Convergence of finite volume schemes for the Euler equations via dissipative measure–valued solutions

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Abstract

The Cauchy problem for the complete Euler system is in general ill–posed in the class of admissible (entropy producing) weak solutions. This suggests that there might be sequences of approximate solutions that develop fine scale oscillations. Accordingly, the concept of *measure–valued solution* that captures possible oscillations is more suitable for analysis. We study the convergence of a class of entropy stable finite volume schemes for the barotropic and complete compressible Euler equations in the multidimensional case. We establish suitable stability and consistency estimates and show that the Young measure generated by numerical solutions represents a dissipative measure–valued solution of the Euler system. Here dissipative means that a suitable form of the Second law of thermodynamics is incorporated in the definition of the measure–valued solutions. In particular, using the recently established weak-strong uniqueness principle, we show that the numerical solutions converge pointwise to the regular solution of the limit systems at least on the lifespan of the latter.

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Contents

1 Introduction

The Euler equations of compressible fluid flow represent the simplest possible model that incorporates all fundamental principles of thermodynamics including the Second law usually expressed in terms of the entropy balance appended as an *admissibility condition* to the system. The entropy should be produced by any physically realistic process and this criterion is supposed to rule out the unphysical solutions that may still satisfy the basic system in the sense of distributions. In addition, the entropy balance provides crucial *a priori* bounds, in particular, positivity of the pressure when the system is written in the so–called conservative variables.

Another characteristic feature of the Euler system is that discontinuities may develop after a finite time even if the initial data are smooth. It is therefore quite natural to look for a weaker representation of solutions, for instance the weak solutions that satisfy the underlying equations in the sense of distributions, see [19, 30, 31, 36, 41, 44, 50] and the references therein. It is also a well-known fact that such weak solutions may fail to be unique, and, consequently, the Second law of thermodynamics has been proposed as a selection criterion. Although the entropy production principle has been efficient in the case of scalar multidimensional hyperbolic conservation laws as well as the one-dimensional systems, see [5, 6, 17, 40], it completely fails for multidimensional systems. Recently, it has been shown by De Lellis and Székelyhidi [21] and by Chiodaroli et al. [14] that infinitely many weak entropy solutions can be constructed for the multidimensional barotropic Euler equations, see also [20] for similar non-uniqueness results for the incompressible Euler equations. These results has been extended in Feireisl et al. [29] to the complete multidimensional Euler system in the class of L^∞ weak admissible solutions. In particular, these solutions satisfy the energy balance together with the entropy inequality; whence they are compatible with both the First and the Second law of thermodynamics.

Inspired by the previous results as well as by the numerical analysis performed in [33], we examine stability and convergence of certain numerical schemes in the class of so-called *dissipative measure–valued solutions*. The concept of measure–valued solutions for conservation laws is not new, see, e.g., [12, 24, 25, 42, 46, 49, 51] and the references therein. However, the recently introduced class of dissipative measure–valued solutions is particularly suitable since the weak–strong uniqueness holds and the dissipative measure–valued solution coincides with the classical solution as far as the latter exists [12, 13, 37]. Similar concept has been adopted by Tzavaras et al. [15, 22], in the context of elastodynamics, thermoelasticity, and other related problems. In the context of incompressible Euler equations we would like to mention the related results by DiPerna and Majda [26] where the measure–valued solutions have been used to study the vanishing viscosity limit of the Leray–Hopf weak solutions, by Brenier, De Lellis and Székelyhidi [11] for the measure–valued–strong principle and by Lions [45] where the concept of dissipative solutions has been introduced.

As is well known, the entropy stability of a numerical scheme plays a crucial role in the convergence analysis of numerical solutions. Construction of entropy conservative schemes has been introduced by Tadmor in a seminal paper [52]. This concept has been later used to study the entropy stability of numerical schemes, we refer the reader to [1, 7, 8, 16, 33, 35, 43, 53] and the references therein.

There is a considerable body of literature dealing with the convergence of numerical schemes for multidimensional hyperbolic conservation laws. Though the chosen techniques depend on the assumptions imposed on exact solutions, a certain form of the discrete entropy inequality is indispensable. Let us mention, for example, the results of Bouchut and Betherlin [7,8,10], where the kinetic flux–splitting method has been used. Relying on the fully discrete entropy inequality and applying the method of DiPerna [24] and Tartar's results on compensated compactness they proved strong convergence of fully discrete kinetic flux–splitting scheme to the bounded weak entropy solution of isentropic Euler equations (or the shallow water equations [9]) provided numerical solutions satisfies L^∞-bounds and the vacuum does not appear.

In [38] Jovanović and Rohde assumed the existence of a classical solution to the Cauchy problem of a general multidimensional hyperbolic conservation law. Applying the stability result for classical solutions in the class of entropy solutions due to Dafermos [18] and DiPerna's method [23, 24], they derived error estimates for the explicit finite volume schemes satisfying the discrete entropy inequality and thus proved that the numerical solutions convergence strongly to the exact classical solution.

In view of the fact that the classical solutions of hyperbolic conservation laws may not exist in general and in view of the recent results on non–uniqueness of weak entropy solutions [14, 20, 21], Fjordholm, Mishra and Tadmor revisited recently the question of convergence and proved that the semi–discrete entropy stable finite volume schemes converge to a measure–valued solution provided numerical solutions satisfy L^∞-bounds, coefficients of numerical viscosity are uniformly bounded from below by a positive constant, and the entropy Hessian is strictly positive definite, see [32–35].

In contrast with the above works that are mostly devoted to general hyperbolic systems, we focus on the specific problems in fluid mechanics represented through the complete Euler system, or its simplified barotropic analogue. Our framework are the dissipative measure–valued solutions introduced in [27] and [12, 13] for the compressible Navier–Stokes and the Euler equations, respectively, see also the related numerical study for the isentropic Navier–Stokes equations [28]. In comparison with the previously used concept of measure–valued solutions, the existence of which is conditioned by mostly rather unrealistic assumptions of boundedness of certain physical quantities and the corresponding fluxes, the new framework accommodates the solutions generated by approximate sequences satisfying only the general energy bounds. Indeed, assuming only uniform lower bound on the density and uniform upper bound on the energy we show that the Lax–Friedrichs–type finite volume schemes generate a dissipative measure–valued solution to the complete Euler equations.

The rest of the paper is organized as follows. In Section 2 we introduce the class of dissipative measure–valued (DMV) solutions to the barotropic and complete Euler systems and formulate the corresponding (DMV)– strong uniqueness results. In Section 3 we recall a general concept of entropy stable finite volume schemes and introduce the local and global Lax-Friedrichs-type finite volume methods for the barotropic and complete Euler systems, respectively. Positivity of the pressure is studied in Section 4. Sections 5 and 6 are devoted to the stability and consistency of our numerical schemes. Finally, the limiting process is studied in Section 7. We will show that the numerical solutions generate a weakly-($*$) convergent subsequence and a Young measure that represents a (DMV) solution to the corresponding Euler system. Moreover, employing the (DMV)–strong uniqueness principle, we will obtain strong (pointwise) convergence to the unique classical solution as long as the latter exists.

2 Dissipative measure–valued solutions for the Euler system

We consider the complete Euler system describing the time evolution of a general compressible fluid and its isentropic (or more general barotropic) analogue that may be seen as the particular case when the entropy of the system is constant. We start with the simpler barotropic system. For the sake of simplicity, we will *systematically* use the space–periodic boundary conditions throughout the whole text. This means the underlying spatial domain can be identified with the flat torus

$$\Omega = \Big([0,1]|_{\{0,1\}} \Big)^N, \ N = 1,2,3. \tag{2.1}$$

Note that, on a bounded domain the physically more relevant impermeability or no-flux boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

can be accommodated in a direct fashion.

2.1 Dissipative measure–valued solutions for the barotropic Euler system

Neglecting the influence of temperature fluctuations we can describe the motion of a compressible fluid by means of only two basic state variables, the mass density $\varrho = \varrho(t,x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$. The resulting *barotropic Euler system* reads

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= 0, \end{aligned} \tag{2.2}$$

where $p = p(\varrho)$ is the pressure. In what follows we focus on the polytropic pressure–density state equation

$$p(\varrho) = a\varrho^\gamma, \ \gamma > 1. \tag{2.3}$$

Moreover, it is more convenient to study (2.2) in the *conservative variables* $[\varrho, \mathbf{m} = \varrho \mathbf{u}]$:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) &= 0. \end{aligned} \tag{2.4}$$

Here, the well known problem is that there are basically no *a priori* bounds for the velocity itself but rather for the momentum $\mathbf{m}$. To recover $\mathbf{u}$, a lower bound on ϱ must be available. We will discuss this issue later in Section 4.

2.1.1 Weak formulation

The *weak formulation* of problem (2.2), (2.1) written in the conservative variables reads:

$$\begin{aligned} &\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x \, \mathrm{d}t \\ &\qquad \text{for any } \tau \in [0,T], \ \varphi \in C^1([0,T] \times \Omega); \\ &\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t \\ &\qquad \text{for any } \tau \in [0,T], \ \boldsymbol{\varphi} \in C^1([0,T] \times \Omega; R^N). \end{aligned} \tag{2.5}$$

Remark 2.1. We tacitly assume that ϱ, $\mathbf{m}$ are weakly continuous in time. Note that the weak formulation (2.5) already includes satisfaction of the initial conditions

$$\varrho(0, \cdot) = \varrho^0, \ \mathbf{m}(0, \cdot) = \mathbf{m}^0 \tag{2.6}$$

for a given sufficiently regular pair of functions ϱ^0, $\mathbf{m}^0$.

Let

$$P(\varrho) := \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z \tag{2.7}$$

be the so–called pressure potential. The weak formulation (2.5), (2.6) is usually supplemented by the *energy inequality*

$$\begin{aligned} &\left[\int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &\leq \int_0^\tau \int_\Omega \left[\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \partial_t \varphi + \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi + p(\varrho) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

for a.a. $\tau \in [0,T]$ and any $\varphi \in C^1([0,T] \times \Omega)$, $\varphi \geq 0$.

It is easy to deduce, taking $\varphi \equiv 1$ in the first equation in (2.5), that the total mass,

$$\int_\Omega \varrho(\tau, \cdot)\ \mathrm{d}x = \int_\Omega \varrho^0\ \mathrm{d}x,\ \ \tau \in [0,T]$$

is a conserved quantity. In particular, one may replace P, given by (2.7), by

$$\frac{a}{\gamma - 1}\varrho^\gamma$$

in the energy inequality as long as the flow is isentropic.

2.1.2 Dissipative measure–valued solutions

The concept of *measure–valued solution* to (2.4) was introduced by Gwiazda, Świerczewska-Gwiazda, and Wiedemann [37] in the framework of Alibert and Bouchitté [2]. There is also a general framework for a hyperbolic system assuming $L^\infty-$*a priori* bounds by Brenier et al. [4]. Here, we prefer a simpler and more versatile approach proposed in [27]. Although the measure–valued solutions are generally thought of as the Young measures, with the associated concentration defect, linked to sequences of approximate/exact solutions, we do not insist on this interpretation and introduce (DMV) solutions as objects independent of any approximating sequence.

Definition 2.2. Let

$$\mathcal{F} = \left\{ [\varrho, \mathbf{m}] \ \middle|\ \varrho \geq 0,\ \mathbf{m} \in R^N \right\}.$$

We say that a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ defined on the space $\mathcal{F}$ is a *dissipative measure–valued (DMV) solution* of problem (2.2) with the initial conditions

$$\mathcal{V}_{0,x} \in \mathcal{P}(\mathcal{F}),$$

$\mathcal{P}$ denoting the set of (Borel) probability measures, if

- $$(t,x) \mapsto \mathcal{V}_{t,x} \text{ is weakly-(*) measurable mapping from the physical space } (0,T)\times\Omega \text{ into } \mathcal{P}(\mathcal{F});$$

- $$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho\rangle\, \varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho\rangle\, \partial_t\varphi + \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \nabla_x \varphi\right]\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x\varphi \cdot \mathrm{d}\mu_C^1 \tag{2.8}$$

  for a.a. $\tau \in (0,T)$, $\varphi \in C^1([0,T]\times\Omega)$;

  $$\begin{aligned}&\left[\int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\varrho)\rangle\, \mathrm{div}_x \boldsymbol{\varphi}\right]\ \mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \nabla_x \varphi : \mathrm{d}\mu_C^2\end{aligned}$$

for a.a. $\tau \in (0,T)$, $\boldsymbol{\varphi} \in C^1([0,T] \times \Omega; R^N)$, where

$$\mu_C^1 \in \mathcal{M}([0,T] \times \Omega; R^N),\ \ \mu_C^2 \in \mathcal{M}([0,T] \times \Omega; R^{N\times N})$$

are signed vector–valued *concentration measures* defined on the physical space $[0,T] \times \Omega$;

- the energy inequality

$$\left[\int_\Omega \left\langle \mathcal{V}_{t,x}; \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} \leq 0 \tag{2.9}$$

  holds for a.a. $\tau \in (0,T)$; we denote

$$\mathcal{D}(\tau) := - \left[\int_\Omega \left\langle \mathcal{V}_{t,x}; \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

  the *dissipation defect* - a non–negative L^∞ function;

- the dissipation defect dominates the concentration measures μ_C^1, μ_C^2:

$$\int_\Omega 1 \ \mathrm{d}|\mu_C^1| \ + \int_\Omega 1 \ \mathrm{d}|\mu_C^2| \lesssim \mathcal{D} \ \text{ a.a. in } (0,T). \tag{2.10}$$

Here and hereafter the symbol $A \lesssim B$ means $A \leq cB$ for a generic positive constant c.

Remark 2.3. The precise meaning of (2.10) is

$$\sup_{\|\boldsymbol{\varphi}\|_{C(\Omega;R^N)} \leq 1} \int_0^T \int_\Omega \psi \boldsymbol{\varphi} \cdot \mathrm{d}\mu_C^1 + \sup_{\|\boldsymbol{\varphi}\|_{C(\Omega;R^{N\times N})} \leq 1} \int_0^T \int_\Omega \psi \boldsymbol{\varphi} : \mathrm{d}\mu_C^2 \lesssim \int_0^T \mathcal{D}\psi \, \mathrm{d}t$$

for any $\psi \in C[0,T]$, $\psi \geq 0$. Relation (2.10) can be replaced by a weaker stipulation

$$\int_0^\tau \int_\Omega 1 \ \mathrm{d}|\mu_C^1| \ + \int_0^\tau \int_\Omega 1 \ \mathrm{d}|\mu_C^2| \lesssim \int_0^\tau \mathcal{D} \ \mathrm{d}t \quad \text{for any } \tau \in (0,T).$$

Remark 2.4. We tacitly assume that all expressions in (2.8)–(2.9) are at least integrable on the physical space $(0,T) \times \Omega$.

