ON THE SHAPE DERIVATIVE OF POLYGONAL INCLUSIONS IN THE CONDUCTIVITY PROBLEM

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Abstract. We consider the conductivity problem for a homogeneous body with an inclusion of a different, but known, conductivity. Our interest concerns the associated shape derivative, i.e., the derivative of the corresponding electrostatic potential with respect to the shape of the inclusion. For a smooth inclusion it is known that the shape derivative is the solution of a specific inhomogeneous transmission problem. We show that this characterization of the shape derivative is also valid when the inclusion is a polygonal domain, but due to singularities at the vertices of the polygon, the shape derivative fails to belong to H^1 in this case.

Key words. Impedance tomography, Lipschitz domain, transmission problem, shape optimization, corner singularities

AMS subject classifications. 35B65, 49N60, 49Q10, 65N21

1. Introduction. The inverse conductivity problem aims at information about the spatial electric conductivity distribution within a body, which is only accessible to electrostatic or quasi-electrostatic measurements at its boundary. For example, in electric impedance tomography a series of probing currents is applied through electrodes attached to the surface of the body, and the resulting boundary potentials are measured (or the other way round). As shown by Astala and Päivärinta [2] the full set of these current/voltage pairs completely determines an isotropic conductivity distribution in two space dimensions. See the monographs by Mueller and Siltanen [26] and Kirsch [20] for an exposition of electric impedance tomography. Except for the D-bar [21] and the layer stripping methods [30] most of the pertinent inversion algorithms are iterative in nature: The spatial conductivity in question is represented by an element from the nonnegative cone of a reasonable infinite dimensional vector space of functions, and in each iteration the gradient of some loss function is used to update the current approximation of the conductivity distribution; cf., e.g., [7, 10, 27] for some early references. This is the method of choice when the material is very inhomogeneous.

When the object is homogeneous, except for an inclusion of some other known material say, then one can hope that only one or a few pairs of Cauchy data may suffice to determine the location and the shape of the inclusion. Sophisticated noniterative reconstruction methods like pole fitting methods [3, 19, 13] or the computation of the convex source support [15] have been developed for this setting, but iterative methods can also be an option, provided they exploit gradient information with respect to the shape of the inclusion (or its parameterization).

Such derivatives have been termed *shape derivative* or *domain derivative*, and have originally been developed in the optimization community, cf., e.g., [29, 16]. The idea is to analyze the impact of a pointwise perturbation of the boundary of the inclusion in the direction of a vector field that is attached to it. Then a gradient descent or Newton type method can be used to shift and deform an approximate inclusion so as to match the given data as good as possible.

Shape derivatives for inverse conductivity problems have been determined in the late 90's of the previous century. However, up to recently their rigorous foundation

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has been limited to inclusions with a certain smoothness, e.g., $C^{1,1}$-domains. For this smooth case it has been shown by Hettlich and Rundell [17] that the shape derivative of an inclusion with prescribed conductivity can be characterized via the solution of a transmission problem with distributed sources sitting on the boundary of the inclusion: The solution of this differential equation provides the perturbation of the electric potential at each individual point within and on the boundary of the body under consideration. More details and references follow in the subsequent section.

In the optimization community shape derivatives for less regular domains, e.g., Lipschitz domains, have been studied, e.g., by Delfour and Zolésio [9], Lamboley, Novruzi, and Pierre [24], and Laurain [23]; these works focus on the derivative of certain integrated shape functionals. In the application that we are interested in, such a functional can be, for example, the inner product of the given data with a certain test function on the boundary of the two-dimensional body. For this particular case Beretta, Francini, and Vessella [5] have determined in a *tour de force* the impact of a perturbation of a conducting polygonal inclusion. In a subsequent paper, Beretta, Micheletti, Perotto, and Santacesaria [6] gave a somewhat more elegant derivation of the same result using the established shape derivative theory. Still, this functional only provides the shape derivative of the given data in a weak (i.e., variational) form, which lacks the interpretation of [17] via an underlying transmission problem. Further, the impact of the perturbation of the polygon on the electric potential in the interior of the body (i.e., the state variable) remained unresolved.

