and hence,

$$n_\mathrm{r}(t) \;=\; g(t)\int_0^t \frac{d_4(\tau)x_\mathrm{c}^{3/2}(\tau)}{g(\tau)}\,\mathrm{d}\tau \qquad \text{with} \qquad g(t) \;=\; \exp\Bigl(-\int_0^t \lambda v(s)\,\mathrm{d}s\Bigr)\,.$$

From (2.4), (3.10a), and (3.14) we therefore obtain the second equation in (3.10b) by Taylor expansion, and there holds

$$u_\mathrm{n}(0) \;=\; u_\mathrm{n}^\circ \;=\; \frac{2}{3n+5}\,d_4^\circ\,(u_\mathrm{c}^\circ)^{3/2}\,. \tag{3.15}$$

Concerning q_r we combine equations (3.3b) and (3.3c) – and use the fact that $y \geqslant 0$ in $(0,t_0')$ – to conclude that

$$\dot q_\mathrm{r} \;+\; \dot q_\mathrm{c} \;=\; d_1 y q_\mathrm{c}^{1/3} \;-\; v q_\mathrm{r}\,,$$

which implies that

$$\begin{aligned} q_\mathrm{r}(t) \;&=\; -q_\mathrm{c}(t) \;+\; \int_0^t \bigl(d_1(\tau)y(\tau)q_\mathrm{c}^{1/3}(\tau) \;-\; v(\tau)q_\mathrm{r}(\tau)\bigr)\,\mathrm{d}\tau \\ &\leqslant\; \int_0^t d_1(\tau)y(\tau)x_\mathrm{c}^{1/2}(\tau)\,\mathrm{d}\tau\,, \end{aligned}$$

where we have used (3.8) and the nonnegativity of q_c and q_r for the final step. It therefore follows from (3.10a) that

$$q_\mathrm{r}(t) \;=\; O(t^{(3n+3)/2})\,, \qquad t \to 0\,. \tag{3.16}$$

We further conclude from (3.3c) and (3.8) that

$$q_\mathrm{r}(t) \;\leqslant\; \int_0^t \bigl(d_2(\tau)x_\mathrm{c}^3(\tau) \;+\; d_3(\tau)v(\tau)x_\mathrm{c}^{3/2}(\tau)q_\mathrm{r}^{2/3}(\tau)n_\mathrm{r}^{1/3}(\tau)\bigr)\,\mathrm{d}\tau\,,$$

and hence, the previously established asymptotics (3.10) and the preliminary estimate (3.16) imply that

$$q_\mathrm{r}(t) \;\leqslant\; \frac{d_2^\circ(u_\mathrm{c}^\circ)^3}{3n+4}\,t^{3n+4} \;+\; O(t^{3n+13/3})\,, \qquad t \to 0\,.$$

Inserting this back into (3.3c) and using that $y_{-} = 0$ we also obtain the matching lower bound

$$q_{\mathrm{r}}(t) \geqslant \int_0^t \big(d_2(\tau)x_{\mathrm{c}}^3(\tau) - v(\tau)q_{\mathrm{r}}(\tau)\big)\,\mathrm{d}\tau \geqslant \frac{d_2^\circ(u_{\mathrm{c}}^\circ)^3}{3n+4}\,t^{3n+4} - O(t^{3n+5}),$$

from which the first representation in (3.10b) follows with

$$u_{\mathrm{q}}(0) = u_{\mathrm{q}}^\circ = \frac{d_2^\circ(u_{\mathrm{c}}^\circ)^3}{3n+4}. \tag{3.17}$$

We have thus succeeded in establishing all four equations of (3.10).

For the second part of the proof (existence and uniqueness) we restrict ourselves to the case that $y^\circ = 0$, i.e., that n of (3.11) is positive, and assume that $q_{\mathrm{r}}^\circ = n_{\mathrm{r}}^\circ = 0$. The other cases are left to the reader; they are easier and can be treated in the same way. We will use the short-hand notation $\boldsymbol{u} = (u_{\mathrm{s}}, u_{\mathrm{c}}, u_{\mathrm{q}}, u_{\mathrm{n}})$, and consider a time interval $[0, t_0']$ for some $t_0' > 0$ sufficiently small, and a closed interval $\Omega \subset \mathbb{R}_+^6$ for the values of $(p, T, \boldsymbol{u})$, assuming that Ω contains the initial data $(p^\circ, T^\circ, u_{\mathrm{s}}^\circ, u_{\mathrm{c}}^\circ, u_{\mathrm{q}}^\circ, u_{\mathrm{n}}^\circ)$ in its interior; the latter four values are specified in (3.13), (3.14), (3.17), and (3.15).