The key result is the (DMV)–strong uniqueness principle shown in Gwiazda et al. [37]:

Proposition 2.5. *Let the initial data $\{\mathcal{V}_{0,x}\}_{x\in\Omega}$ be given as*

$$\mathcal{V}_{0,x} = \delta_{\varrho^0(x),\mathbf{m}^0(x)} \text{ for a.a. } x \in \Omega;$$

where

$$\varrho^0 \in C^1(\Omega),\ \mathbf{m}^0 \in C^1(\Omega; R^N),\ \varrho^0(x) > 0 \text{ for all } x \in \Omega.$$

Suppose that the problem (2.2), (2.1) admits a strong solution $\varrho \in C^1([0,T] \times \Omega)$, $\mathbf{m} \in C^1([0,T] \times \Omega; R^N)$ defined in $[0,T]$, with the initial data ϱ^0, $\mathbf{m}^0$. Let $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ be a (DMV) solution of the same problem in the sense specified in Definition 2.2, with the initial data $\mathcal{V}_{0,x}$.

Then

$$\mathcal{V}_{t,x} = \delta_{\varrho(t,x),\mathbf{m}(t,x)} \text{ for a.a. } (t,x) \in (0,T) \times \Omega.$$

Remark 2.6. Strictly speaking Proposition 2.5 was proved for the measure–valued solutions in the sense of Alibert and Bouchitté [2], but the result formulated above directly follows also for the (DMV) solutions in the sense of Definition 2.2. We refer a reader to [55] for a nice overview on the weak–strong uniqueness results for the Euler equations.

2.2 Dissipative measure–valued solutions for the complete Euler system

Similarly to the preceding section, we may introduce (DMV) solutions for the *complete Euler system*

$$\begin{aligned}\partial_t \varrho + \operatorname{div}_x(\varrho\mathbf{u}) &= 0,\\ \partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho,\vartheta) &= 0,\\ \partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right) + \operatorname{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\mathbf{u}\right] + \operatorname{div}_x(p(\varrho,\vartheta)\mathbf{u}) &= 0\end{aligned} \tag{2.11}$$

supplemented with the *periodic boundary conditions*, meaning Ω can be identified with the flat torus

$$\Omega = \Big([0,1]|_{\{0,1\}}\Big)^N. \tag{2.12}$$

Here, the new variable is the absolute temperature ϑ. The third equation in (2.11) expresses the conservation of the total energy, where $e = e(\varrho,\vartheta)$ is the specific internal energy. In addition, we suppose that p and e are interrelated to the *specific entropy* $s = s(\varrho,\vartheta)$ via Gibbs' equation

$$\vartheta Ds = De + PD\left(\frac{1}{\varrho}\right). \tag{2.13}$$

Accordingly, if all quantities in (2.11) are smooth, the entropy satisfies a conservation law

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s\mathbf{u}) = 0.$$

In the context of *weak* solutions, the entropy equation is replaced by an inequality

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s\mathbf{u}) \geq 0$$

that may be seen as a mathematical formulation of the Second law of thermodynamics.

Similarly to the preceding section, the concept of (DMV) solution uses the conservative variables: the density ϱ, the momentum $\mathbf{m} = \varrho\mathbf{u}$, and the total energy $E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)$. In addition, we suppose a relation between the pressure and the internal energy,

$$p = (\gamma - 1)\varrho e, \text{ with } \gamma > 1. \tag{2.14}$$

Under these circumstances, we have

$$s = S\left(\frac{(\gamma-1)e}{\varrho^{\gamma-1}}\right) = S\left(\frac{p}{\varrho^{\gamma}}\right)$$

for a certain function S. Accordingly, the system (2.11) rewrites as

$$\begin{aligned}\partial_t \varrho + \mathrm{div}_x \mathbf{m} &= 0,\\ \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + (\gamma - 1) \nabla_x \left(E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right) &= 0,\\ \partial_t E + \mathrm{div}_x \left[\left(E + (\gamma - 1) \left(E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right) \right) \frac{\mathbf{m}}{\varrho} \right] &= 0,\end{aligned} \tag{2.15}$$

together with the associated entropy inequality

$$\partial_t \left(\varrho S \left((\gamma - 1) \frac{E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma} \right) \right) + \mathrm{div}_x \left[S \left((\gamma - 1) \frac{E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma} \right) \mathbf{m} \right] \geq 0. \tag{2.16}$$

In addition, we may use, formally, the equation of continuity, to replace (2.16) by a more restrictive stipulation

$$\partial_t \left(\varrho \mathcal{S}_\chi \left((\gamma - 1) \frac{E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma} \right) \right) + \mathrm{div}_x \left[\mathcal{S}_\chi \left((\gamma - 1) \frac{E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma} \right) \mathbf{m} \right] \geq 0, \tag{2.17}$$

where

$$\mathcal{S}_\chi = \chi \circ S,\ \chi : R \to R \ \text{ is an increasing concave function, } \chi \leq \overline{\chi}. \tag{2.18}$$

Inequality (2.17) may be seen as a *renormalized* variant of (2.16), see also Harten [39] where a similar entropy renormalization for the polytropic Euler equations with a slightly different condition on χ was firstly introduced. For the sake of simplicity, we focus on the constitutive equations of a perfect gas, specifically

$$p(\varrho, \vartheta) = \varrho \vartheta,\ e(\varrho, \vartheta) = c_v \vartheta,\ s(\varrho, \vartheta) = \log \left(\frac{\vartheta^{c_v}}{\varrho} \right), \tag{2.19}$$

where $c_v = \frac{1}{\gamma - 1}$ is the (constant) specific heat at constant volume. Consequently,

$$S(Z) = \frac{1}{\gamma - 1} \log (Z), \text{ and entropies } \eta = \varrho \chi \left(\frac{1}{\gamma - 1} \log \left(\frac{p}{\varrho^\gamma} \right) \right) \tag{2.20}$$

for χ as in (2.18). We are ready to state the definition of a (DMV) solution for the complete Euler system (2.15) with (2.12), cf. [12].

Definition 2.7. Let

$$\mathcal{F} = \left\{ [\varrho, \mathbf{m}, E] \ \middle|\ \varrho \geq 0,\ \mathbf{m} \in R^N, E \geq 0 \right\}.$$

We say that a parameterized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ defined on the space $\mathcal{F}$ is a *dissipative measure–valued (DMV) solution* of problem (2.15), (2.12) with the initial conditions

$$\mathcal{V}_{0,x} \in \mathcal{P}(\mathcal{F})$$

if

- $$(t,x) \mapsto \mathcal{V}_{t,x} \ \text{ is weakly-(*) measurable mapping from the physical space } (0,T)\times\Omega \ \text{ into } \ \mathcal{P}(\mathcal{F});$$

- $$\left[\int_\Omega \langle \mathcal{V}_{0,x}; \varrho\rangle\, \varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho\rangle\, \partial_t\varphi + \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \nabla_x \varphi\right] \ \mathrm{d}x\,\mathrm{d}t$$

  for a.a. $\tau \in (0,T)$, $\varphi \in C^1([0,T]\times\Omega)$;

- $$\left[\int_\Omega \langle \mathcal{V}_{0,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
  $$= \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right\rangle : \nabla_x \boldsymbol{\varphi} + (\gamma-1)\left\langle \mathcal{V}_{t,x}; E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right\rangle \mathrm{div}_x \boldsymbol{\varphi}\right] \ \mathrm{d}x\,\mathrm{d}t$$
  $$+ \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mu_C$$

  for a.a. $\tau \in (0,T)$, $\boldsymbol{\varphi} \in C^1([0,T]\times\Omega; R^N)$, where μ_C is a (vectorial) signed *concentration measure* on the physical space $[0,T]\times\Omega$;

- the energy inequality

  $$\int_\Omega \langle \mathcal{V}_{\tau,x}; E\rangle \ \mathrm{d}x \le \int_\Omega \langle \mathcal{V}_{0,x}; E\rangle \ \mathrm{d}x \ \text{ holds for a.a. } \tau \in (0,T);$$

  the *dissipation defect* is given by

  $$\mathcal{D}(\tau) = -\left[\int_\Omega \langle \mathcal{V}_{\tau,x}; E\rangle \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_\Omega \Big[\, \langle \mathcal{V}_{0,x}; E\rangle - \langle \mathcal{V}_{\tau,x}; E\rangle \,\Big] \ \mathrm{d}x;$$

- $$\left[\int_\Omega \langle \mathcal{V}_{0,x}; \varrho \mathcal{S}_\chi(\varrho,\mathbf{m},E)\rangle\, \varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
  $$\ge \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \mathcal{S}_\chi(\varrho,\mathbf{m},E)\rangle\, \partial_t\varphi + \langle \mathcal{V}_{t,x}; \mathcal{S}_\chi(\varrho,\mathbf{m},E)\mathbf{m}\rangle \cdot \nabla_x \varphi\right] \ \mathrm{d}x\,\mathrm{d}t$$

  for a.a. $\tau \in (0,T)$, $\varphi \in C^1([0,T]\times\Omega)$, $\varphi \ge 0$, and any χ defined on R, increasing concave such that $\chi(Z) \le \overline{\chi}$ for all Z;

- the dissipation defect dominates the concentration measures μ_C :

$$\int_0^\tau \int_\Omega 1 d\, |\mu_C| \leq c(N,\gamma) \int_0^\tau \int_\Omega \Big[\langle \mathcal{V}_{0,x}; E \rangle - \langle \mathcal{V}_{t,x}; E \rangle \Big] \, \mathrm{d}x \, \mathrm{d}t \quad \text{for a.a. } \tau \in (0,T).$$

Finally, we formulate an analogue of the (DMV)–strong uniqueness result stated in Proposition 2.5. To this end, we recall the hypothesis of thermodynamic stability:

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho, \vartheta > 0, \tag{2.21}$$

or, in terms of the conservative variables,

$$(\varrho, \mathbf{m}, E) \mapsto \varrho S \left((\gamma - 1) \frac{E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma} \right) \text{ is a concave upper semi–continuous function on } \mathcal{F},$$

see [13] for details.

Remark 2.8. It follows from [13] that the entropy $\eta = \varrho \mathcal{S}_\chi$ with $\mathcal{S}_\chi$ as in (2.18) is concave for any function S satisfying

$$(\gamma - 1) S'(Z) + \gamma S''(Z) Z < 0 \text{ for all } Z > 0.$$

In particular it holds for S in (2.20).

We are ready to state the (DMV)–strong uniqueness result, see [12, Theorem 3.3].

Proposition 2.9. *Let the thermodynamic functions p, e, and s satisfy the hypotheses (2.13), (2.14), (2.21). Suppose that the Euler system (2.11), (2.12) admits a continuously differentiable solution $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$ in $[0,T] \times \Omega$ emanating from the initial data*

$$\tilde{\varrho}^0 > 0, \ \tilde{\vartheta}^0 > 0 \ \textit{in } \Omega.$$

Assume that $\{\mathcal{V}_{t,x}\}_{(t,x) \in (0,T) \times \Omega}$ is a (DMV) solution of the system (2.15), (2.12) in the sense specified in Definition 2.7, such that

$$\mathcal{V}_{0,x} = \delta_{\tilde{\varrho}^0(x), \tilde{\varrho}^0 \tilde{\mathbf{u}}^0(x), \frac{1}{2}\tilde{\varrho}^0(x)|\tilde{\mathbf{u}}^0(x)|^2 + \tilde{\varrho}^0 e(\tilde{\varrho}^0, \tilde{\vartheta}^0)(x)} \ \textit{for a.a. } x \in \Omega.$$

Then

$$\mathcal{V}_{t,x} = \delta_{\tilde{\varrho}(t,x), \tilde{\varrho}\tilde{\mathbf{u}}(t,x), \frac{1}{2}\tilde{\varrho}(x)|\tilde{\mathbf{u}}(x)|^2 + \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta})(t,x)} \ \textit{for a.a. } (t,x) \in (0,T) \times \Omega.$$

Remark 2.10. We would like to point out that the concept of (DMV) solutions introduced in Definitions 2.2 and 2.7 for the barotropic and the complete Euler system, respectively, requires minimum conditions that imply the (DMV)–strong uniqueness results. Thus, for the complete Euler system we need to require that the entropy inequality holds. On the other hand in both cases we may relax energy equation asking only that the total energy dissipates in time.

3 Entropy stable finite volume schemes for conservation laws

We start with recalling the concept of entropy stable finite volume schemes for a general multidimensional system of hyperbolic conservation laws

$$
\begin{aligned}
\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{f}(\mathbf{U}) &= 0, \qquad \text{in } \Omega \times (0,T) \\
\mathbf{U}(0,\cdot) &= \mathbf{U}^0, \quad \text{in } \Omega.
\end{aligned}
\tag{3.1}
$$

Here $\mathbf{U}$, $\mathbf{f}(\mathbf{U})$ denote the vectors of conservative variables and the flux function, respectively. The system (3.1) is usually accompanied with suitable boundary conditions. As agreed above, we will exclusively use the periodic boundary conditions. Throughout the paper we will confine ourselves to semi–discrete schemes. Specifically, the time will remain continuous, the discretization applied to the space variable only. The question of time discretization is more subtle. As is well known the *implicit* time discretization gives rise to the entropy production and thus the correct sign in the entropy inequality. Consequently, the resulting fully implicit scheme will be entropy stable once its semi–discrete variant was entropy stable. On the other hand, the *explicit* time discretization which is a natural choice for hyperbolic conservation laws may actually reduce the (physical) entropy, and the interplay between the spatial entropy production and temporal entropy dissipation has to be taken into account in practical applications, see, e.g., [8, 43, 53].

3.1 Spatial discretization

The relevant domain for the space discretization is $\Omega \equiv \Omega_h \subset R^N$, $N = 1, 2, 3$, where $\Omega_h := [0, \ell]^N$, $\ell > 0$, being divided into finite volume cells K, i.e.,

$$
\overline{\Omega}_h := \bigcup_{K \in \mathcal{T}_h} \overline{K}.
$$

Mesh $\mathcal{T}_h$ is a regular quadrilateral grid. For instance, in two space dimensions, cell K, its center S_K, and the uniform mesh size h are given by

$$
K := \left[x_{i+\frac{1}{2},j}, x_{i-\frac{1}{2},j}\right) \times \left[y_{i,j+\frac{1}{2}}, y_{i,j-\frac{1}{2}}\right), \quad S_K := (x_i, y_j) = \left(\frac{x_{i-\frac{1}{2},j} + x_{i+\frac{1}{2},j}}{h}, \frac{y_{i,j-\frac{1}{2}} + y_{i,j+\frac{1}{2}}}{h} \right),
$$

and $h := x_{i+\frac{1}{2},j} - x_{i-\frac{1}{2},j} = y_{i,j+\frac{1}{2}} - y_{i,j-\frac{1}{2}}$, respectively.