In this paper we show that the same transmission problem as in the smooth case describes the sensitivity of the electric potential at each point within the body; its trace on the outer boundary provides an explicit definition of the shape derivative of the electrostatic measurements. Due to singularities at the vertices of the polygon, the solution of this transmission problem is less regular than in the smooth case. This presents some difficulties in proving the existence of a solution, which can be overcome by using a technique which goes back to Kondratiev [22] (see also Grisvard [12]), and which has also been employed in [23].

Our results are relevant for investigating the stability of numerical algorithms for solving the inverse problem. As we will show in a companion paper [14], the outcome of the present work implies that the determination of a polygon from two pairs of current/voltage boundary measurements – which is possible according to Seo [28] – is actually well-posed, i.e., the inverse operator is locally Lipschitz continuous.

The outline of this paper is as follows. In Section 2 we set up the problem under consideration and review known results about the associated shape derivative. After that we focus on polygonal inclusions only. In Section 3 we recapitulate the properties of the electric potential for such inclusions. Then, in Section 4 we derive some estimates for the shape derivative of the electrostatic boundary measurements determined in [5, 6], and provide preliminary results about the shape derivative of the electric potential within the body. In Section 5 we turn to the transmission problem formulated in [17], and show that it has a unique solution when the inclusion is a polygon, and we determine its regularity. That this solution coincides with the shape derivative of the electric potential is the topic of Section 6. In the final section we briefly discuss the case of an insulating or a perfectly conducting polygonal inclusion; all our results essentially extend to these two degenerate cases.

2. The conductivity problem and associated shape derivatives. Let $\Omega \subset \mathbb{R}^2$ be a simply connected Lipschitz domain, and $\mathscr{D}$ be another Lipschitz domain with simply connected closure $\overline{\mathscr{D}} \subset \Omega$. The outer normal vector on $\partial\mathscr{D}$ and $\partial\Omega$, respectively, will be denoted by ν. We call $\mathscr{D}$ the inclusion, and we assume that the conductivity in Ω is given by

$$\sigma(x) \;=\; \begin{cases} k\,, & \text{for } x \in \mathscr{D}\,,\\ 1\,, & \text{for } x \in \Omega\backslash\overline{\mathscr{D}}\,, \end{cases} \tag{2.1}$$

where k is taken to be greater than zero and different from one; in Section 7, we will briefly treat the degenerate cases $k = 0$ and $k = +\infty$, respectively.

Given a fixed (nontrivial) driving boundary current

$$f \;\in\; L^2_\diamond(\partial\Omega) \;=\; \bigl\{\, f \in L^2(\partial\Omega) \;:\; \int_{\partial\Omega} f \,\mathrm{d}s = 0 \,\bigr\}\,,$$

the induced quasistatic electric potential

$$u \;\in\; H^1_\diamond(\Omega) \;=\; \bigl\{\, u \in H^1(\Omega) \;:\; \int_{\partial\Omega} u \,\mathrm{d}s = 0 \,\bigr\}$$

is the unique solution of the conductivity equation

$$\nabla\cdot(\sigma\nabla u) \;=\; 0 \quad \text{in } \Omega\,, \qquad\quad \frac{\partial}{\partial\nu}u \;=\; f \quad \text{on } \partial\Omega\,, \tag{2.2}$$

with vanishing mean on $\partial\Omega$. The restrictions $u^- = u|_{\mathscr{D}}$ and $u^+ = u|_{\Omega\backslash\overline{\mathscr{D}}}$ are both harmonic (smooth) functions, which satisfy the transmission conditions

$$\bigl[u\bigr]_{\partial\mathscr{D}} = u^+|_{\partial\mathscr{D}} - u^-|_{\partial\mathscr{D}} = 0 \qquad \text{and} \qquad \bigl[D_\nu u\bigr]_{\partial\mathscr{D}} = \frac{\partial}{\partial\nu}u^+ \;-\; k\,\frac{\partial}{\partial\nu}u^- = 0 \tag{2.3}$$

on $\partial\mathscr{D}$. Alternatively, u can be characterized as the unique solution in $H^1_\diamond(\Omega)$ of the variational problem

$$\int_\Omega \sigma\,\nabla u\cdot\nabla w \,\mathrm{d}x \;=\; \int_{\partial\Omega} f w \,\mathrm{d}s \qquad \text{for all } w \in H^1(\Omega)\,. \tag{2.4}$$