Inserting (3.10) into (3.9), (3.3a), and (3.3c)–(3.3f) we obtain an equivalent system of integral equations

$$\begin{bmatrix} p \\ T \\ u_{\mathrm{s}} \\ u_{\mathrm{c}} \\ u_{\mathrm{q}} \\ u_{\mathrm{n}} \end{bmatrix} = \begin{bmatrix} \phi_p(p,T,\boldsymbol{u}) \\ \phi_T(p,T,\boldsymbol{u}) \\ \phi_{\mathrm{s}}(p,T,\boldsymbol{u}) \\ \phi_{\mathrm{c}}(p,T,\boldsymbol{u}) \\ \phi_{\mathrm{q}}(p,T,\boldsymbol{u}) \\ \phi_{\mathrm{n}}(p,T,\boldsymbol{u}) \end{bmatrix} = \Phi(p,T,\boldsymbol{u}) \tag{3.18}$$

for p, T, and $\boldsymbol{u}$; the details of this derivation will be provided only for u_{n}, because this equation constitutes a representative case.

From (3.10) and (3.8) it follows by integrating (3.3d) that

$$\begin{aligned} t^{(3n+5)/2}u_{\mathrm{n}}(t) &= \int_0^t \Big(d_4(\tau)\tau^{(3n+3)/2}u_{\mathrm{c}}^{3/2}(\tau) - \lambda v(\tau)\tau^{(3n+5)/2}u_{\mathrm{n}}(\tau)\Big)\,\mathrm{d}\tau \\ &= \int_0^1 \Big((st)^{(3n+3)/2}d_4(st)u_{\mathrm{c}}^{3/2}(st) - (st)^{(3n+5)/2}\lambda v(st)u_{\mathrm{n}}(st)\Big)t\,\mathrm{d}s \end{aligned}$$

for $t \in [0, t_0']$, where we have used that $n_{\mathrm{r}}(0) = 0$ and $y(t) \geqslant 0$ by virtue of (3.10). We have thus arrived at the bottom identity of (3.18) with

$$\begin{aligned} \big(\phi_{\mathrm{n}}(p,T,\boldsymbol{u})\big)(t) = &\int_0^1 s^{(3n+3)/2}d_4(st)u_{\mathrm{c}}^{3/2}(st)\,\mathrm{d}s \\ &- t\int_0^1 s^{(3n+5)/2}\lambda v(st)u_{\mathrm{n}}(st)\,\mathrm{d}s. \end{aligned} \tag{3.19}$$

Vice versa, if continuous functions p, T, and $\boldsymbol{u}$ solve (3.18) in $[0, t_0']$, then n_{r} of (3.10) belongs to $C([0, t_0']) \cap C^1((0, t_0'))$ and solves the differential equation (3.3d).

Recall that d_4 is a bounded Lipschitz continuous function of p, T, and u_{c}, and v of (2.3) can be rewritten as

$$v = d_7\Big(\frac{mu_{\mathrm{q}}}{mu_{\mathrm{n}} + t^{(3n+3)/2}u_{\mathrm{q}}}\Big)^\mu t^{\mu(3n+3)/2}$$

by virtue of (3.10); accordingly, v is a bounded Lipschitz continuous function of p, T, u_{c}, u_{q}, and u_{n} for $t \in [0, t_0']$ and $(p, T, \boldsymbol{u}) \in \Omega$. It thus follows from (3.19) that there exists a constant $L > 0$, such that

$$\begin{aligned} \big\| \phi_{\mathrm{n}}(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) \;-\; \phi_{\mathrm{n}}(p, T, \boldsymbol{u}) \big\| \;\leqslant\; & L\Big(\|\widetilde{p} - p\| + \|\widetilde{T} - T\| + \|\widetilde{u}_{\mathrm{c}} - u_{\mathrm{c}}\| \Big) \\ & + \; L t_0' \Big(\|\widetilde{u}_{\mathrm{q}} - u_{\mathrm{q}}\| + \|\widetilde{u}_{\mathrm{n}} - u_{\mathrm{n}}\| \Big) \end{aligned} \tag{3.20}$$