Remark 3.1. Note that the usual relabelling $(x_1, x_2) \mapsto (x, y)$ has been taken into account in the above example. It is also possible to consider the rectangular cells with $h_x = ch_y$, where c is a positive constant and h_x, h_y are fixed mesh sizes in $x-$ and $y-$direction, respectively. An analogous generalization of the three mesh sizes h_x, h_y, h_z is applicable for $N = 3$ as well. For the sake of simplicity we keep the mesh size fixed in all space directions.

Let $X(\mathcal{T}_h)$ denote the space of piecewise constant functions defined on mesh $\mathcal{T}_h$. For $g_h \in X(\mathcal{T}_h)$ we set $g_K := g_{h_{|_K}}$. Then it holds that

$$\int_\Omega g_h \,\mathrm{d}x = h^N \sum_{K\in\mathcal{T}_h} g_K.$$

Further, we define the projection

$$\Pi_h : L^1(\Omega) \to X(\mathcal{T}_h), \quad (\Pi_h(\phi))_K := \frac{1}{h^N}\int_K \phi(x)\,dx.$$

Boundary ∂K of a cell K is created by faces σ. The face between two neighbouring cells K and L shall be denoted by $\sigma = K|L$. By $\mathcal{E}$ we denote the set of all faces σ of all cells $K \in \mathcal{T}_h$. The value of G_h on the face σ shall be denoted by G_σ, and analogously for faces $\sigma, s\pm$ of cell K in $\pm\mathbf{e}_s$ direction. Note that $\mathbf{e}_s$ is the unit basis vector in the $s-$th space direction, $s = 1,\dots,N$. For $g_h, G_h \in X(\mathcal{T}_h)$ we define the following discrete operators

$$\left(\widetilde{\partial_h^s} g_h\right)_K := \frac{g_L - g_J}{2h},\ \left(\partial_h^{s+} g_h\right)_K := \frac{g_L - g_K}{h},\ \left(\partial_h^{s-} g_h\right)_K := \frac{g_K - g_J}{h}, \quad L = K + h\mathbf{e}_s, J = K - h\mathbf{e}_s,$$
$$(\partial_h^s G_h)_K := \frac{G_{\sigma,s+} - G_{\sigma,s-}}{h}, \quad s = 1,\dots,N.$$

Let $\mathcal{N}(K)$ denote the set of all neighbouring cells of the cell K. The discrete Laplace and divergence operators are defined as follows

$$(\Delta_h\, g_h)_K := \frac{1}{h^2}\sum_{L\in\mathcal{N}(K)} (g_L - g_K) = \sum_{s=1}^N (\Delta_h^s\, g_h)_K\,,$$
$$\left(\widetilde{\mathrm{div}}_h\, \mathbf{g}_h\right)_K := \sum_{s=1}^N \left(\widetilde{\partial_h^s} g_h^s\right)_K, \quad (\mathrm{div}_h\, \mathbf{G}_h)_K := \sum_{s=1}^N (\partial_h^s G_h^s)_K\,.$$

Furthermore, on a face $\sigma = K|L \in \mathcal{E}$ we define the jump and mean value operators

$$[\![g_h]\!]_\sigma := g_L \mathbf{n}_K^+ + g_K \mathbf{n}_K^-, \quad (\overline{g_h})_\sigma := \frac{g_K + g_L}{2}, \quad L = K + h\mathbf{e}_s,\ s = 1,\dots,N,$$

respectively. Here $\mathbf{n}_K^+$, $\mathbf{n}_K^- \equiv \mathbf{n}_L^+$ denote the unit outer normal to K and L, respectively. Note that in our case the mesh is a regular quadrilateral grid, and thus $\mathbf{n}_K^\pm || \mathbf{e}_s$ for some $s = 1,\dots,N$. Finally, we introduce the mean value of $g_h \in X(\mathcal{T}_h)$ in a cell K in the direction of $\mathbf{e}_s$ by

$$(\widetilde{g_h})_K^s := \frac{g_L + g_J}{2}, \quad L = K + h\mathbf{e}_s,\ J = K - h\mathbf{e}_s.$$

3.2 Entropy stable numerical scheme

By $\mathbf{U}_h(t) \in X(\mathcal{T}_h)^M$, $M > 1$, we denote the solution of a semi–discrete finite volume scheme

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_K(t) + (\mathrm{div}_h\,\mathbf{F}_h(t))_K = 0, \quad t > 0, \quad K \in \mathcal{T}_h, \\ \mathbf{U}_K(0) = (\Pi_h(\mathbf{U}^0))_K, \quad K \in \mathcal{T}_h. \end{aligned} \tag{3.2}$$

Recall that $\mathbf{U}_h(t)_{|K} = \mathbf{U}_K(t)$ is the value of a finite volume approximation $\mathbf{U}_h(t)$ in a cell K. The numerical flux function $\mathbf{F}_h$ quantifies the flux across the interfaces $\sigma \in \mathcal{E}$. For $\sigma = K|L$ we have $\mathbf{F}_\sigma \equiv \mathbf{F}_h(\mathbf{U}_K, \mathbf{U}_L)$. In what follows we formulate assumptions on admissible numerical fluxes.

Firstly, the numerical flux $\mathbf{F}_h$ is assumed to be *consistent* with the physical flux $\mathbf{f}$ in the sense that $\mathbf{F}_h(\mathbf{w}, \mathbf{w}) = \mathbf{f}(\mathbf{w})$ for all $\mathbf{w} \in R^M$. Moreover, it is assumed to be *locally Lipschitz continuous*, i.e., for every compact set $D \subset R^M$ there exists a $C > 0$ such that

$$\|\mathbf{F}_\sigma(t) - \mathbf{f}(\mathbf{U}_K(t))\| \equiv \|\mathbf{F}_h(\mathbf{U}_K(t), \mathbf{U}_L(t)) - \mathbf{f}(\mathbf{U}_K(t))\| \le C\|\mathbf{U}_K(t) - \mathbf{U}_L(t)\|,\ \sigma = K|L,$$

whenever $\mathbf{U}_K(t)$, $\mathbf{U}_L(t) \in D$ for $t \in [0, T]$. Note that all numerical fluxes discussed below are consistent and locally Lipschitz continuous.

The discrete entropy inequality plays a crucial role in obtaining stability results for $\mathbf{U}_h(t)$. Let $(\eta, \mathbf{q})$ be an *entropy pair* associated with system (3.1), i.e., $(\eta, \mathbf{q}) : R^M \to R \times R^N$ such that η is concave and $\mathbf{q}$ satisfies for all $\mathbf{w} \in R^M$ the compatibility condition

$$\nabla_{\mathbf{w}} q^s(\mathbf{w})^T = \nabla_{\mathbf{w}}\eta(\mathbf{w})^T \nabla_{\mathbf{w}} f^s(\mathbf{w}), \quad s = 1, \ldots, N.$$

Scheme (3.2) is then said to be *entropy stable* if it satisfies the discrete entropy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(\mathbf{U}_K(t)) + (\mathrm{div}_h\,\mathbf{Q}_h(t))_K \ge 0, \quad K \in \mathcal{T}_h,\ t > 0. \tag{3.3}$$

If, in particular, equality holds in (3.3), we say the scheme (3.2) is *entropy conservative*. Here $\mathbf{Q}_h$ denotes the numerical entropy flux function that is a function of two neighbouring values, i.e., $\mathbf{Q}_\sigma \equiv \mathbf{Q}_h(\mathbf{U}_K, \mathbf{U}_L)$ for $\sigma = K|L$. It is assumed to be *consistent* with the differential entropy flux $\mathbf{q}$, i.e., $\mathbf{Q}_h(\mathbf{w}, \mathbf{w}) = \mathbf{q}(\mathbf{w})$ for all $\mathbf{w} \in R^M$. Following the work of Tadmor et al. [33, 53], entropy flux $\mathbf{Q}_h$ can be explicitly written in terms of the vector of entropy variables $\mathbf{V}$, the numerical flux $\mathbf{F}_h$ and the potential function $\psi = \psi(\mathbf{U}(\mathbf{V}))$, as

$$\mathbf{Q}_\sigma := (\overline{\mathbf{V}_h})_\sigma \mathbf{F}_\sigma - (\overline{\psi(\mathbf{V}_h)})_\sigma. \tag{3.4}$$

We shall omit the dependence on time whenever there is no confusion. Further, we say that solution $\mathbf{U}_h(t)$ of scheme (3.2) satisfies the *weak BV (bounded variation)* condition if

$$\int_0^T \sum_{\sigma \in \mathcal{E}} \lambda_\sigma \Big| [\![\mathbf{U}_h(t)]\!]_\sigma \Big| h^N \,\mathrm{d}t \to 0 \quad \text{as} \quad h \to 0^+, \tag{3.5}$$

where λ_σ is the numerical viscosity coefficient that will be introduced in (3.6).

Remark 3.2. In the literature (mathematical) convex entropy, $-\eta$, is often used, see, e.g., [33,53]. Here we prefer to work with (physical) entropy that is a concave function on its effective domain, cf. Remark 2.8.

Remark 3.3. For the complete Euler system (2.15) the vector of entropy variables is given in terms of conservative variables $\mathbf{U}$ by

$$\mathbf{V} := \nabla_{\mathbf{U}}\eta(\mathbf{U}) = \frac{\chi'(S(\mathbf{U}))}{p}\begin{pmatrix} E + \frac{p}{\gamma-1}\Big((\gamma-1)\frac{\chi(S(\mathbf{U}))}{\chi'(S(\mathbf{U}))} - \gamma - 1\Big) \\ -\mathbf{m} \\ \varrho \end{pmatrix}.$$

Substituting for pressure $p = (\gamma-1)\left(E - \dfrac{1}{2}\dfrac{|\mathbf{m}|^2}{\varrho}\right)$ we obtain

$$\mathbf{V} = \frac{\chi'(S(\mathbf{U}))}{(\gamma-1)\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right)}\begin{pmatrix} E\left((\gamma-1)\frac{\chi(S(\mathbf{U}))}{\chi'(S(\mathbf{U}))} - \gamma\right) - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\Big((\gamma-1)\frac{\chi(S(\mathbf{U}))}{\chi'(S(\mathbf{U}))} - \gamma - 1\Big) \\ -\mathbf{m} \\ \varrho \end{pmatrix}.$$

The potential function for the complete Euler system reads $\psi(\mathbf{U}(\mathbf{V})) = -\chi'(S(\mathbf{U}))\mathbf{m}$. For the barotropic Euler system the corresponding entropy variables and the entropy potential are given by

$$\mathbf{V} = \begin{pmatrix} \frac{a\gamma}{\gamma-1}\varrho^{\gamma-1} - \frac{|\mathbf{m}|^2}{2\varrho^2} \\ \frac{\mathbf{m}}{\varrho} \end{pmatrix}, \qquad \psi(\mathbf{U}(\mathbf{V})) = a\gamma\varrho^{\gamma-1}\mathbf{m}.$$

The specific form of $\mathbf{V}$, as well as the flux function used in the discretization of the complete Euler system discussed below, immediately reveals a peculiar difficulty connected with the development of the vacuum state $\varrho = 0$ in finite time. Indeed the fluxes are not correctly defined as soon as $\varrho = 0$, while the corresponding Lipschitz constant may blow up for $\varrho \to 0$. We discuss this problem in Section 4 below.

3.2.1 Examples of entropy stable numerical schemes

- Rusanov / Lax–Friedrichs schemes
  Following [53] the Rusanov scheme with the following numerical flux is entropy stable.

$$\mathbf{F}_\sigma := (\overline{\mathbf{f}(\mathbf{U}_h)})_\sigma - d_\sigma[\![\mathbf{U}_h]\!]_\sigma,$$

  where $d_\sigma = \dfrac{1}{2}\max\limits_{s=1,\ldots,N}(|\lambda^s(\mathbf{U}_K)|, |\lambda^s(\mathbf{U}_L)|)$, $\sigma = K|L$ and λ^s is the $s-$th eigenvalue of the corresponding Jacobian matrix $\mathbf{f}'(\mathbf{U}_h)$. In the case that $d_\sigma = \dfrac{1}{2}\max\limits_{s=1,\ldots,N}\max\limits_{K\in\mathcal{T}_h}|\lambda^s(\mathbf{U}_K)|$ we obtain the Lax–Friedrichs scheme that is entropy stable, too.

- entropy stable Roe scheme
  The following entropy stable version of the Roe scheme has been proposed in [53]
  $$\mathbf{F}_\sigma := (\overline{\mathbf{f}(\mathbf{U}_h)})_\sigma - D_\sigma [\![\mathbf{U}_h]\!]_\sigma.$$
  Denoting $\overline{A}_\sigma$ the Roe matrix, that satisfies $[\![\mathbf{F}_h]\!]_\sigma \equiv \overline{A}_\sigma [\![\mathbf{U}_h]\!]_\sigma$, we define the viscosity matrix $D_\sigma = d(\overline{A}_\sigma)$ with the function $d(\overline{\lambda}^s) = \max(|\overline{\lambda}^s|, kC_\sigma [\![\mathbf{U}_h]\!]_\sigma)$. Here $k > 0$ is the upper bound of $\frac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2}$ and C_σ is chosen such that $\min_s(\lambda^s(Q_\sigma)) \geq C_\sigma |[\![\mathbf{V}_h]\!]_\sigma|$, Q_σ is the viscosity matrix with respect to the entropy variables $\mathbf{V}_h$, see [53, Theorem 5.3, Example 5.8].

- Lax–Wendroff scheme
  In [35] the entropy stable Lax–Wendroff scheme has been presented. The numerical flux reads
  $$\mathbf{F}_\sigma := \tilde{\mathbf{F}}^r_\sigma - d_\sigma |[\![\mathbf{V}_h]\!]_\sigma|^{r-1} [\![\mathbf{V}_h]\!]_\sigma,$$
  where $\tilde{\mathbf{F}}^r_\sigma$ is a $r-$th order entropy conservative numerical flux, see [53], d_σ is a positive number. In [32] it has been shown that this scheme is formally $r-$th order accurate, entropy stable and under the assumptions that $\frac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2} \geq \underline{\eta} > 0$ (for convex mathematical entropy) and $d_\sigma \geq c > 0$ the scheme satisfies the weak BV estimates (3.5) with $\lambda_\sigma \equiv 1$.

- TeCNO scheme
  In [34] essential non–oscillatory entropy stable (TeCNO) schemes for system of conservation laws have been introduced. The numerical flux has the form
  $$\mathbf{F}_\sigma := \tilde{\mathbf{F}}^r_\sigma - \frac{1}{2} D_\sigma (\mathbf{V}^-_L - \mathbf{V}^+_K),$$
  where $\tilde{\mathbf{F}}^r_\sigma$ is a $r-$th order entropy conservative numerical flux as above, D_σ is a positive definite matrix and $\mathbf{V}^-_L$, $\mathbf{V}^+_K$ are the cell interface values of a $r-$th order accurate ENO reconstruction. The scheme is formally $r-$th order accurate, entropy stable and satisfies weak BV estimates (3.5) under the above mentioned assumptions on $\frac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2}$, see [34, 35].