Assuming that the (normalized) background conductivity and its anomalous value $k > 0$ in the inclusion are fixed (and known), we are interested in the sensitivity of the boundary data

$$u|_{\partial\Omega} \;=:\; \Lambda_f(\mathscr{D}) \;\in\; L^2_\diamond(\partial\Omega)$$

of the electrostatic potential with respect to perturbations of $\mathscr{D}$. To be specific, assume that a vector field $h : \partial\mathscr{D} \to \mathbb{R}^2$ is prescribed on the Lipschitz boundary of $\mathscr{D}$, say, and define the perturbation

$$\Gamma_h \;=\; \bigl\{x + h(x) \;:\; x \in \partial\mathscr{D}\bigr\}$$

of $\partial\mathscr{D}$ in the direction of h. When h belongs to $W^{1,\infty}(\partial\mathscr{D})$ then Γ_h is a Jordan curve in Ω for h sufficiently small, and we denote its interior domain by $\mathscr{D}_h$. Let u_h be the solution of the corresponding problem (2.2) with $\mathscr{D}$ replaced by $\mathscr{D}_h$; then the so-called shape derivative u'_h of the electric potential $u = u(\mathscr{D})$ in direction h is given by

$$u'_h \;=\; \lim_{t\to 0} \frac{u_{th} - u}{t} \qquad \text{in } \Omega\,, \tag{2.5}$$

provided this limit exists. Likewise,

$$\partial\Lambda_f(\mathscr{D})h \;=\; \lim_{t\to 0}\frac{1}{t}\big(\Lambda_f(\mathscr{D}_{th}) - \Lambda_f(\mathscr{D})\big)$$

is the shape derivative of $\Lambda_f(\mathscr{D})$ in direction h, if the latter limit exists.

Under the assumption that the boundary of $\mathscr{D}$ is smooth, Hettlich and Rundell [17] (see also Afraites, Dambrine, and Kateb [1]) established the existence of u'_h and showed that it is the unique solution of the inhomogeneous transmission problem

$$\begin{aligned} &\Delta u'_h \;=\; 0 \quad \text{in } \Omega\backslash\partial\mathscr{D}\,, \qquad \frac{\partial}{\partial\nu}u'_h \;=\; 0 \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} u'_h\,\mathrm{d}s \;=\; 0\,, \\ &\big[u'_h\big]_{\partial\mathscr{D}} \;=\; (1-k)(h\cdot\nu)\frac{\partial}{\partial\nu}u^-\,, \quad \big[D_\nu u'_h\big]_{\partial\mathscr{D}} \;=\; (1-k)\,\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big)\,. \end{aligned} \tag{2.6}$$

Due to the H^2-regularity of the forward problem at either side of the smooth boundary of $\mathscr{D}$, compare, e.g., McLean [25, Theorem 4.20], the inhomogeneous data for this problem belong to the appropriate Sobelev spaces $H^{\pm 1/2}(\partial\mathscr{D})$, and this garantees existence of a unique weak solution of (2.6), whose restriction to $\Omega\backslash\overline{\mathscr{D}}$ belongs to H^1, so that its trace $u'|_{\partial\Omega} \in L^2_\diamond(\partial\Omega)$ is well-defined. If $\mathscr{D}$ is merely a Lipschitz domain, then this chain of arguments is no longer valid, and it is not immediately clear, whether the transmission problem (2.6) admits a solution, and if so, in which space. And even if there is a unique solution of (2.6), one may still wonder whether this solution is the shape derivative (2.5) of u.