for every $(p, T, \boldsymbol{u}), (\widetilde{p}, \widetilde{T} \widetilde{\boldsymbol{u}}) \in \Omega$, where $\|\cdot\|$ refers to the supremum norm over $[0, t_0']$. Note that the power of t_0' in (3.20) is not optimal, but this suffices for our purposes. Further, if p, T, and $\boldsymbol{u}$ attain the stipulated initial data, then

$$\big(\phi_{\mathrm{n}}(p, T, \boldsymbol{u})\big)(0) \;=\; \int_0^1 s^{(3n+3)/2} d_4^\circ (u_{\mathrm{c}}^\circ)^{3/2} \,\mathrm{d}s \;=\; \frac{2}{3n+5}\, d_4^\circ (u_{\mathrm{c}}^\circ)^{3/2} \;=\; u_{\mathrm{n}}^\circ \,,$$

compare (3.15), and hence we have

$$\begin{aligned} \big\| \phi_{\mathrm{n}}(p, T, \boldsymbol{u}) - u_{\mathrm{n}}^\circ \big\| \;&\leqslant\; \int_0^1 s^{(3n+3)/2} \Big\| d_4 u_{\mathrm{c}}^{3/2} - d_4^\circ (u_{\mathrm{c}}^\circ)^{3/2} \Big\| \,\mathrm{d}s \;+\; t_0' \int_0^1 s^{(3n+5)/2} \lambda \,\big\| v u_{\mathrm{n}} \big\| \,\mathrm{d}s \\ &\leqslant\; L\Big(\|p - p^\circ\| + \|T - T^\circ\| + \|u_{\mathrm{c}} - u_{\mathrm{c}}^\circ\| \Big) \;+\; L t_0' \end{aligned}$$

after adjusting the constant L, when necessary. Accordingly, by choosing the Ω-components of the p, T, and u_{c} variables sufficiently narrow and t_0' sufficiently small, the values of $\phi_{\mathrm{n}}(p, T, \boldsymbol{u})$ belong to the u_{n}-component of Ω.

Proceeding in a similar fashion for the other variables one can show that the nonlinear integral operator Φ of (3.18) maps

$$\mathcal{X}^\circ \;=\; \Big\{ (p, T, \boldsymbol{u}) \in C\big([0, t_0'], \Omega\big) \;:\; p(0) = p^\circ,\, T(0) = T^\circ,\, \boldsymbol{u}(0) = (u_{\mathrm{s}}^\circ, u_{\mathrm{c}}^\circ, u_{\mathrm{q}}^\circ, u_{\mathrm{n}}^\circ) \Big\}$$

to itself, provided that t_0' is sufficiently small and Ω is sufficiently narrow. Moreover, there holds

$$\begin{bmatrix} \|\phi_p(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \phi_p(p, T, \boldsymbol{u})\| \\ \|\phi_T(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \phi_T(p, T, \boldsymbol{u})\| \\ \|\phi_{\mathrm{s}}(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \phi_{\mathrm{s}}(p, T, \boldsymbol{u})\| \\ \|\phi_{\mathrm{c}}(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \phi_{\mathrm{c}}(p, T, \boldsymbol{u})\| \\ \|\phi_{\mathrm{q}}(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \phi_{\mathrm{q}}(p, T, \boldsymbol{u})\| \\ \|\phi_{\mathrm{n}}(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \phi_{\mathrm{n}}(p, T, \boldsymbol{u})\| \end{bmatrix} \leqslant \begin{bmatrix} Lt_0' & Lt_0' & 0 & Lt_0' & 0 & 0 \\ Lt_0' & Lt_0' & Lt_0' & Lt_0' & 0 & 0 \\ L & L & Lt_0' & Lt_0' & 0 & 0 \\ L & L & L & Lt_0' & Lt_0' & Lt_0' \\ L & L & 0 & L & Lt_0' & Lt_0' \\ L & L & 0 & L & Lt_0' & Lt_0' \end{bmatrix} \begin{bmatrix} \|\widetilde{p} - p\| \\ \|\widetilde{T} - T\| \\ \|\widetilde{u}_{\mathrm{s}} - u_{\mathrm{s}}\| \\ \|\widetilde{u}_{\mathrm{c}} - u_{\mathrm{c}}\| \\ \|\widetilde{u}_{\mathrm{q}} - u_{\mathrm{q}}\| \\ \|\widetilde{u}_{\mathrm{n}} - u_{\mathrm{n}}\| \end{bmatrix}, \tag{3.21}$$

where the inequality sign is meant componentwise. Again, no attempt has been made to come up with optimal powers of t_0' in the matrix $A_{t_0'}$ on the right-hand side of (3.21).