3.3 Numerical schemes for the barotropic Euler system

Our aim is to prove the convergence of some entropy stable finite volume schemes for the multidimensional Euler equations. More precisely, we show that a sequence of numerical solutions generates a Young measure that represents a dissipative measure–valued solution. To illustrate the ideas we will consider scheme (3.2) with a Lax–Friedrichs–type numerical flux $\mathbf{F}_h$ whose value on a face $\sigma = K|L$ is given by

$$\mathbf{F}_\sigma := (\overline{\mathbf{f}(\mathbf{U}_h)})_\sigma - \lambda_\sigma [\![\mathbf{U}_h]\!]_\sigma. \tag{3.6}$$

Here the global diffusion coefficient is $\lambda_\sigma \equiv \lambda := \max_{s=1,\ldots,N} \max_{K \in \mathcal{T}_h} |\lambda^s(\mathbf{U}_K)|$, while the local diffusion coefficient is $\lambda_\sigma := \max_{s=1,\ldots,N} \max(|\lambda^s(\mathbf{U}_K)|, |\lambda^s(\mathbf{U}_L)|)$. As already mentioned above λ^s is the $s-$th eigenvalue of the corresponding Jacobian matrix $\mathbf{f}'(\mathbf{U}_h)$. Finite volume scheme with the local diffusion coefficient is in the literature also called the Rusanov scheme.

Substituting $\mathbf{U} = [\varrho, \mathbf{m}]^T$ and $\mathbf{f}(\mathbf{U}) = [\mathbf{m}, \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p\mathbb{I}]^T$, $p = a\varrho^\gamma$, into (3.6) we derive the semi–discrete finite volume scheme for the barotropic Euler system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) + \left(\widetilde{\mathrm{div}}_h\, \mathbf{m}_h(t)\right)_K - \frac{1}{h}\sum_{\sigma\in\partial K} \lambda_\sigma [\![\varrho_h(t)]\!]_\sigma (\mathbf{n}_K^+ \cdot \mathbf{e}_s) = 0, \tag{3.7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m}_K(t) + \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\right)\right)_K - \frac{1}{h}\sum_{\sigma\in\partial K} \lambda_\sigma [\![\mathbf{m}_h(t)]\!]_\sigma (\mathbf{n}_K^+ \cdot \mathbf{e}_s) = 0,\ t>0,\ K \in \mathcal{T}_h. \tag{3.7b}$$

Note that $(\mathbf{n}_K^+ \cdot \mathbf{e}_s)$ determines whether the jump belongs to in– or outgoing fluxes. For the global numerical diffusion coefficient (3.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) + \left(\widetilde{\mathrm{div}}_h\, \mathbf{m}_h(t)\right)_K - \lambda h \left(\Delta_h\, \varrho_h(t)\right)_K = 0, \tag{3.8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m}_K(t) + \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\right)\right)_K - \lambda h \left(\Delta_h\, \mathbf{m}_h(t)\right)_K = 0,\ t>0\ K \in \mathcal{T}_h. \tag{3.8b}$$

Recall that $p_h(t) = p(\varrho_h(t)) = a\varrho_h^\gamma(t)$, $\gamma > 1$, $a > 0$, cf. (2.3). The initial conditions for the schemes (3.7) and (3.8) are prescribed as follows

$$(\varrho_K(0), \mathbf{m}_K(0))^T = ((\Pi_h\varrho^0)_K, (\Pi_h\mathbf{m}^0)_K)^T, \quad K \in \mathcal{T}_h.$$

3.4 Numerical schemes for the complete Euler system

Analogously as above, we insert the corresponding vector of conservative variables $\mathbf{U} = [\varrho, \mathbf{m}, E]^T$ and the flux function $\mathbf{f}(\mathbf{U}) = \left[\mathbf{m}, \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p\mathbb{I}, \frac{\mathbf{m}}{\varrho}(E+p)\right]^T$, $p = (\gamma-1)(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho})$, into the definition of the Lax–Friedrichs–type numerical flux (3.6) to obtain the finite volume scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) + \left(\widetilde{\mathrm{div}}_h\, \mathbf{m}_h(t)\right)_K - \frac{1}{h}\sum_{\sigma\in\partial K} \lambda_\sigma [\![\varrho_h(t)]\!]_\sigma (\mathbf{n}_K^+ \cdot \mathbf{e}_s) = 0, \tag{3.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m}_K(t) + \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\right)\right)_K - \frac{1}{h}\sum_{\sigma\in\partial K} \lambda_\sigma [\![\mathbf{m}_h(t)]\!]_\sigma (\mathbf{n}_K^+ \cdot \mathbf{e}_s) = 0, \tag{3.9b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_K(t) + \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)}{\varrho_h(t)}(E_h(t) + p_h(t))\right)\right)_K - \frac{1}{h}\sum_{\sigma\in\partial K} \lambda_\sigma [\![E_h(t)]\!]_\sigma (\mathbf{n}_K^+ \cdot \mathbf{e}_s) = 0,\ t>0,\ K \in \mathcal{T}_h. \tag{3.9c}$$

The global numerical viscosity coefficient yields analogously as above

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) + \left(\widetilde{\mathrm{div}}_h\, \mathbf{m}_h(t)\right)_K - \lambda h \left(\Delta_h\, \varrho_h(t)\right)_K = 0, \tag{3.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m}_K(t) + \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\right)\right)_K - \lambda h\left(\Delta_h\,\mathbf{m}_h(t)\right)_K = 0, \tag{3.10b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_K(t) + \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)}{\varrho_h(t)}(E_h(t)+p_h(t))\right)\right)_K - \lambda h\left(\Delta_h\,E_h(t)\right)_K = 0,\ t>0,\ K\in\mathcal{T}_h. \tag{3.10c}$$

Recall that $p_h(t) = (\gamma-1)\left(E_h(t) - \frac{1}{2}\frac{|\mathbf{m}_h(t)|^2}{\varrho_h(t)}\right)$. Finite volume schemes (3.9) and (3.10) are equipped with the initial conditions

$$(\varrho_K(0), \mathbf{m}_K(0), E_K(0))^T = ((\Pi_h\varrho^0)_K, (\Pi_h\mathbf{m}^0)_K, (\Pi_h E^0)_K)^T, \quad K\in\mathcal{T}_h.$$

Note that all finite volume schemes for the Euler systems defined above require the positivity of $\varrho_h(t)$, $t>0$.

4 Positivity of the discrete density and pressure

As observed above, *positivity* of the discrete density is necessary for the scheme to be properly defined. Starting from a positive initial density $\varrho_h(0)>0$, the semi–discrete scheme admits the unique solution defined on a maximal time interval $[0,T_{\max})$, $T_{\max}>0$. In general, $T_{\max}$ may even depend on h and shrink to zero for $h\to 0$. In order to avoid this difficulty, suitable *a priori* bounds that would guarantee $\varrho_h(t)$ being bounded below away from zero must be established. This problem has been treated for the relevant *fully discrete* schemes by, e.g., Perthame and Shu [48]. Note that these results are always *conditioned* by a kind of CFL stability condition or other relevant restrictions. Seen from this perspective, the existence of an *unconditional* result for the semi–discrete scheme seems to be out of reach both at the discrete level and for the limit Euler system. To eliminate this problem, we shall therefore impose positivity of ϱ_h as our principal working *hypothesis*:

$$\varrho_h(t) \geq \underline{\varrho} > 0 \text{ uniformly for } t\in[0,T],\ h\to 0 \tag{4.1}$$

for a positive constant $\underline{\varrho}$.

Positivity of the density at the *discrete level*, meaning with the lower bound $\underline{\varrho}_h$ depending on the discretization parameter h, can be achieved by adding lower order "damping" terms to the right–hand side of the momentum equation (3.9b) and the energy equation (3.9c), namely,

$$-h^\alpha\frac{\mathbf{m}_h(t)}{\varrho_h(t)} \text{ and } -h^\alpha\left|\frac{\mathbf{m}_h(t)}{\varrho_h(t)}\right|^2.$$

Indeed adding these terms would:

- leave the entropy balance in the same form;
- produce a uniform upper–bound on the discrete velocity

$$\mathbf{u}_h(t) \equiv \frac{\mathbf{m}_h(t)}{\varrho_h(t)},\ \text{specifically } \mathbf{u}_h\in L^2(0,T;L^\infty(\Omega;R^N)), \tag{4.2}$$

resulting from boundedness of the discrete total energy $E_h(t)$.

In the next section we show how the positivity of the discrete density can be obtained under the hypothesis (4.2).

4.1 Conditional positivity of the discrete density

In this section, we show the positivity of the discrete density under the extra hypothesis on the approximate velocity,

$$\mathbf{u}_h \equiv \frac{\mathbf{m}_h(t)}{\varrho_h(t)} \in L^2(0,T;L^\infty(\Omega)). \tag{4.3}$$

We restrict ourselves to the case of constant numerical viscosities. Thus the first two equations of the numerical scheme for the Euler system read,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) + \left(\widetilde{\operatorname{div}}_h\left(\varrho_h(t)\mathbf{u}_h(t)\right)\right)_K - \lambda h\left(\Delta_h\,\varrho_h(t)\right)_K = 0, \tag{4.4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varrho_K(t)\mathbf{u}_K(t)) + \left(\widetilde{\operatorname{div}}_h\left(\varrho_h(t)(\mathbf{u}_h(t)\otimes\mathbf{u}_h(t)) + p_h(t)\mathbb{I}\right)\right)_K - \lambda h\left(\Delta_h\left(\varrho_h(t)\mathbf{u}_h(t)\right)\right)_K = 0, \tag{4.4b}$$

equipped with the relevant initial conditions.

Lemma 4.1. *Let $\varrho_h(0) > 0$, and let a couple $(\varrho_h(t), \mathbf{u}_h(t))$, $t > 0$, satisfy the discrete continuity equation* (4.4a)*, where $\mathbf{u}_h$ belongs to the class* (4.3)*.*
Then

$$\varrho_K(t) > \underline{\varrho}_h > 0, \quad t\in[0,T],\ K\in\mathcal{T}_h.$$

Proof. Let $\varrho_K(t)$ be such that $\varrho_K(t) \le \varrho_L(t)$ for all $L\in\mathcal{T}_h$. Equation (4.4a) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) = -\sum_{s=1}^{N}\left(\widetilde{\partial_h^s}\varrho_h\right)_K(\widetilde{u_h^s})_K^s - \varrho_K\left(\widetilde{\operatorname{div}}_h\,\mathbf{u}_h\right)_K - \sum_{s=1}^{N}\left(\Delta_h^s\,\varrho\right)_K\left(\frac{h^2}{2}\left(\widetilde{\partial_h^s}u_h^s\right)_K - \lambda h\right). \tag{4.5}$$

By the definition of λ and the minimality of $\varrho_K(t)$ we can conclude that

$$\begin{aligned}
-\left(\widetilde{\partial_h^s}\varrho_h\right)_K(\widetilde{u_h^s})_K^s &= -\frac{1}{2}\left[\left(\partial_h^{s+}\varrho_h\right)_K + \left(\partial_h^{s-}\varrho_h\right)_K\right](\widetilde{u_h^s})_K^s \\
&\ge -\frac{\lambda}{2}\left[\left(\partial_h^{s+}\varrho_h\right)_K - \left(\partial_h^{s-}\varrho_h\right)_K\right] = -\frac{\lambda h}{2}\left(\Delta_h^s\,\varrho_h\right)_K, \\
-\left(\Delta_h^s\,\varrho_h\right)_K\left(\frac{h^2}{2}\left(\widetilde{\partial_h^s}u_h^s\right)_K - \lambda h + \frac{\lambda h}{2}\right) &= -\frac{h}{4}\left(\Delta_h^s\,\varrho_h\right)_K(u_L^s - \lambda) + \frac{h}{4}\left(\Delta_h^s\,\varrho_h\right)_K(u_J^s + \lambda) \ge 0,
\end{aligned}$$

and, consequently, equation (4.5) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_K(t) \ge -\varrho_K\left(\widetilde{\operatorname{div}}_h\,\mathbf{u}_h\right)_K.$$

As $\mathbf{u}_h$ satisfies (4.3), we easily deduce a bound on the discrete divergence,

$$\left(\widetilde{\mathrm{div}}_h\,\mathbf{u}_h\right)_K \in L^2(0,T;L^\infty(\Omega)).$$

Thus the Gronwall inequality together with the assumption $\varrho_K(0) > 0$, $K \in \mathcal{T}_h$, finally yields for all $L \in \mathcal{T}_h$ that $\varrho_L(t) \geq \varrho_K(t) > 0$, $t \in [0,T]$. $\square$

Under the hypothesis (4.3), setting $\mathbf{m}_h \equiv \varrho_h \mathbf{u}_h$ and comparing (4.4a) with (3.8a) or (3.10a), we realize that both formulations are equivalent. Analogous results hold for the schemes (3.7) and (3.9) with the local Lax–Friedrichs flux for both Euler systems, respectively.

4.2 Positivity of the discrete pressure

Recall the entropy $\eta(\mathbf{U}_h) = \varrho_h \mathcal{S}_\chi(\mathbf{U}_h)$ is a concave function as mentioned in Remark 3.2. The discrete entropy inequality (3.3) holds, cf. [39], and may be used similarly to [54] for showing the minimal entropy principle. In particular, the relation between the initial density and temperature is time invariant and gives rise to the positivity of pressure.

Lemma 4.2. *Let the initial density and temperature for the complete Euler system satisfy*

$$0 < \varrho_K(0) \leq \overline{C}(\vartheta_K(0))^{1/(\gamma-1)},\ \overline{C} > 0, \quad \textit{for all } K \in \mathcal{T}_h, \tag{4.6}$$

where $\vartheta_K(0) = \dfrac{(\gamma-1)}{\varrho_K(0)}\left(E_K(0) - \dfrac{1}{2}\dfrac{|\mathbf{m}_K(0)|^2}{\varrho_K(0)}\right)$.
Then, for all $K \in \mathcal{T}_h$*, it holds that*

$$0 < \varrho_K(t) \leq \overline{C}(\vartheta_K(t))^{1/(\gamma-1)}, \quad t \in [0,T], \tag{4.7}$$

where $\vartheta_K(t) = \dfrac{(\gamma-1)}{\varrho_K(t)}\left(E_K(t) - \dfrac{1}{2}\dfrac{|\mathbf{m}_K(t)|^2}{\varrho_K(t)}\right)$*. In particular,* $p_K(t) = \varrho_K(t)\vartheta_K(t) > 0$, $t \in [0,T]$.