As a step towards less regular inclusions Beretta, Francini, and Vessella [5] studied the conductivity equation (2.2) with a polygonal anomaly and a perturbation field h, whose vector components are linear splines over $\partial\mathscr{D}$ with their nodes attached to the vertices of $\mathscr{D}$; i.e., both components of h are affine linear on each of the edges of $\mathscr{D}$ and continuous at the vertices. With such a field the perturbed domains $\mathscr{D}_h$ also are polygons – at least for h sufficiently small. It is proved in [5] that the associated shape derivative $\partial\Lambda_f(\mathscr{D})$ of Λ_f at $\mathscr{D}$ exists in a Fréchet sense; more precisely they showed that

$$\|\Lambda_f(\mathscr{D}_h) - \Lambda_f(\mathscr{D}) - \partial\Lambda_f(\mathscr{D})h\|_{L^2(\partial\Omega)} \;\leqslant\; C\|h\|^{1+\delta} \tag{2.7}$$

for h sufficiently small, where the constants $C > 0$ and $\delta \in (0,1)$ are independent of the particular choice of h. (Note that in (2.7) it is irrelevant which norm of h is used, because the admissible spline functions from [5] constitute a finite dimensional vector space.) Finally, the authors of [5] came up with a weak (variational) definition of this shape derivative, namely*

$$\langle\, \partial\Lambda_f(\mathscr{D})h, g\,\rangle_{L^2(\Omega)} \;=\; (1-k)\int_{\partial\mathscr{D}} (h\cdot\nu)\,\nabla u^- \cdot (M\nabla v_g^-)\,\mathrm{d}s \tag{2.8}$$

for all $g \in L^2_\diamond(\partial\Omega)$, where

$$v_g \;=\; \begin{cases} v_g^- & \text{in } \mathscr{D}\,, \\ v_g^+ & \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \end{cases}$$

is the corresponding solution in $H^1_\diamond(\Omega)$ of the conductivity equation

$$\nabla\cdot(\sigma\nabla v_g) \;=\; 0 \quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial\nu}v_g \;=\; g \quad \text{on } \partial\Omega\,, \tag{2.9}$$

*Beware of the sign error in [5, Theorem 4.6].

with driving current g, and M is the symmetric 2×2-matrix with eigenvalues 1 and k and corresponding eigenvectors in tangential and normal directions, respectively; see (3.10) below for a justification that the expression (2.8) is well-defined.

We mention that the analysis in [5] avoids the use of the transmission problem (2.6) and the general theory of shape derivatives. An alternative derivation of (2.8) on the grounds of this latter theory was subsequently handed in by Beretta, Micheletti, Perotto, and Santacesaria [6], who determined the so-called material derivative $\dot{u}$ of u in Ω, cf. (4.3) below, but the existence of the shape derivative u' of u remained unsettled. In Section 4 we will also exploit the material derivative to show that the estimate (2.7) for the Taylor remainder of the shape derivative of the boundary data can be improved to the order $\|h\|^2$.

In Sections 5 and 6 we analyze the transmission problem (2.6) for polygonal inclusions as treated in [5, 6], and we prove that it has a unique solution u'_h. In contrast to the smooth case its restrictions to $\mathscr{D}$ and to $\Omega\backslash\overline{\mathscr{D}}$ merely belong to some Sobolev space H^γ, where $\gamma \in (1/2, 1)$ depends on the interior angles of the vertices of $\mathscr{D}$. We further show that u'_h has a well-defined trace which satisfies the same variational problem (2.8) as $\partial\Lambda_f(\mathscr{D})h$; from this we then conclude that the solution of the transmission problem is the shape derivative of u in Ω, i.e., that the main result by Hettlich and Rundell carries over to polygonal inclusions.

3. The conductivity problem with polygonal inclusions. We recapitulate some basic facts about the two-dimensional conductivity problem (2.2) with a polygonal inclusion $\mathscr{D}$ with n vertices, $n \geqslant 3$, and a simply connected closure $\overline{\mathscr{D}} \subset \Omega$.