In order to complete the proof we need to show that (3.18) has a unique fixed point $(p, T, \boldsymbol{u})$ in the complete metric space $\mathcal{X}^\circ$, equipped with its natural metric given by the norm $\|\cdot\|_{\mathcal{X}}$ of $\mathcal{X} = C([0, t_0'], \mathbb{R}^6)$. It is obvious from (3.21) that Φ is Lipschitz continuous with respect to this topology, but it is also readily seen from (3.20) that Φ fails to be a contraction in general. On the other hand, when turning to $\Phi^4 = \Phi \circ \Phi \circ \Phi \circ \Phi$ we conclude from (3.21) that

$$\big\| \Phi^4(\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - \Phi^4(p, T, \boldsymbol{u}) \big\|_{\mathcal{X}} \;\leqslant\; \|A_{t_0'}^4\|_\infty \, \big\| (\widetilde{p}, \widetilde{T}, \widetilde{\boldsymbol{u}}) - (p, T, \boldsymbol{u}) \big\|_{\mathcal{X}} \,,$$

where $\|A^4_{t_0'}\|_\infty$ denotes the row sum norm of $A^4_{t_0'}$, and this norm can be made smaller than one by choosing t_0' sufficiently small: This is easily seen by observing that for $t_0' = 0$ there holds

$$A_0 \,=\, L \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A_0^2 = L^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and hence,

$$\lim_{t_0' \to 0} \|A^4_{t_0'}\|_\infty \,=\, \|(A_0^2)^2\|_\infty \,=\, 0\,.$$

We thus have shown that Φ is a Lipschitz map from $\mathcal{X}^\circ$ to itself and that Φ^4 is a contraction, provided that t_0' is sufficiently small and Ω is sufficiently narrow. Therefore, one can make use of Weissinger's fixed point theorem [14, 12] to verify that (3.18) has a unique fixed point in $\mathcal{X}^\circ$. In other words, the initial value problem (3.3a), (3.9), (3.3c)–(3.3f) has a unique solution of the form (3.10).

With $q_\mathrm{c} = x_\mathrm{c}^{3/2}$, compare (3.8), we thus obtain the unique physically consistent solution of (3.3) for the given initial data. Moreover, according to (3.10), q_c, q_r, and n_r are positive in $(0, t_0')$. □

Remark 3.3. The assumption in Theorem 3.2 that w be analytic can be relaxed to merely requiring that w be continuous with

$$w(t) \,=\, a_\nu t^\nu \,+\, o(t^\nu)\,, \qquad t \to 0\,,$$

for some $\nu \geqslant 0$ and $a_\nu > 0$, compare (3.7).

So far we did not consider critical initial data with $q_\mathrm{c}^\circ \neq 0$, because this case is of little relevance in practice, as we will see in Section 4. For the ease of completeness we nevertheless state the corresponding result.

Theorem 3.4. *The system* (3.3) *with initial data*

$$q_\mathrm{c}^\circ > 0\,, \qquad q_\mathrm{r}^\circ = n_\mathrm{r}^\circ = 0\,, \qquad p^\circ > 0\,, \qquad T^\circ > 0\,,$$

and $y^\circ \geqslant -q_\mathrm{vs}(p^\circ, T^\circ)$ *admits a unique local solution, and the correponding components* q_r *and* n_r *are positive in some open time interval* $(0, t_0')$.

By definition this solution is physically consistent. We omit the proof of Theorem 3.4, which can follow the lines of the preceeding one. No assumption on w besides continuity is required for this result.