Proof. Recall that the renormalized entropy in our case, cf. (2.18) - (2.20), can be rewritten as

$$\eta = \varrho \mathcal{S}_\chi = \varrho \chi\left(\log\left(\frac{(\gamma-1)}{\varrho^\gamma}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right)\right)\right).$$

Following [12] we now take a function χ satisfying (2.18) to be such that

$$\chi'(z) \geq 0, \quad \chi(z)\begin{cases} < 0, & z < z_0 \\ = 0, & z \geq z_0, \end{cases}, \quad z_0 = (\gamma-1)\ln(1/\overline{C}). \tag{4.8}$$

Under the assumption (4.6) it holds that

$$\log\left(\frac{(\gamma-1)}{\varrho_K(0)^\gamma}\left(E_K(0) - \frac{1}{2}\frac{|\mathbf{m}_K(0)|^2}{\varrho_K(0)}\right)\right) = \log\left(\frac{(\vartheta_K(0))^{1/(\gamma-1)}}{\varrho_K(0)}\right) \geq z_0,$$

which combined with (4.8) implies $\eta(\mathbf{U}_K(0)) = 0$. Thus, the sum of the discrete entropy inequality (3.3) integrated in time yields

$$\sum_{K\in\mathcal{T}_h} \eta(\mathbf{U}_K(t)) \geq \sum_{K\in\mathcal{T}_h} \eta(\mathbf{U}_K(0)) = 0, \quad t \in [0,T]. \tag{4.9}$$

From inequality (4.9) it directly follows that

$$\sum_{K\in\mathcal{T}_h} \varrho_K(t)\chi\left(\log\left(\frac{(\gamma-1)}{\varrho_K(t)^\gamma}\left(E_K(t) - \frac{1}{2}\frac{|\mathbf{m}_K(t)|^2}{\varrho_K(t)}\right)\right)\right) = \sum_{K\in\mathcal{T}_h} \varrho_K(t)\chi\left(\log\left(\frac{(\vartheta_K(t))^{1/(\gamma-1)}}{\varrho_K(t)}\right)\right) \geq 0.$$

Consequently, employing (4.8) and the positivity of $\varrho_K(t)$, we get that

$$\log\left(\frac{(\gamma-1)}{\varrho_K(t)^\gamma}\left(E_K(t) - \frac{1}{2}\frac{|\mathbf{m}_K(t)|^2}{\varrho_K(t)}\right)\right) = \log\left(\frac{(\vartheta_K(t))^{1/(\gamma-1)}}{\varrho_K(t)}\right) \geq z_0, \quad t \in [0,T],$$

which concludes the proof. □

Lemma 4.3. *Let $\mathbf{U}_h = [\varrho_h, \mathbf{m}_h, E_h]$ be a solution of the complete Euler system constructed via the numerical schemes* (3.9) *or* (3.10)*. In addition, suppose that*

$$0 < \underline{\varrho} \leq \varrho_h(t),\ E_h(t) \leq \overline{E} \ \textit{uniformly for } h \to 0,\ t \in [0,T]$$

for some constants $\underline{\varrho}$, $\overline{E}$.
Then there exist constants $\overline{\varrho}$, $\underline{\vartheta}$, $\overline{\vartheta}$, $\underline{p}$, $\overline{p}$, $\overline{\mathbf{m}}$ such that

$$\varrho_h(t) \leq \overline{\varrho},\ |\mathbf{m}_h(t)| \leq \overline{\mathbf{m}}, \quad 0 < \underline{\vartheta} \leq \vartheta_h(t) \leq \overline{\vartheta},\ 0 < \underline{p} \leq p_h(t) \leq \overline{p} \ \ \textit{uniformly for } h \to 0,\ t \in [0,T]. \tag{4.10}$$

Proof. Since we already know that the pressure p_h is positive, we have

$$0 < p_h = (\gamma-1)\left(E_h - \frac{1}{2}\frac{|\mathbf{m}_h|^2}{\varrho_h}\right) \leq \overline{E},$$

which yields the existence of $\overline{p}$ satisfying (4.10). From Lemma 4.2 we also have

$$0 < \varrho_h \leq \overline{C}(\vartheta_h)^{1/(\gamma-1)}.$$

Therefore,

$$0 < \underline{\varrho}^\gamma \leq \varrho_h^\gamma \leq \overline{C}^{\gamma-1}\varrho_h\vartheta_h = \overline{C}^{\gamma-1}p_h \leq \overline{C}^{\gamma-1}\overline{E},$$

which gives the existence of $\overline{\varrho}$, $\underline{p}$, $\underline{\vartheta}$, $\overline{\vartheta}$. Finally,

$$|\mathbf{m}_h|^2 \leq 2\varrho_h E_h \leq 2\overline{\varrho}\overline{E}.$$

□

5 Stability of numerical schemes

We show the stability of the numerical schemes defined in Section 3 by deriving a priori estimates.

5.1 A priori estimates for the barotropic Euler system

Firstly, we sum up the continuity equation (3.8a) (or (3.7a)) multiplied by h^N for all $K \in \mathcal{T}_h$ and integrate in time to get

$$\int_\Omega \varrho_h(t)\ \mathrm{d}x = \int_\Omega \varrho_h(0)\ \mathrm{d}x.$$

The positivity of $\varrho_h(t)$ then indicates $\varrho_h \in L^\infty(0,T;L^1(\Omega))$. Further we know that our entropy stable finite volume scheme (3.8) directly yields the discrete entropy inequality. It is important to point out that for barotropic flow the energy plays the role of the entropy (with a negative sign). Denoting

$$\eta(\mathbf{U}_K) = \frac{1}{2}\frac{|\mathbf{m}_K|^2}{\varrho_K} + P(\varrho_K) \tag{5.1}$$

we obtain for the entropy stable finite volume schemes the discrete energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(\mathbf{U}_K(t)) + (\mathrm{div}_h\,\mathbf{Q}_h(t))_K \le 0,\ \ K \in \mathcal{T}_h. \tag{5.2}$$

Since the numerical entropy flux given by (3.4) is conservative, i.e., $\sum\limits_{K\in\mathcal{T}_h} (\mathrm{div}_h\,\mathbf{Q}_h)_K = 0$, the integral of (5.2) yields

$$\int_\Omega \eta(\mathbf{U}_h(t))\ \mathrm{d}x \le \int_\Omega \eta(\mathbf{U}_h(0))\ \mathrm{d}x.$$

Similarly as above, the latter inequality gives rise to $\eta(\mathbf{U}_h) \in L^\infty(0,T;L^1(\Omega))$. Noting also (2.3) and (2.7), we conclude the a priori estimates for the barotropic Euler equations:

$$\begin{aligned} \varrho_h \in L^\infty(0,T;L^\gamma(\Omega)),\ \gamma > 1, \qquad p_h \in L^\infty(0,T;L^1(\Omega)),\\ \sqrt{\varrho_h}\mathbf{u}_h \in L^\infty(0,T;L^2(\Omega)),\ \ \text{and}\ \ \mathbf{m}_h = \varrho_h\mathbf{u}_h \in L^\infty(0,T;L^r(\Omega)),\ r = \frac{2\gamma}{1+\gamma} > 1. \end{aligned} \tag{5.3}$$

5.2 A priori estimates for the complete Euler system

We sum up equation of continuity (3.10a) (or (3.9a)) and energy equation (3.10c) (or (3.9c)) multiplied by h^N over $K \in \mathcal{T}_h$. Due to the periodic boundary conditions we get

$$\int_\Omega \varrho_h(t)\ \mathrm{d}x = \int_\Omega \varrho_h(0)\ \mathrm{d}x, \quad \int_\Omega E_h(t)\ \mathrm{d}x = \int_\Omega E_h(0)\ \mathrm{d}x. \tag{5.4}$$

In Section 4 we have shown that $\varrho_h(t),\ p_h(t) > 0$, and thus also $E_h(t) > 0$ for $t \in [0,T]$. The conservation of mass and energy (5.4) combined with (4.7) imply the a priori estimates for the complete Euler system. Namely,

$$
\begin{aligned}
\varrho_h \in L^\infty(0,T;L^\gamma(\Omega)),\ \gamma > 1,\ p_h \in L^\infty(0,T;L^1(\Omega)),\ E_h \in L^\infty(0,T;L^1(\Omega))\\
\sqrt{\varrho_h}\mathbf{u}_h \in L^\infty(0,T;L^2(\Omega)), \quad \text{and} \quad \mathbf{m}_h = \varrho_h\mathbf{u}_h \in L^\infty(0,T;L^r(\Omega)),\ r = \frac{2\gamma}{1+\gamma} > 1.
\end{aligned}
\tag{5.5}
$$

6 Consistency

In this section our aim is to show consistency of the entropy stable finite volume schemes (3.7), (3.8) and (3.9), (3.10). We derive suitable formulations of the continuity and momentum equations that are the same for the barotropic and the complete Euler systems. In addition, for the complete Euler system, we also show consistency of the entropy inequality.

6.1 Consistency formulation of continuity and momentum equations

Let us multiply the continuity equations (3.7a) or (3.8a) (for the barotropic Euler) and (3.9a) or (3.10a) (for the complete Euler) by $h^N(\Pi_h\varphi(t))_K$, with $\varphi \in C^3([0,T]\times\Omega)$, and the momentum equations (3.7b) or (3.8b) (for the barotropic Euler) and (3.9b) or (3.10b) (for the complete Euler) by $h^N(\Pi_h\boldsymbol{\varphi}(t))_K$, with $\boldsymbol{\varphi} \in C^3([0,T]\times\Omega;R^N)$. We sum the resulting equations over $K \in \mathcal{T}_h$ and integrate in time. The a priori estimates (5.3) or (5.5) for both the barotropic and the complete Euler systems combined with some boundedness assumptions specified below shall allow us to show the consistency.

Time derivative

Integration by parts with respect to time leads to

$$
\begin{aligned}
h^N \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \sum_{K\in\mathcal{T}_h} \varrho_K(t)(\Pi_h\varphi(t))_K \,\mathrm{d}t &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_K(t)\varphi(t,x)\ \mathrm{d}x\,\mathrm{d}t\\
&= \left[\int_\Omega \varrho_h(\tau)\varphi(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{\tau=T} - \int_0^T\int_\Omega \varrho_h(t)\partial_t\,\varphi(t,x)\ \mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

in the continuity equations, and similarly to

$$
h^N \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \sum_{K\in\mathcal{T}_h} \mathbf{m}_K(t)\cdot(\Pi_h\boldsymbol{\varphi}(t))_K \,\mathrm{d}t = \left[\int_\Omega \mathbf{m}_h(\tau)\cdot\boldsymbol{\varphi}(\tau,x)\ \mathrm{d}x\right]_{\tau=0}^{\tau=T} - \int_0^T\int_{\Omega_h} \mathbf{m}_h(t)\cdot\partial_t\,\boldsymbol{\varphi}(t,x)\ \mathrm{d}x\,\mathrm{d}t
$$

in the momentum equations.

Convective terms

To treat the convective terms in the continuity equations we use the discrete integration by parts and the Taylor expansion to get

$$
\begin{aligned}
& h^N \int_0^T \sum_{K \in \mathcal{T}_h} \left(\widetilde{\operatorname{div}}_h \, \mathbf{m}_h(t) \right)_K (\Pi_h \varphi(t))_K \, \mathrm{d}t \\
& = -h^N \int_0^T \sum_{K \in \mathcal{T}_h} \sum_{s=1}^N m_K^s(t) \left(\int_K \frac{\varphi(t, x + h\mathbf{e}_s) - \varphi(t, x - h\mathbf{e}_s)}{2h} \, \mathrm{d}x \right) \mathrm{d}t \\
& = - \int_0^T \int_\Omega \mathbf{m}_h(t) \cdot \nabla_x \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t + r_1,
\end{aligned}
$$

where term r_1 is estimated as follows

$$
r_1 \lesssim h \left\| \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2}(\hat{x}) \right\|_{C(0,T)} \|\mathbf{m}_h\|_{L^\infty(L^1)}, \quad \text{where} \quad \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} := \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right)_{i,j=1}^N . \tag{6.1}
$$

Point $\hat{x}$ appears in the remainder of the Taylor expansion and lies either between the points $x + h\mathbf{e}_s$ and x or the points x and $x - h\mathbf{e}_s$.

We proceed analogously with the convective term in the momentum equations, i.e.,

$$
\begin{aligned}
& h^N \int_0^T \sum_{K \in \mathcal{T}_h} \left(\widetilde{\operatorname{div}}_h \left(\frac{\mathbf{m}_h(t) \otimes \mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I} \right) \right)_K (\Pi_h \boldsymbol{\varphi}(\boldsymbol{t}))_K \, \mathrm{d}t \\
& = -h^N \int_0^T \sum_{K \in \mathcal{T}_h} \sum_{s=1}^N \sum_{z=1}^N \left(\frac{m_h^s(t) m_h^z(t)}{\varrho_h(t)} + p_h(t) \right) \left(\int_K \frac{\varphi^z(t, x + h\mathbf{e}_s) - \varphi^z(t, x - h\mathbf{e}_s)}{2h} \, \mathrm{d}x \right) \mathrm{d}t \\
& = - \int_0^T \int_\Omega \left(\frac{\mathbf{m}_h(t) \otimes \mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I} \right) \cdot \nabla_x \boldsymbol{\varphi}(t,x) \, \mathrm{d}x \, \mathrm{d}t + r_2,
\end{aligned}
$$

where term r_2 is bounded by

$$
r_2 \lesssim h \left\| \frac{\mathrm{d}^2 \boldsymbol{\varphi}}{\mathrm{d}x^2}(\hat{x}) \right\|_{C(0,T)} \left\{ \left\| \sqrt{\varrho_h(t)} \mathbf{u}_h(t) \right\|_{L^\infty(L^2)} + \|p_h(t)\|_{L^\infty(L^1)} \right\}.
$$

Numerical diffusion

Diffusive terms of the numerical schemes (3.7), (3.8) and (3.9), (3.10) will be computed separately for the global and the local numerical diffusion coefficients λ and λ_σ, respectively.

For the global numerical diffusion coefficient we can write

$$
h^{N+1} \int_0^T \lambda \sum_{K \in \mathcal{T}_h} (\Delta_h \, \mathbf{U}_h(t))_K \, (\Pi_h \boldsymbol{\varphi}(t))_K \, \mathrm{d}t
$$

$$= h^{N+1}\int_0^T \lambda \sum_{K\in\mathcal{T}_h} \mathbf{U}_K(t)\left(\int_K \sum_{s=1}^N \frac{\boldsymbol{\varphi}(x+h\mathbf{e}_s)-2\boldsymbol{\varphi}(x)+\boldsymbol{\varphi}(x-h\mathbf{e}_s)}{h^2}\,\mathrm{d}x\right)\mathrm{d}t$$

$$= h^N \int_0^T \lambda \int_\Omega \mathbf{U}_h(t)\Delta_x\boldsymbol{\varphi}(t,x)\ \mathrm{d}x\,\mathrm{d}t + r_3.$$

Similarly as in (6.1) the remainders of the Taylor expansions result in term r_3 that is bounded by

$$r_3 \lesssim h \left\|\frac{\mathrm{d}^3\boldsymbol{\varphi}}{\mathrm{d}x^3}(\tilde{x})\right\|_{C(0,T)} \|\mathbf{U}_h\|_{L^\infty(L^1)} \int_0^T \lambda\,\mathrm{d}t.$$

Moreover, the term stemming from the numerical diffusion is of order $\mathcal{O}(h)$. Indeed, we have

$$h\int_0^T \lambda \int_\Omega \mathbf{U}_h(t)\Delta_x\boldsymbol{\varphi}(t,x)\ \mathrm{d}x\,\mathrm{d}t \le hT\,\|\Delta_x\boldsymbol{\varphi}\|_\infty \|\mathbf{U}_h\|_{L^\infty(L^1)} \int_0^T \lambda\,\mathrm{d}t.$$

Assuming a finite speed of waves propagation, i.e.,

$$\text{there exists } \overline{\lambda} > 0 \text{ such that } \lambda \le \overline{\lambda},$$

the latter term goes to 0 as $h \to 0$.