We start with some notation: We denote the (relatively open) edges of $\mathscr{D}$ by Γ_i and its vertices by x_i, $i = 1, \ldots, n$, with the convention that x_i connects Γ_i and Γ_{i+1}, where we identify Γ_{n+1} with Γ_1; likewise we identify x_{n+1} with x_1, when necessary. On Γ_i the tangent vector τ is pointing in the direction of x_i. The interior angle of $\mathscr{D}$ at x_i is denoted by $\alpha_i \in (0, 2\pi)\backslash\{\pi\}$, and we use a local coordinate system

$$x \;=\; x_i \;+\; \bigl(r\cos(\theta_i + \theta), r\sin(\theta_i + \theta)\bigr)\,, \qquad 0 < r < r_i\,,\; 0 \leqslant \theta < 2\pi\,, \tag{3.1}$$

near $x = x_i$, where θ_i is such that $\theta = 0$ corresponds to points on Γ_{i+1}, $\theta = \alpha_i$ for $x \in \Gamma_i$, and the range $0 < \theta < \alpha_i$ corresponds to points $x \in \mathscr{D}$, while the interval $\alpha_i < \theta < 2\pi$ corresponds to points $x \in \Omega\backslash\overline{\mathscr{D}}$. We will also make use of cut-off functions $\chi_i = \chi_i(x) \in C_0^\infty(\mathbb{R}^2)$, which only depend on the distance $|x - x_i|$, are one near $x = x_i$ and zero for $|x - x_i| \geqslant r_i$. Without loss of generality we assume that the radius r_i is so small that the supports of any two cut-off functions χ_i and χ_j with $i \neq j$ have no points in common, and that the intersection of $\partial\mathscr{D}$ with the support of χ_i is a subset of $\Gamma_i \cup \{x_i\} \cup \Gamma_{i+1}$.

Since the two components $u^\pm$ of the solution u of (2.2) are harmonic and satisfy the transmission conditions (2.3) they can both be continued as harmonic functions across each of the edges Γ_i of $\mathscr{D}$ by appropriate reflections. This proves that all their derivatives extend continuously to every edge from either side. In particular this implies that $u^+|_{\Gamma_i} = u^-|_{\Gamma_i}$ is a well-defined C^∞ function; we therefore drop the superscripts $\pm$ when dealing with this trace.

We quote from Bellout, Friedman, and Isakov [4] that near a fixed vertex x_i of $\mathscr{D}$, $i \in \{1, \ldots, n\}$, the potential u satisfies

$$u(x) \;=\; u(x_i) \;+\; \sum_{j=1}^{\infty} \beta_{ij}\, y_{ij}(\theta)\, r^{\gamma_{ij}}\,, \tag{3.2}$$

when using the local coordinate system (3.1) for

$$x \in \mathcal{B}_{r_i}(x_i) \;=\; \bigl\{x \,:\, |x - x_i| < r_i\bigr\}\,.$$

The functions $y_{ij} \in H^1(0,2\pi)$ in (3.2) are the eigenfunctions of the eigenvalue problem

$$-y'' = \gamma^2 y \qquad \text{in } (0,2\pi)\backslash\{\alpha_i\}\,, \tag{3.3a}$$

subject to the transmission conditions

$$\begin{aligned} y(0+) &= y(2\pi-)\,, & y(\alpha_i-) &= y(\alpha_i+)\,,\\ k\,y'(0+) &= y'(2\pi-)\,, & k\,y'(\alpha_i-) &= y'(\alpha_i+)\,, \end{aligned} \tag{3.3b}$$

γ_{ij}^2 are the corresponding eigenvalues in increasing order, and γ_{ij} its square roots. The latter are the nonnegative solutions of the nonlinear equation

$$\bigl|\sin\gamma_{ij}(\alpha_i - \pi)\bigr| \;=\; \lambda\,\bigl|\sin\gamma_{ij}\pi\bigr|\,, \qquad \lambda \;=\; \Bigl|\frac{k+1}{k-1}\Bigr|\,. \tag{3.4}$$