The unique physically consistent solution for given initial data can be extended as in the classical Lipschitz continuous case (at least) up to the point when its trajectory reaches the boundary of the particular domain, in which the right-hand side of (3.3) is well-defined and Lipschitz. Since we already know that any admissible solution of (3.3) is bounded, this continuation ends (if at all) at a point where one of its components q_v, q_c, q_r, or n_r turns zero. As we have mentioned before, q_v cannot become zero, because q_vs is always positive, cf. (3.1), and hence, $\dot{q}_\mathrm{v} > 0$ whenever $0 < q_\mathrm{v} < q_\mathrm{vs}$ by virtue of (2.1a). At zeros of q_c, on the other hand, we can use Theorems 3.1 and 3.2 to extend the solution (uniquely) in a physically consistent way. It thus remains to consider the case that n_r or q_r become zero.

4. Zeros of $q_{\rm r}$ and $n_{\rm r}$: Self-consistency of the model. In reality rain drops are either present or they are not; in a realistic numerical model the associated quantities $q_{\rm r}$ and $n_{\rm r}$ should therefore either be both zero or both nonzero. We have already seen in the previous section that $q_{\rm r}$ and $n_{\rm r}$ always switch simultaneously from zero to nonzero depending on the level of saturation. Here we show that if $q_{\rm r}$ or $n_{\rm r}$ becomes zero in the course of the time evolution, then both quantities do indeed vanish simultaneously.

PROPOSITION 4.1. *Consider an admissible solution of* (3.3) *in some time interval* $\mathcal{I} = (t_0', t_0) \subset \mathbb{R}^+$. *If its component* $q_{\rm r}$ *satisfies* $q_{\rm r}(t) > 0$ *for* $t \in \mathcal{I}$ *and* $q_{\rm r}(t) \to 0$ *for* $t \to t_0$, *or if* $n_{\rm r}(t) > 0$ *for* $t \in \mathcal{I}$ *and* $n_{\rm r}(t) \to 0$ *for* $t \to t_0$, *then the solution extends continuously to* $t = t_0$ *with*

$$q_{\rm r}(t_0) \;=\; n_{\rm r}(t_0) \;=\; q_{\rm c}(t_0) \;=\; 0 \qquad \textit{and} \qquad y(t_0) \;\leqslant\; 0\,. \tag{4.1}$$

Moreover, $q_{\rm c}(t) > 0$ *for* t *near* t_0, $t < t_0$.

Proof. As observed in Section 2 all components of an admissible solution are bounded from above for $t_0' < t < t_0$, and so is the right-hand side of (3.3), with the possible exception of (3.3d). It follows that y, $q_{\rm c}$, $q_{\rm r}$, p, and T are Lipschitz continuous function of $t \in (t_0', t_0)$, and hence they extend continuously to $t = t_0$.

Consider first the case that $\mathcal{I} = (t_0', t_0)$ is such that $n_{\rm r}(t) > 0$ for $t \in \mathcal{I}$ and $n_{\rm r}(t) \to 0$ for $t \to t_0$. Let us assume that $q_{\rm r}(t_0) \neq 0$. Then there exists $\delta > 0$ and some time interval $[t_0'', t_0) \subset \mathcal{I}$ such that $q_{\rm r}(t) \geqslant \delta$ for all $t \in [t_0'', t_0)$. Using (3.3d) we have

$$y_{-}E \;\leqslant\; \frac{\dot n_{\rm r}}{n_{\rm r}}\, q_{\rm r} + \lambda v q_{\rm r}\,,$$

and inserting this into (3.3c) we conclude that

$$\frac{\dot q_{\rm r}}{q_{\rm r}} \;\leqslant\; \frac{\dot n_{\rm r}}{n_{\rm r}} \;+\; (\lambda-1)v \;+\; d_2\frac{q_{\rm c}^2}{q_{\rm r}} \;+\; d_3 v q_{\rm c}\Big(\frac{n_{\rm r}}{q_{\rm r}}\Big)^{1/3}$$

in $[t_0'', t_0)$. This implies that

$$\log\frac{n_{\rm r}(t)}{n_{\rm r}(t_0'')} \;\geqslant\; \log\frac{q_{\rm r}(t)}{q_{\rm r}(t_0'')} \;-\; \int_{t_0''}^{t}\Big((\lambda-1)v \;+\; d_2\frac{q_{\rm c}^2}{q_{\rm r}} \;+\; d_3 v q_{\rm c}\Big(\frac{n_{\rm r}}{q_{\rm r}}\Big)^{1/3}\Big)\,{\rm d}\tau$$

for all $t \in (t_0'', t_0)$. Since $q_{\rm r}$ is assumed to be bounded from below by δ, the right-hand side stays bounded as $t \to t_0$, whereas the left-hand side of this inquality tends to $-\infty$. This contradiction shows that $q_{\rm r}(t) \to 0$ as $t \to t_0$.