For the local numerical diffusion coefficient we are able to prove consistency of the numerical diffusion term without the assumption on the finite speed of propagation. Indeed, considering the diffusion terms we obtain

$$h^{N-1}\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\partial K} \lambda_\sigma [\![\mathbf{U}_h(t)]\!]_\sigma (\mathbf{n}_K^+\cdot\mathbf{e}_s)\,(\Pi_h\boldsymbol{\varphi}(t))_K\,\mathrm{d}t. \tag{6.2}$$

The terms belonging to an arbitrary but fixed face $\sigma = K|L$ are

$$\frac{1}{h}\int_0^T \left(\lambda_\sigma[\![\mathbf{U}_h(t)]\!]_\sigma \int_K \boldsymbol{\varphi}(t)\,\mathrm{d}x - \lambda_\sigma[\![\mathbf{U}_h(t)]\!]_\sigma \int_L \boldsymbol{\varphi}(t)\,\mathrm{d}x\right)\mathrm{d}t. \tag{6.3}$$

Let us now consider an arbitrary but fixed point $\tilde{x}\in\sigma$; w.l.o.g. let $\tilde{x} = (\tilde{x}_s, x')$, $x'\in\mathbb{R}^{N-1}$, $s = 1,\ldots,N$. The Taylor expansion for $x = (x_s, x')\in K$ with respect to $(\tilde{x}_s, x')$ gives

$$\boldsymbol{\varphi}(x_s,x') = \boldsymbol{\varphi}(\tilde{x}_s,x') - \xi\partial_s\boldsymbol{\varphi}(\tilde{x}_s,x') + \mathcal{O}(h^2),$$

where $\xi\in(0,h)$. Analogously, we have for $x = (\tilde{x}_s, x')\in L$

$$\boldsymbol{\varphi}(x_s,x') = \boldsymbol{\varphi}(\tilde{x}_s,x') + \xi\partial_s\boldsymbol{\varphi}(\tilde{x}_s,x') + \mathcal{O}(h^2).$$

Substituting the above Taylor expansions in (6.3) we directly see that the terms multiplied by $\boldsymbol{\varphi}(\tilde{x}_s,x')$ vanish. The resulting terms give

$$\left|\int_0^T -\frac{1}{h}\lambda_\sigma[\![\mathbf{U}_h]\!]_\sigma \int_0^h \int_\sigma \xi\partial_s\boldsymbol{\varphi}(\tilde{x}_s,x')\mathrm{d}\xi\mathrm{d}S_{x'} + \frac{1}{h}\lambda_\sigma[\![\mathbf{U}_h]\!]_\sigma \int_0^h\int_\sigma -\xi\partial_s\boldsymbol{\varphi}(\tilde{x}_s,x')\mathrm{d}\xi\mathrm{d}S_{x'}\,\mathrm{d}t\right|$$

$$
\begin{aligned}
&\leq \frac{2}{h}\int_0^T \left|\lambda_\sigma [\![\mathbf{U}_h]\!]_\sigma \int_0^h \int_\sigma \xi \partial_s \boldsymbol{\varphi}(\tilde{x}_s, x')\mathrm{d}\xi \mathrm{d}S_{x'}\right| \mathrm{d}t \\
&\lesssim h^N \int_0^T \lambda_\sigma \left|[\![\mathbf{U}_h]\!]_\sigma\right| dt \, \|\boldsymbol{\varphi}\|_{C^1([0,T]\times\Omega)} \to 0 \qquad \text{for} \quad h \to 0.
\end{aligned}
$$

The last convergence follows from the weak BV property (3.5) and implies the consistency of the numerical diffusion term (6.2).

Remark 6.1 (weak BV (3.5) holds for the finite volume schemes (3.7), (3.9))**.**
In what follows we show that the finite volume schemes (3.7) and (3.9) with the local numerical diffusion satisfy the weak BV estimate (3.5). To unify the argumentation we set in this remark $\eta := -\varrho\mathcal{S}_\chi$ for the complete Euler equations in order to work with the convex entropy for both the barotropic and complete Euler systems. Let us assume that

- no vacuum appears, i.e.
$$\exists\, \underline{\varrho} > 0 : \varrho_h(t) \geq \underline{\varrho} \tag{6.4}$$
- entropy Hessian is strictly positive definite, i.e.
$$\exists\, \underline{\eta} > 0 : \frac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2} \geq \underline{\eta}\mathbb{I}, \quad \mathbb{I} \text{ is a unit matrix.} \tag{6.5}$$

The entropy residual r_σ arising in the discrete entropy inequality, that is obtained by multiplying the conservation law (3.1) by $\nabla_{\mathbf{U}}\eta(\mathbf{U})$, reads, see, e.g., [32, 53],

$$r_\sigma = -\delta_\sigma \, [\![\mathbf{U}]\!]_\sigma \, [\![\mathbf{V}]\!]_\sigma.$$

Here $\delta_\sigma > \lambda_\sigma/2 > 0$. For the Euler equations it holds that $\lambda_\sigma = \max(|\mathbf{u}_K| + c_K, |\mathbf{u}_L| + c_L)$, $\sigma = K|L$. Furthermore, we have for the barotropic and the complete Euler equations $c = \sqrt{\gamma\varrho^{\gamma-1}}$ and $c = \sqrt{\gamma p/\varrho}$, respectively.

It follows from the construction of the entropy stable schemes that the entropy residual is negative, see [32, 52, 53]. Consequently, integrating the discrete entropy inequality over Ω and over time interval $(0, T)$ yields

$$\int_\Omega \eta(\mathbf{U}_h(T))\,\mathrm{d}x - \int_0^T \sum_{\sigma\in\mathcal{E}} h^{N-1} r_\sigma \;\mathrm{d}t \leq \int_\Omega \eta(\mathbf{U}_h(0))\,\mathrm{d}x \leq \text{const.}$$

Furthermore, it holds that $\eta(\mathbf{U}_h(t)) \geq \tilde{\eta}, \;\; t \in (0,T)$. Indeed, for the barotropic Euler system this bound holds due to (6.4) and thus (6.6) follows. For the complete Euler system it holds for any $\eta = -\varrho S_\chi$ since χ is bounded from above and (6.4) holds. Thus, passing to the limit with $\chi(Z) \to Z$ in the entropy inequality we obtain

$$-\int_0^T \sum_{\sigma\in\mathcal{E}} h^N r_\sigma \;\mathrm{d}t \to 0 \qquad \text{for} \quad h \to 0. \tag{6.6}$$

Assumption (6.5) and the mean value theorem imply

$$[\![\mathbf{U}]\!]_\sigma = \mathbf{U}'(\tilde{\mathbf{V}})[\![\mathbf{V}]\!]_\sigma = \left(\frac{d^2(\eta(\tilde{\mathbf{U}}))}{d\mathbf{U}^2}\right)^{-1}[\![\mathbf{V}]\!]_\sigma$$

and thus

$$\underline{\eta}[\![\mathbf{U}_h]\!]_\sigma \le [\![\mathbf{V}_h]\!]_\sigma.$$

Consequently, we have

$$\frac{\underline{\eta}}{2}\int_0^T \sum_{\sigma\in\mathcal{E}} h^N \lambda_\sigma [\![\mathbf{U}]\!]_\sigma^2 \,\mathrm{d}t \le \int_0^T \sum_{\sigma\in\mathcal{E}} h^N \delta_\sigma [\![\mathbf{U}]\!]_\sigma [\![\mathbf{V}]\!]_\sigma \,\mathrm{d}t, \tag{6.7}$$

where the last term tends to 0 for $h \to 0$ according to (6.6). It remains to show that the weak BV estimate (3.5) holds. Indeed,

$$\int_0^T \sum_{\sigma\in\mathcal{E}} h^N \lambda_\sigma |[\![\mathbf{U}]\!]_\sigma| \,\mathrm{d}t \le \left(\int_0^T \sum_{\sigma\in\mathcal{E}} h^N\lambda_\sigma \,\mathrm{d}t\right)^{1/2}\left(\int_0^T \sum_{\sigma\in\mathcal{E}} h^N \lambda_\sigma |[\![\mathbf{U}]\!]_\sigma|^2 \,\mathrm{d}t\right)^{1/2}. \tag{6.8}$$

The second term on the right–hand side of (6.8) tends to 0 due to (6.7) and (6.6). To show the boundedness of the first term we apply the discrete trace inequality, cf., e.g., [28], that holds for arbitrary piecewise constant function f_h,

$$\|f_h\|_{L^p(\partial K)} \le h^{-1/p}\|f_h\|_{L^p(K)}, \qquad 1 \le p \le \infty.$$

Thus,

$$\begin{aligned}\int_0^T \sum_{\sigma\in\mathcal{E}_{in}} h^N \lambda_\sigma \,\mathrm{d}t &\le h\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\partial K}\int_\sigma \lambda_\sigma \mathrm{dS}\,\mathrm{d}t \\ &\lesssim h\int_0^T \sum_{K\in\mathcal{T}_h}\frac{1}{h}\int_K |\lambda(U_K)|\,\mathrm{d}x\,\mathrm{d}t \le \text{const.}, \quad \lambda(U_K) = |\mathbf{u}_K| + c_K.\end{aligned}$$

The last inequality follows from the assumption (6.4) and from a priori estimates (5.3) and (5.5) for the barotropic and the complete Euler equations, respectively. In conclusion, the weak BV estimate (3.5) holds for the finite volume schemes (3.7) and (3.9) provided there is no vacuum and the entropy Hessian is strictly positive definite for barotropic and strictly negative definite for the complete Euler equations, respectively.

6.2 Consistency formulation of the entropy inequality for the complete Euler system

For the complete Euler system we shall also derive a suitable consistency formulation of the discrete entropy inequality (3.3) for

$$\eta(\mathbf{U}_h) = \varrho_h \chi\left(\frac{1}{\gamma-1}\log\left((\gamma-1)\frac{E_h - \frac{1}{2}\frac{|\mathbf{m}_h|^2}{\varrho_h}}{\varrho_h^\gamma}\right)\right). \tag{6.9}$$

Due to a priori estimates (5.5), Lemma 4.2 and assumptions (2.18) on χ we know that $\eta(\mathbf{U}_h) \in L^\infty(0,T;L^\gamma(\Omega))$. By the same token we know that

$$\mathbf{q}(\mathbf{U}_h) = \mathbf{m}_h \chi\left(\frac{1}{\gamma-1}\log\left((\gamma-1)\frac{E_h - \frac{1}{2}\frac{|\mathbf{m}_h|^2}{\varrho_h}}{\varrho_h^\gamma}\right)\right) \in L^\infty(0,T;L^r(\Omega)),\ r = \frac{2\gamma}{\gamma+1}. \tag{6.10}$$

In what follows we assume that the numerical entropy flux $\mathbf{Q}_h$ is *globally Lipschitz-continuous*, i.e., there exists a $C_L > 0$ such that for any $\sigma = K|L$ it holds that

$$\|\mathbf{Q}_\sigma(t) - \mathbf{q}(\mathbf{U}_K(t))\| \equiv \|\mathbf{Q}_h(\mathbf{U}_K(t),\mathbf{U}_L(t)) - \mathbf{q}(\mathbf{U}_K(t))\| \le C_L\|\mathbf{U}_K(t) - \mathbf{U}_L(t)\|, \quad L = K + h\mathbf{e}_s. \tag{6.11}$$

To derive the consistency formulation of the discrete renormalized entropy inequality we multiple (3.3) by $h^N(\Pi_h\varphi(t))_K$, for any $\varphi \in C^2([0,T]\times\Omega)$, $\varphi \ge 0$, and integrate in time to get:

Time derivative:

$$\begin{aligned}
&h^N \int_0^T \sum_{K\in\mathcal{T}_h} \frac{\mathrm{d}}{\mathrm{d}t}\eta(\mathbf{U}_K(t))(\Pi_h\varphi(t))_K \,\mathrm{d}t \\
&= \left[\int_\Omega \eta(\mathbf{U}_K(\tau))\varphi(\tau,\cdot)\ \mathrm{d}x\right]_{\tau=0}^{\tau=T} - \int_0^T\int_\Omega \eta(\mathbf{U}_K(t))\partial_t\,\varphi(t,x)\ \mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Convective term: discrete integration by parts yields

$$\begin{aligned}
&h^N \int_0^T \sum_{K\in\mathcal{T}_h} (\mathrm{div}_h\,\mathbf{Q}_h(t))_K\,(\Pi_h\varphi(t))_K \,\mathrm{d}t \\
&= -h^N \int_0^T \sum_{s=1}^N \sum_{\sigma\in\mathcal{E}} Q_\sigma^s(t)\left(\partial_h^{s+}(\Pi_h\varphi(t))\right)_\sigma \mathrm{d}t = \\
&= -h^N \int_0^T \sum_{s=1}^N \sum_{\sigma\in\mathcal{E}} \left(Q_\sigma(t) - q^s(\mathbf{U}_K(t))\right)\left(\partial_h^{s+}(\Pi_h\varphi(t))\right)_K \mathrm{d}t - \\
&- \int_0^T\int_\Omega \mathbf{q}(\mathbf{U}_K(t))\cdot\nabla_x\varphi(t,x)\ \mathrm{d}x\,\mathrm{d}t + R,
\end{aligned}$$

where the last two terms with

$$R \lesssim h\,\|\nabla_x\varphi(\hat{x})\|_{C(0,T)}\,\|\mathbf{q}(\mathbf{U}_h)\|_{L^\infty(L^r)}$$

appeared as a result of the identity

$$h^N\left(\partial_h^{s+}(\Pi_h\varphi(t))\right)_K = \int_K \frac{\varphi(t,x+h\mathbf{e}_s) - \varphi(t,x)}{h}\,\mathrm{d}x = \int_K \nabla_x\varphi(t,x) - \frac{h}{2}\frac{\mathrm{d}^2\varphi(\hat{x})}{\mathrm{d}x^2}\,\mathrm{d}x.$$