Consider next the other case that $q_{\rm r}(t) > 0$ in $\mathcal{I}$ with $q_{\rm r}(t) \to 0$ as $t \to t_0$, and let us assume that $n_{\rm r}(t) \geqslant \delta$ for some $\delta > 0$ and all $t \in [t_0'', t_0)$, where $t_0' \leqslant t_0'' < t_0$. We can use (3.3c) to estimate

$$y_{-}E \;\leqslant\; \dot q_{\rm r} + v q_{\rm r}\,,$$

and it thus follows from (3.3d) that

$$\frac{\dot n_{\rm r}}{n_{\rm r}} \;\leqslant\; \frac{\dot q_{\rm r}}{q_{\rm r}} \;+\; (1-\lambda)v \;+\; d_4\frac{q_{\rm c}}{n_{\rm r}}$$

in $[t_0'', t_0)$. Integrating the latter inequality from t_0'' to $t \in (t_0'', t_0)$ we conclude that

$$\log\frac{q_{\rm r}(t)}{q_{\rm r}(t_0'')} \;\geqslant\; \log\frac{n_{\rm r}(t)}{n_{\rm r}(t_0'')} \;-\; \int_{t_0''}^{t}\Big((1-\lambda)v \;+\; d_4\frac{q_{\rm c}}{n_{\rm r}}\Big)\,{\rm d}\tau\,.$$

By assumption, the right-hand side of this inequality stays bounded whereas the left-hand side tends to $-\infty$ as $t \to t_0$, so that we have arrived at a contradiction.

This means that there is some sequence $(t_k)_{k\in\mathbb{N}} \subset (t_0', t_0)$ with $t_k \to t_0$ and $n_\mathrm{r}(t_k) \to 0$, and it thus follows from (2.10) that

$$n_\mathrm{r}(t) \;\leqslant\; n_\mathrm{r}(t_k) \;+\; \int_{t_k}^{t} d_4 q_\mathrm{c}\,\mathrm{d}\tau\,, \qquad t_k \leqslant t < t_0\,.$$

Since the integrand on the right-hand side of this inequality is bounded, we conclude that

$$\limsup_{t\to t_0} n_\mathrm{r}(t) \;\leqslant\; 0\,,$$

and since n_r is nonnegative, this proves that $n_\mathrm{r}(t) \to 0$ for $t \to t_0$.

So, in either of the two cases the solution extends continuously to $t = t_0$ and the first two equations of (4.1) hold true.

Consider q_c next, and assume first that its limit $q_\mathrm{c}(t_0)$ is positive. Then $\dot q_\mathrm{r} > 0$ near $t = t_0$ according to (3.3c) and (2.2), and hence, q_r is strictly increasing near $t = t_0$, in contradiction to what we have already shown. Therefore $q_\mathrm{c}(t_0) = 0$. Assume next that $q_\mathrm{c}(t) = 0$ for all $t \in [t_0'', t_0]$ with some $t_0'' \in (t_0', t_0)$. In this case the differential equations for q_r and n_r simplify,

$$\dot q_\mathrm{r} \;=\; y_- E \;-\; v q_\mathrm{r} \qquad \text{and} \qquad \dot n_\mathrm{r} \;=\; \frac{n_\mathrm{r}}{q_\mathrm{r}} y_- E \;-\; \lambda v n_\mathrm{r} \tag{4.2}$$

in $[t_0'', t_0)$, cf. (3.6). Further note that we can choose t_0'' so close to t_0 that neither n_r nor q_r have a zero in $[t_0'', t_0)$, by what we have already shown. We thus conclude from (4.2) that

$$\frac{\dot n_\mathrm{r}}{n_\mathrm{r}} \;=\; \frac{\dot q_\mathrm{r}}{q_\mathrm{r}} \;+\; (1-\lambda)v\,,$$

and upon integration,

$$\log\frac{n_\mathrm{r}(t)}{n_\mathrm{r}(t_0'')} \;=\; \log\frac{q_\mathrm{r}(t)}{q_\mathrm{r}(t_0'')} \;+\; \int_{t_0''}^{t} (1-\lambda)v\,\mathrm{d}\tau\,, \qquad t_0'' < t < t_0\,.$$

As the integrand on the right-hand side is bounded, it follows that there exists some constant $C > 0$, such that

$$n_\mathrm{r}(t) \;\leqslant\; C q_\mathrm{r}(t)\,, \qquad t_0'' \leqslant t < t_0\,.$$

According to (2.2) this implies that $E \leqslant C' q_\mathrm{r}$ for some other constant $C' > 0$, and hence, (3.6b) yields

$$\dot q_\mathrm{r} \;\geqslant\; -\bigl(C'|y_-| + v\bigr) q_\mathrm{r}$$

in $[t_0'', t_0)$. From this we readily obtain the inequality

$$\log\frac{q_\mathrm{r}(t)}{q_\mathrm{r}(t_0'')} \;\geqslant\; -\int_{t_0''}^{t} \bigl(C'|y_-| + v\bigr)\,\mathrm{d}\tau\,,$$

which gives the desired contradiction as $t \to t_0$, because the integral on the right-hand side is bounded. This proves that $q_\mathrm{c}(t)$ is positive for t near t_0, $t < t_0$.

Since, on the other hand, $q_c(t) \to 0$ for $t \to t_0$, we necessarily have $y(t_0) \leqslant 0$, for otherwise the dominating term on the right-hand side of (3.3b) is given by $d_1 y q_c^{1/3} > 0$ for t close to t_0, and this would imply that q_c is strictly increasing near $t = t_0$. □

We thus have proved that the cloud model is self-contained in the sense that it cannot happen that one of the two rain drop quantities of an admissible solution of (3.3) is turning zero while the other one is not. We further have seen that the droplet mass will also vanish at this point – but not earlier. Note that if the cloud parcel is subsaturated and all cloud droplets have vaporized while rain drops are still present, then the proof of Proposition 4.1 shows that from this point onwards the rain drop population decreases exponentially, but does not exhale completely.

We have further seen that an extinction of cloud droplets is only possible in the subsaturated regime or exactly at saturation. Thereafter cloud droplets and rain drops reappear when the system returns into a supersaturated state, cf. Theorem 3.2.

5. Concluding remarks. This paper deals with a mathematical model for warm cloud parcels suggested in [7]. We have shown that the corresponding system of ordinary differential equations admits a unique physically consistent solution provided that the external forcing velocity of the air package is an analytic function of time. We have further demonstrated that the model is self-consistent in that the two rain drop quantities of the model are either simultaneously zero or nonzero.

Our result relies on the positivity of the physical parameter $c = c(p, T)$ of (3.5) for realistic values of pressure and temperature. As pointed out by Spichtinger [11] this assumption can be supported by physical arguments when interpreting the two competing terms in (3.5) as the scale heights of the pressure p and the saturation vapor pressure $p_s(T)$, respectively. The fact that the former is much larger than the latter in the relevant temperature regime is due to the relatively large latent heat of vaporization, which in turn is a consequence of the anomaly of water due to the strong hydrogen bonds.

We finally mention that the paper [7] not only contains the details for the derivation of this cloud model but also develops a sophisticated numerical integration scheme, which respects the nonnegativity of the masses of the different water species and the rain drop number; at the same time this numerical scheme conserves the total mass of water (up to precipitation, of course) and dry air.

Appendix. To facilitate the comparison with [7] we provide the explicit specifications of the parameters d_i of (2.5) in terms of the constants and coefficients tabulated in [7, Appendix A]:

$$\begin{aligned}
d_1 &= d\rho n_c^{2/3}\,, \\
d_2 &= k_1\rho/\rho_l\,, \\
d_3 &= k_2\pi\big(3/(4\pi\rho_l)\big)^{2/3}(h/c_q)\rho\,, \\
d_4 &= k_1\rho n_c/(2\rho_l)\,, \\
d_5 &= da_E\rho\,, \\
d_6 &= db_E(h/c_q)^{1/2}\rho\,, \\
d_7 &= \alpha(c_q/h)(\rho_*/\rho)^{1/2}\,, \\
d_{10} &= g\rho\,, \\
d_{11} &= \gamma\,, \\
d_{12} &= L/c_p\,,
\end{aligned}$$

where ρ is the density of dry air, which can be eliminated by using the ideal gas law

$$\rho = \frac{p}{R_\mathrm{a} T},$$

and n_c is the total number of cloud droplets (see below). Finally, the recommended value of λ to be used in (2.1d) is given by

$$\lambda = c_n / c_q$$

according to [7].