What remains is to show that

$$-h^N \int_0^T \sum_{s=1}^N \sum_{\sigma\in\mathcal{E}} \Big(Q_\sigma(t) - q^s(\mathbf{U}_K(t))\Big) \Big(\partial_h^{s+}(\Pi_h\varphi(t))\Big)_K \,\mathrm{d}t \to 0 \quad \text{as } h\to 0.$$

Due to the global Lipschitz continuity of $\mathbf{Q}_h$, cf. (6.11), we get the following inequality

$$\begin{aligned}
&- h^N \int_0^T \sum_{s=1}^N \sum_{\sigma\in\mathcal{E}} \Big(Q_\sigma(t) - q^s(\mathbf{U}_K(t))\Big) \Big(\partial_h^{s+}(\Pi_h\varphi(t))\Big)_K \,\mathrm{d}t \\
&\leq C_L h^N \int_0^T \sum_{K\in\mathcal{T}_h} \left[\Big\|\mathbf{U}_K(t) - \mathbf{U}_L(t)\Big\| \sum_{s=1}^N \Big| \Big(\partial_h^{s+}(\Pi_h\varphi(t))\Big)_K \Big| \right] \mathrm{d}t \\
&\leq C_L \left(h^N \int_0^T \sum_{K\in\mathcal{T}_h} \Big\|\mathbf{U}_K(t) - \mathbf{U}_L(t)\Big\|^2 \mathrm{d}t \right)^{1/2} \left(\int_0^T \int_\Omega \sum_{s=1}^N \left| \frac{\mathrm{d}\varphi}{\mathrm{d}x_s}(x) - \frac{h}{2}\frac{\mathrm{d}^2\varphi(\tilde{x})}{\mathrm{d}{x_s}^2} \right|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\lesssim C_L \left(\int_0^T \sum_{\sigma\in\mathcal{E}} |[\![\mathbf{U}_h(t)]\!]_\sigma|^2 \, h^N \,\mathrm{d}t \right)^{1/2} \left\{ \|\nabla_x\varphi\|_\infty + h \left\| \frac{\mathrm{d}^2\varphi(\tilde{x})}{\mathrm{d}x^2} \right\|_{C(0,T)} \right\}.
\end{aligned} \tag{6.12}$$

To show that the first term in (6.12) goes to zero, we follow analogous arguments as in Remark 6.1. We assume strict positivity of the density (6.4). Furthermore, for the physical entropy we assume its uniform concavity, i.e. strict positive definiteness of the Hessian for the mathematical entropy, cf. (6.5). Applying Lemma 4.2 we obtain from the control of the entropy residual r_σ that there exists $\underline{\lambda} > 0$, such that

$$\frac{\underline{\lambda}\eta}{2} \int_0^T \sum_{\sigma\in\mathcal{E}} h^N [\![\mathbf{U}_h(t)]\!]_\sigma^2 \,\mathrm{d}t \leq \int_0^T \sum_{\sigma\in\mathcal{E}} h^N \delta_\sigma [\![\mathbf{U}_h(t)]\!]_\sigma [\![\mathbf{V}_h(t)]\!]_\sigma \,\mathrm{d}t \to 0.$$

Finally, we have shown

$$- h^N \int_0^T \sum_{s=1}^N \sum_{\sigma\in\mathcal{E}} \Big(Q_\sigma(t) - q^s(\mathbf{U}_K(t))\Big) \Big(\partial_h^{s+}(\Pi_h\varphi(t))\Big)_K \,\mathrm{d}t \to 0 \text{ as } h\to 0.$$

6.3 Summary of consistency results

Let us summarize the consistency results derived above.

Consistency formulation for the barotropic Euler system

The consistency formulation of the numerical schemes (3.7) and (3.8) for the barotropic Euler equations reads

$$
\begin{aligned}
-\int_\Omega \varrho_h(0)\varphi(0,\cdot)\ \mathrm{d}x &= \int_0^T \int_\Omega \varrho_h \partial_t \varphi + \mathbf{m}_h \cdot \nabla_x \varphi \ \mathrm{d}x\, \mathrm{d}t + \mathcal{O}(h) \\
&\quad \text{for any } \varphi \in C_c^3([0,T)\times\Omega); \\
-\int_\Omega \mathbf{m}_h(0)\cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x &= \int_0^T \int_\Omega \mathbf{m}_h \cdot \partial_t \boldsymbol{\varphi} \ \mathrm{d}x\, \mathrm{d}t + \\
&\quad + \int_0^T \int_\Omega \left(\frac{\mathbf{m}_h \otimes \mathbf{m}_h}{\varrho_h} + p_h \mathbb{I} \right) \cdot \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x\, \mathrm{d}t + \mathcal{O}(h) \\
&\quad \text{for any } \boldsymbol{\varphi} \in C_c^3([0,T)\times\Omega; R^N); \\
\left[\int_\Omega \eta(\mathbf{U}_h(t))\ \mathrm{d}x \right]_{t=0}^{t=\tau} &\le 0, \quad \text{for a.a. } 0 \le \tau \le T \ \text{ with } \ \eta(\mathbf{U}_h) = \frac{1}{2}\frac{|\mathbf{m}_h|^2}{\varrho_h} - P(\varrho_h).
\end{aligned}
\tag{6.13}
$$

Lemma 6.2. *Let us assume that*

(A1) no vacuum appears, i.e., there exists $\underline{\varrho} > 0$, such that $\varrho_h(t) \ge \underline{\varrho}$, cf. (4.1)

(A2) if $1 < \gamma < 3$ then there exists $\overline{\varrho} > 0$, such that $\varrho_h(t) \le \overline{\varrho}$.

Then the local Lax–Friedrichs scheme (3.7) is consistent with the barotropic Euler equations (2.2) and the consistency formulation (6.13) holds. If we assume that

(A1) no vacuum appears, i.e., there exists $\underline{\varrho} > 0$, such that $\varrho_h(t) \ge \underline{\varrho}$, cf. (4.1)

(A3) finite speed of propagation holds, i.e., there exists $\overline{\lambda} > 0$, such that $\lambda(\mathbf{U}_h(t)) \le \overline{\lambda}$ uniformly for $t \in [0,T]$ and $h \to 0$,

then the global Lax–Friedrichs scheme (3.8) is consistent with the barotropic Euler equations (2.2) and the consistency formulation (6.13) holds.

Proof. The only point to verify is to show that (A1) and (A2) imply strict positive definiteness of the entropy Hessian. Indeed, we have for the barotropic Euler systems that

$$
\frac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2} = \begin{pmatrix} a\gamma\varrho^{\gamma-2} + \frac{|\mathbf{m}|^2}{\varrho^3} & -\frac{|\mathbf{m}|}{\varrho^2} \\ -\frac{|\mathbf{m}|}{\varrho^2} & \frac{1}{\varrho} \end{pmatrix}.
$$

Direct calculation yields the determinant and the trace of the entropy Hessian, i.e. $\det = a\gamma\varrho^{\gamma-3}$ and $\mathrm{tr} = a\gamma\varrho^{\gamma-2} + \dfrac{|\mathbf{m}|^2}{\varrho^3} + \dfrac{1}{\varrho}$, respectively. Consequently, for $\gamma \ge 3$ the Hessian is uniformly strictly positive if (A1) holds, for $1 < \gamma < 3$ we need to require (A1) and (A2). $\square$

Consistency formulation for the complete Euler system

The consistency formulation of the numerical schemes (3.9) and (3.10) for the complete Euler equations reads

$$
\begin{aligned}
-\int_\Omega \varrho_h(0)\varphi(0,\cdot)\,\mathrm{d}x &= \int_0^T\int_\Omega \varrho_h(t)\partial_t\varphi(t,x) + \mathbf{m}_h(t)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(h)\\
&\quad \text{for any } \varphi\in C_c^3([0,T)\times\Omega);\\
-\int_\Omega \mathbf{m}_h(0)\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x &= \int_0^T\int_\Omega \mathbf{m}_h(t)\cdot\partial_t\boldsymbol{\varphi}(t,x)\,\mathrm{d}x\,\mathrm{d}t +\\
&\quad + \int_0^T\int_\Omega\left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\right)\cdot\nabla_x\boldsymbol{\varphi}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(h)\\
&\quad \text{for any } \boldsymbol{\varphi}\in C_c^3([0,T)\times\Omega;R^N);\\
\left[\int_\Omega E_h(t)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= 0, \quad \text{for a.a. } 0\le\tau\le T;\\
-\int_\Omega \eta(\mathbf{U}_h(0))\varphi(0,\cdot)\,\mathrm{d}x &\ge \int_0^T\int_\Omega \eta(\mathbf{U}_h(t))\cdot\partial_t\varphi(t,x) + \mathbf{q}_h(t)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(h),\\
&\quad \text{with } \ \eta(\mathbf{U}_h) = \varrho_h\chi\left(\log\left(\frac{(\gamma-1)}{\varrho_h^\gamma}\left(E_h - \frac{1}{2}\frac{|\mathbf{m}_h|^2}{\varrho_h}\right)\right)\right)
\end{aligned}
$$

for any $\varphi\in C_c^2([0,T)\times\Omega)$, $\varphi\ge 0$, and any increasing concave χ defined on R, $\chi(Z)\le\overline{\chi}$ for all Z. (6.14)

Lemma 6.3. *Let us assume that*

(A1) no vacuum appears, i.e., there exists $\underline{\varrho}>0$*, such that* $\varrho_h(t)\ge\underline{\varrho}$*, cf. (4.1)*

(A2) entropy Hessian is strictly negative definite, i.e., there exists $\underline{\eta}>0$*, such that* $\dfrac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2}\le -\underline{\eta}\mathbb{I}$

(A3) numerical entropy flux $\mathbf{Q}_h$ *is globally Lipschitz continuous, cf.* (6.11)*.*

Then the local Lax–Friedrichs scheme (3.9) is consistent with the complete Euler system (2.15) and the consistency formulation (6.14) holds. If we assume that

(A1) no vacuum appears, i.e., there exists $\underline{\varrho}>0$*, such that* $\varrho_h(t)\ge\underline{\varrho}$*, cf. (4.1)*

(A2) entropy Hessian is strictly negative definite, i.e., there exists $\underline{\eta}>0$*, such that* $\dfrac{d^2\eta(\mathbf{U})}{d\mathbf{U}^2}\le -\underline{\eta}\mathbb{I}$

(A3) numerical entropy flux $\mathbf{Q}_h$ *is globally Lipschitz continuous, cf.* (6.11)

(A4) finite speed of propagation holds, i.e., there exists $\overline{\lambda}>0$ *such that* $\lambda(\mathbf{U}_h(t))\le\overline{\lambda}$ *uniformly for* $t\in[0,T]$ *and* $h\to 0$*,*

then the global Lax–Friedrichs scheme (3.10) is consistent with the complete Euler system (2.15) and the consistency formulation (6.14) holds.

Recalling Lemmas 4.3, 6.2 and 6.3 we derive the following results.

Corollary 6.4. *Let* $\mathbf{U}_h = [\varrho_h, \mathbf{m}_h]$ *be a numerical solution of the barotropic Euler system constructed by the global Lax–Friedrichs scheme* (3.8)*. Suppose that there exist positive constants* $\underline{\varrho}$, $\overline{\varrho}$, $\overline{\mathbf{m}} > 0$ *such that*

$$0 < \underline{\varrho} \leq \varrho_h \leq \overline{\varrho}, \ |\mathbf{m}_h| \leq \overline{\mathbf{m}}, \ \textit{uniformly for } h \to 0.$$

Then the assumptions (A1), (A3) of Lemma 6.2 are satisfied.

Let $\mathbf{U}_h = [\varrho_h, \mathbf{m}_h]$ *be a numerical solution of the local Lax–Friedrichs scheme* (3.7)*. For* $\gamma \geq 3$ *we suppose that there exists constant* $\underline{\varrho}$*, such that*

$$0 < \underline{\varrho} \leq \varrho_h \quad \textit{uniformly for } h \to 0,$$

for $1 < \gamma < 3$ *we suppose that there exist constants* $\underline{\varrho}$, $\overline{\varrho}$*, such that*

$$0 < \underline{\varrho} \leq \varrho_h \leq \overline{\varrho}, \quad \textit{uniformly for } h \to 0.$$

Then the assumptions (A1), (A2) of Lemma 6.2 are satisfied. Consequently, the global and the local Lax–Friedrichs schemes for the barotropic Euler equations satisfy the consistency formulation (6.13)*.*

Corollary 6.5. *Let* $\mathbf{U}_h = [\varrho_h, \mathbf{m}_h, E_h]$ *be a numerical solution of the complete Euler system constructed by the schemes* (3.9) *or* (3.10)*. Suppose that there exist constants* $\underline{\varrho}$, $\overline{E} > 0$ *such that*

$$\underline{\varrho} \leq \varrho_h, \ E_h \leq \overline{E}, \ \textit{uniformly for } h \to 0.$$

Then the assumptions (A1) – (A4) of Lemma 6.3 are satisfied. In particular, the global and the local Lax–Friedrichs schemes for the complete Euler equations satisfy the consistency formulation (6.14)*.*

7 Limit process

Recall that for simplicity we assume that the computational domain is the flat torus $\Omega = \Big([0,1]|_{\{0,1\}}\Big)^N$, $N = 1, 2, 3$, meaning we focus on spatially periodic solutions. In addition, we prescribe regular initial data,

$$\varrho^0 \in C^1(\Omega), \ \varrho^0 > 0, \ \mathbf{m}^0 = C^1(\Omega; R^N), \ E^0 \in C^1(\Omega), \ p^0 = (\gamma - 1)\left(E^0 - \frac{1}{2}\frac{|\mathbf{m}^0|^2}{\varrho^0}\right) > 0. \tag{7.1}$$

Under the perfect gas state equation, the last condition gives rise to the initial temperature,

$$\vartheta^0 = \frac{(\gamma - 1)}{\varrho^0}\left(E^0 - \frac{1}{2}\frac{|\mathbf{m}^0|^2}{\varrho^0}\right).$$

7.1 Generating measure–valued solutions

7.1.1 Equation of continuity, weak limit

Let $\{\varrho_h, \mathbf{m}_h, E_h\}_{h>0}$ be a family of numerical solutions computed by our finite volume schemes. The energy estimates (5.3) and (5.5) can be used to deduce, at least for suitable subsequences,

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),\ \varrho \geq 0,$$
$$\mathbf{m}_h \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0,T;L^r(\Omega;R^N)),\ r = \frac{2\gamma}{\gamma+1} > 1,$$

for both the barotropic and the complete Euler systems. In addition, it may be deduced from (6.13) or (6.14) that the limit functions satisfy the equation of continuity in the form

$$-\int_\Omega \varrho^0 \varphi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi]\ \mathrm{d}x\,\mathrm{d}t \tag{7.2}$$

for any test function $\varphi \in C^1_c([0,T) \times \Omega)$. Clearly,

$$\varrho \in C_{\text{weak}}([0,T];\Omega)$$

and (7.2) can be rewritten in the form

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi]\ \mathrm{d}x\,\mathrm{d}t \tag{7.3}$$

for any $0 \leq \tau \leq T$ and any $\varphi \in C^1([0,T] \times \Omega)$.