A constitutive algebraic constraint has been utilized in [7] to couple the droplet number n_c with their mass q_c; this constraint is given in terms of an explicit strictly positive and bounded Lipschitz continuous function $n_\mathrm{c} = n_\mathrm{c}(q_\mathrm{c})$. We like to use the occasion to advocate a slight modification of this function, namely

$$n_\mathrm{c}(q_\mathrm{c}) = \frac{1 + \kappa q_\mathrm{c}}{1 + \kappa q_\mathrm{c} + \kappa^2 q_\mathrm{c}^2} \frac{q_\mathrm{c}}{m_0} \coth \frac{q_\mathrm{c}}{N_0 m_0}, \qquad \kappa = \frac{1}{N_\infty m_0},$$

where m_0, N_0, and $N_\infty \gg N_0$ are suitable positive parameters (the same as in [7]), and they depend on the particular meteorological scenario to be simulated; see [7] for corresponding examples. The graph of the modified function suggested above is almost identical to the plot shown in [7, Fig. 3], but the present one is more appropriate from the modeling point of view because the resulting droplet number is strictly increasing with droplet mass. We emphasize that the precise definition of this function is irrelevant for the results in our present paper; all we need is that this function is strictly positive, bounded, and Lipschitz continuous.

Finally we note that only d_1, d_4, and d_8 do depend on q_c, because n_c does, and that d_{11} and d_{12} are constants.

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REFERENCES

[1] R.P. Agarwal and V. Lakshmikantham, *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations*, World Scientific, Singapore, 1993.

[2] S.R. Bernfeld, R.D. Driver, and V. Lakshmikantham, Uniqueness for ordinary differential equations, Math. Systems Theory **9** (1976), pp. 359–367.

[3] J.M. Bownds and J.B. Diaz, On restricted uniqueness for systems of ordinary differential equations, Proc. Amer. Math. Soc. **37** (1973), pp. 100–104.

[4] H.G. Kaper and M.K. Kwong, Uniqueness for a class of nonlinear initial value problems, J. Math. Anal. Appl. **130** (1988), pp. 467–473.

[5] S. Kichenassamy, *Fuchsian Reduction*, Birkhäuser, Boston, 2007.

[6] D.M. Murphy and T. Koop, Review of the vapour pressures of ice and supercooled water for atmospheric applications, Q. J. R. Meteorol. Soc. **131** (2005), pp. 1539–1565.

[7] N. Porz, M. Hanke, M. Baumgartner, and P. Spichtinger, A model for warm clouds with implicit droplet activation, avoiding saturation adjustment, Math. Clim. Weather Forecast. **4** (2018), pp. 50–78.

[8] A.D. Rendall and B.G. Schmidt, Existence and properties of spherically symmetric static fluid bodies with a given equation of state, Classical Quantum Gravity **8** (1991), pp. 985–1000.

[9] R.R. Rogers and M.K. Yau, *A Short Course in Cloud Physics*, 3rd ed., Pergamon Press, Oxford, 1989.

[10] A. Seifert and K.D. Beheng, A two-moment cloud microphysics parameterization for mixed-phase clouds. Part I: Model description, Meteorol. Atmos. Phys. **92** (2006), pp. 45–66.
[11] P. Spichtinger, private communication, May 2019.
[12] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, Amer. Math. Soc., Providence, RI, 2012.
[13] W. Walter, *Ordinary Differntial Equations*, Springer, New York, 1998.
[14] J. Weissinger, Zur Theorie und Anwendung des Iterationsverfahrens, Math. Nachr. **8** (1952), pp. 193–212.
[15] D.V.V. Wend, Existence and uniqueness of solutions of ordinary differential equations, Proc. Amer. Math. Soc. **23** (1969), pp. 27–33.