7.1.2 Young measure generated by numerical solutions

The entropy inequality (3.3) along with the consistency formulations (6.13) and (6.14) provide a suitable platform for the use of the theory of dissipative measure–valued solutions developed in [27]. Consider a family of numerical solutions $\{\varrho_h, \mathbf{m}_h\}_{h>0}$ (barotropic Euler) or $\{\varrho_h, \mathbf{m}_h, E_h\}_{h>0}$ (complete Euler). In accordance with the weak convergence statement derived in the preceding part and boundedness of the total energy established in (5.4), these families generate a Young measure - a parametrized measure

$$\mathcal{V}_{t,x} \in L^\infty((0,T) \times \Omega; \mathcal{P}(\mathcal{F})) \text{ for a.a. } (t,x) \in (0,T) \times \Omega,$$

sitting on the phase space $\mathcal{F}$, where the latter is

$$\mathcal{F} = \Big\{ [\varrho, \mathbf{m}] \in [0,\infty) \times R^N \Big\}$$

for the barotropic Euler system, and

$$\mathcal{F} = \left\{ [\varrho, \mathbf{m}, E] \ \middle|\ [0,\infty) \times R^N \times [0,\infty) \right\}$$

for the complete Euler system. Recall that, in accordance with the fundamental theorem of the theory of Young measures (see, e.g., Ball [3] or Pedregal [47]), we have

$$\langle \mathcal{V}_{t,x}, g(\mathbf{U}) \rangle = \overline{g(\mathbf{U})}(t,x) \text{ for a.a. } (t,x) \in (0,T) \times \Omega,$$

whenever $g \in C_c(\mathcal{F})$, and

$$g(\mathbf{U}_h) \to \overline{g(\mathbf{U})} \text{ weakly in } L^1((0,T) \times \Omega).$$

7.1.3 Continuity equation

Accordingly, the equation of continuity (7.3) can be written as

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho \rangle \, \varphi(t,\cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t \tag{7.4}$$

Note that there is no concentration measure in (7.4), i.e., $\mu_C^1 = 0$.

7.1.4 Momentum equation

We apply a similar treatment to the momentum equation (3.8b) and (3.10b). Using a priori bounds (5.3) and (5.5) we obtain that

$$\frac{\mathbf{m}_h \otimes \mathbf{m}_h}{\varrho_h} \text{ is bounded in } L^\infty(0,T; L^1(\Omega; R^{N\times N})),$$

and

$$p_h \text{ is bounded in } L^\infty(0,T; L^1(\Omega)).$$

Recall that the pressure is defined as

$$p_h = \begin{cases} a\varrho_h^\gamma & \text{in the barotropic case,} \\ \\ (\gamma - 1)\left(E_h - \frac{1}{2} \frac{|\mathbf{m}_h|^2}{\varrho_h} \right) & \text{for the complete system.} \end{cases}$$

Thus, passing to subsequences as the case may be, we deduce

$$\frac{\mathbf{m}_h \otimes \mathbf{m}_h}{\varrho_h} + p_h \mathbb{I} \to \overline{\frac{\mathbf{m}_h \otimes \mathbf{m}_h}{\varrho_h} + p_h \mathbb{I}} \ \text{ weakly-(*) in } L^\infty(0,T; \mathcal{M}(\Omega; R^{N\times N})).$$

We set

$$\mu_C^2 := \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p\mathbb{I}} - \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p\mathbb{I} \right\rangle \in L^\infty(0,T; \mathcal{M}(\Omega; R^{N\times N})),$$

which is the concentration measure appearing in the limit momentum equation.

Letting $h \to 0$ in (3.8b) and (3.10b) we conclude

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}, p\rangle \operatorname{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\ \ \mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \mu_C^2 : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\, \mathrm{d}t \tag{7.5}$$

for any $0 \le \tau \le T$, $\boldsymbol{\varphi} \in C^1([0,T]\times\Omega; R^N)$.

7.1.5 Energy inequality for the barotropic Euler system

In the barotropic case the energy plays the role of the entropy, cf. (5.1). A priori estimates (5.3) indicate that the energy

$$\eta(\mathbf{U}_h) = \frac{|\mathbf{m}_h|^2}{2\varrho_h} + P(\varrho_h)$$

is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$. Letting $h \to 0$ in (3.3) for the barotropic Euler system we obtain

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \eta(\mathbf{U}_h(t))\rangle\ \ \mathrm{d}x\right]_{t=0}^{t=\tau} + \mathcal{D}(t) \le 0,$$

with the dissipation defect $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D}(t) \ge 0$, see [27] for details. Moreover, applying [27, Lemma 2.1.] for

$$F(\mathbf{U}_h(t)) = \int_\Omega \frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\ \mathrm{d}x, \quad G(\mathbf{U}_h(t)) = \int_\Omega \eta(\mathbf{U}_h(t))\ \mathrm{d}x,\ \text{a.a. } t \in (0,T),$$

we get the compatibility condition (2.10), specifically

$$\int_\Omega 1\ \mathrm{d}|\mu_C^2| \lesssim \mathcal{D} \text{ a.a. in } (0,T).$$

7.1.6 Entropy inequality and energy balance for the complete Euler system

Entropy inequality

Due to a priori estimates the entropy pair $\{(\eta(\mathbf{U}_h), \mathbf{q}(\mathbf{U}_h))\}_{h>0}$ for the complete Euler system, cf. (6.9) and (6.10), is uniformly bounded in $[L^\infty(0,T;L^\gamma(\Omega))] \times [L^\infty(0,T;L^r(\Omega))]^N$. Therefore we have

$$\eta(\mathbf{U}_h) \to \overline{\eta(\mathbf{U})}\ \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),$$
$$\mathbf{q}(\mathbf{U}_h) \to \overline{\mathbf{q}(\mathbf{U})}\ \text{ weakly-(*) in } L^\infty(0,T;L^r(\Omega)),\ r = \frac{2\gamma}{\gamma+1} > 1.$$

Letting $h \to 0$ in the equation (6.14), we get analogously as before,

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \eta(\mathbf{U})\rangle \cdot \varphi(0,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \eta(\mathbf{U})\rangle \cdot \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{q}(\mathbf{U})\rangle \cdot \nabla_x \varphi\right] \mathrm{d}x\ \mathrm{d}t \tag{7.6}$$

for a.a. $0 \leq \tau \leq T$, and any $\varphi \in C^1([0,T] \times \Omega)$, $\varphi \geq 0$.

Energy balance

Equation (3.10c) of the complete Euler system yields the discrete energy balance

$$\left[\int_\Omega E_h(t)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = 0. \tag{7.7}$$

Letting $h \to 0$ in (7.7) and taking into account that $\{E_h\}_{h>0}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ we obtain

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; E_h(t)\rangle\ \ \mathrm{d}x\right]_{t=0}^{t=\tau} + \mathcal{D}(t) = 0,$$

where $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \geq 0$. We again apply [27, Lemma 2.1.] for

$$F(\mathbf{U}_h(t)) = \int_\Omega \frac{\mathbf{m}_h(t) \otimes \mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\ \mathrm{d}x, \quad G(\mathbf{U}_h(t)) = \int_\Omega E_h(t)\ \mathrm{d}x,\ \text{a.a. } t \in (0,T),$$

to get that

$$\int_\Omega 1\ \mathrm{d}|\mu_C^2| \lesssim \mathcal{D} \text{ a.a. in } (0,T).$$

Summarizing the discussion of this section we are ready to formulate the following result.

Theorem 7.1. *Let the initial data satisfy* (7.1)*. Let* $\mathbf{U}_h = [\varrho_h, \mathbf{m}_h, E_h]$ *be a numerical solution of the complete Euler system constructed by the schemes* (3.9) *or* (3.10)*. In addition, suppose that there exist constants* $\underline{\varrho}$, $\overline{E} > 0$ *such that*

$$\underline{\varrho} \leq \varrho_h,\ E_h \leq \overline{E},\ \text{uniformly for } h \to 0. \tag{7.8}$$

Then $\{\mathbf{U}_h\}_{h>0}$ *up to a subsequence generates a Young measure*

$$\mathcal{V}_{t,x} \in L^\infty_{weak(*)}((0,T) \times \Omega, \mathcal{P}([0,\infty) \times R^N \times [0,\infty)))$$

representing a (DMV) solution of the complete Euler system in the sense of Definition 2.7.

Note that hypothesis (7.8) is considerably weaker than the standard stipulation

$$\|\mathbf{U}_h\|_{L^\infty} \leq C,\ 0 < \underline{\varrho} \leq \varrho_h,\ 0 < \underline{E} \leq E_h, \tag{7.9}$$

cf. [10, 16, 33, 35, 38]. The missing piece of information between (7.8) and (7.9) is provided by the careful analysis of the renormalized entropy inequality in Section 4, see Lemma 4.3.

Similar result can be shown in the context of the barotropic Euler system.

Theorem 7.2. *Let the initial data ϱ^0, $\mathbf{m}^0$ be as in* (7.1)*. Let $\mathbf{U}_h = [\varrho_h, \mathbf{m}_h]$ be a numerical solution of the barotropic Euler system constructed by the schemes* (3.7) *or* (3.8)*. In addition,*

- *if $\mathbf{U}_h$ is generated by the scheme* (3.7) *and $\gamma \geq 3$, we suppose*

$$0 < \underline{\varrho} \leq \varrho_h \quad \textit{uniformly for } h \to 0, \tag{7.10}$$

- *if $\mathbf{U}_h$ is generated by the scheme* (3.7) *and $1 < \gamma < 3$, we suppose*

$$0 < \underline{\varrho} \leq \varrho_h \leq \overline{\varrho} \quad \textit{uniformly for } h \to 0, \tag{7.11}$$

- *if $\mathbf{U}_h$ is generated by the scheme* (3.8)*, we suppose*

$$0 < \underline{\varrho} \leq \varrho_h \leq \overline{\varrho}, \ |\mathbf{m}_h| \leq \overline{\mathbf{m}}, \quad \textit{uniformly for } h \to 0, \tag{7.12}$$

for certain positive constants $\underline{\varrho}$, $\overline{\varrho}$ and $\overline{\mathbf{m}}$.
Then $\{\mathbf{U}_h\}_{h>0}$ up to a subsequence generates a Young measure

$$\mathcal{V}_{t,x} \in L^{\infty}_{weak(*)}((0,T) \times \Omega, \mathcal{P}([0,\infty) \times R^N))$$

representing a (DMV) solution of the barotropic Euler system in the sense of Definition 2.2.

It should be pointed out that for the barotropic Euler system the only available mathematical entropy is the energy, and in addition, its flux can not be controlled in the asymptotic limit for $h \to 0$ unless we assume (7.12).

7.2 Convergence to regular solution

We have proven that the numerical solutions $\{\mathbf{U}_h\}_{h>0}$ to (3.8) and (3.10) for the barotropic and the complete Euler system converges to a dissipative measure–valued solution defined in Definition 2.2 and Definition 2.7, respectively. Employing the corresponding (DMV)–strong uniqueness results from [37] and [12] we can show the strong convergence to the strong solution of the system on its lifespan.

Theorem 7.3. *Suppose that the approximate solutions $\{\mathbf{U}_h\}_{h>0}$ to* (3.9) *or* (3.10) *for the complete Euler system generate a (DMV) solution in the sense of Definition 2.7. In addition, let the Euler equations* (2.15) *possess the unique strong (continuously differentiable) solution $\mathbf{U} = [\varrho, \mathbf{m}, E]$, emanating form the initial data* (7.1)*.*
Then

$$\mathbf{U}_h \to \mathbf{U} \ \textit{strongly in } L^1((0,T) \times \Omega)$$

More precisely,

$$\begin{aligned} \varrho_h &\to \varrho \ \textit{weakly-(*) in } L^{\infty}(0,T; L^{\gamma}(\Omega)) \textit{ and strongly in } L^1((0,T) \times \Omega), \\ \mathbf{m}_h &\to \mathbf{m} \ \textit{weakly-(*) in } L^{\infty}(0,T; L^{2\gamma/(\gamma-1)}(\Omega)) \textit{ and strongly in } L^1((0,T) \times \Omega; R^N)), \\ E_h &\to E \ \textit{weakly-(*) in } L^{\infty}(0,T; L^1(\Omega)) \textit{ and strongly in } L^1((0,T) \times \Omega). \end{aligned} \tag{7.13}$$

Remark 7.4. Recall that the strong solution of the complete Euler system conserves energy, in particular, the dissipation defect $\mathcal{D}$, and, accordingly, the concentration measure μ_C^2 vanish. This also justifies the strong convergence of the total energy claimed in (7.13).

In contrast with Theorem 7.1 the results stated in Theorem 7.3 is unconditional provided that:

- the limit system admits a smooth solution.
- the numerical solution generates a (DMV) solution.

Exactly the same result can be obtained for the barotropic Euler system (2.2) and the entropy stable finite volume schemes (3.7) and (3.8).

Conclusions

We have shown convergence of the Lax–Friedrichs–type finite volume schemes for multidimensional barotropic and complete Euler equations. Since multidimensional Euler equations are ill-posed in the class of weak solutions for L^∞-initial data [29], we propose here to investigate the convergence in the class of dissipative measure–valued (DMV) solutions. The latter has been introduced for the Euler equations recently in [12, 13], see also the related works on the (DMV) solutions of the compressible Navier–Stokes equations [27, 28]. The (DMV) solutions represent the most general class of solutions that still satisfy the weak–strong uniqueness property. Thus, if the strong solution exists, the (DMV) solution coincides with the strong one on its lifespan, cf. [37] and [12] for the barotropic and complete Euler equations, respectively.

We build on the concept of entropy stable schemes that has been introduced by Tadmor [52], see also [53] and the references therein. We work here with the Lax–Friedrichs–type finite volume schemes (3.7), (3.8) and (3.9), (3.10) that are entropy stable. Furthermore, using some refined a priori estimates for the numerical solutions we have shown consistency of our entropy stable schemes. More precisely, assuming only strict positivity of the density and the upper bound on the energy we have proven the consistency for the complete Euler system, cf. Corollary 6.5. On the other hand, the consistency of the local Lax–Friedrichs–type scheme (3.7) for the barotropic Euler equations with $\gamma \geq 3$ can be obtained assuming only the strict positivity of density, cf. Lemma 6.2. In Theorems 7.1, 7.2 we have shown that numerical solutions given by the Lax–Friedrichs–type finite volume schemes generate Young measures representing (DMV) solutions of the complete and barotropic Euler equations, respectively. Employing the corresponding (DMV)–strong uniqueness results we have shown in Theorem 7.3 the strong convergence to the strong solution of the complete Euler system on its lifespan. Analogous strong convergence result holds for the barotropic Euler equations, too.

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