There holds

$$\gamma_{i0} \;=\; 0\,, \quad \frac{1}{2} \;<\; \gamma_{i1} \;<\; 1\,, \quad \text{and} \quad \gamma_{i2} \;>\; 1\,, \tag{3.5}$$

and the eigenfunctions have the form

$$y_{ij}(\theta) \;=\; \begin{cases} A_{ij}^- \cos\gamma_{ij}\theta \;+\; B_{ij}^- \sin\gamma_{ij}\theta\,, & 0 < \theta < \alpha_i\,,\\ A_{ij}^+ \cos\gamma_{ij}\theta \;+\; B_{ij}^+ \sin\gamma_{ij}\theta\,, & \alpha_i < \theta < 2\pi\,, \end{cases}$$

with appropriate values of $A_{ij}^\pm$ and $B_{ij}^\pm$. In particular, y_{i0} is constant in the entire interval $(0,2\pi)$. We normalize these eigenfunctions in such a way that they define an orthonormal basis with respect to the inner product

$$\langle\, y, \widetilde{y}\,\rangle \;=\; \int_0^{\alpha_i} k\,y(\theta)\widetilde{y}(\theta)\,\mathrm{d}\theta \;+\; \int_{\alpha_i}^{2\pi} y(\theta)\widetilde{y}(\theta)\,\mathrm{d}\theta\,.$$

From the transmission conditions (3.3b) it follows that

$$\begin{bmatrix} 1 & 0 & -\cos 2\pi\gamma & -\sin 2\pi\gamma\\ 0 & k & \sin 2\pi\gamma & -\cos 2\pi\gamma\\ \cos\gamma\alpha & \sin\gamma\alpha & -\cos\gamma\alpha & -\sin\gamma\alpha\\ -k\sin\gamma\alpha & k\cos\gamma\alpha & \sin\gamma\alpha & -\cos\gamma\alpha \end{bmatrix} \begin{bmatrix} A^-\\ B^-\\ A^+\\ B^+ \end{bmatrix} \;=\; \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, \tag{3.6}$$

where we have omitted all subscripts i and j for the ease of readability; in order that this system has a nontrivial solution the matrix $Y \in \mathbb{R}^{4\times 4}$ in (3.6) must be singular. In fact, we state the following result, which is implicit in Seo [28].

Lemma 3.1. *Let $Y = Y(\gamma,\alpha)$ be given by the 4×4-matrix in (3.6) with $\alpha \in (0,2\pi)\backslash\{\pi\}$ and $\gamma > 0$. Then Y is singular, if and only if γ is a solution of*

$$\bigl|\sin(\gamma(\alpha - \pi))\bigr| \;=\; \lambda\,\bigl|\sin\gamma\pi\bigr| \tag{3.7}$$

with λ as in (3.4). If α and γ satisfy (3.7) then $\operatorname{rank}(Y) = 3$, unless $\gamma \in \mathbb{N}\backslash\{1\}$ and $\alpha = l\pi/\gamma$ for some $l \in \{1,\dots,2\gamma-1\}\backslash\{\gamma\}$, in which case $\operatorname{rank}(Y) = 2$ and the corresponding eigenvalue problem (3.3) has a two dimensional eigenspace.

Proof. Since we need this result in a slightly different context below (see (5.4)) we provide a sketch of its proof. Introducing

$$D \;=\; \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \qquad \text{and} \qquad R(\omega) \;=\; \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$$

we can write

$$Y \;=\; \begin{bmatrix} D & -R(2\pi\gamma) \\ DR(\gamma\alpha) & -R(\gamma\alpha) \end{bmatrix},$$

and since its $(1,1)$-block D is nonsingular, Y is singular, if and only if the Schur complement

$$\begin{aligned} S \;&=\; DR(\gamma\alpha)D^{-1}R(2\pi\gamma) - R(\gamma\alpha) \\ &=\; DR(\gamma\pi)\Big(R(\gamma(\alpha-\pi))D^{-1}R(\gamma\pi) \;-\; R(-\gamma\pi)D^{-1}R(\gamma(\alpha-\pi)\Big)R(\gamma\pi) \end{aligned}$$

of D in $Y(\gamma,\alpha)$ is singular. Computing the products in the inner paranthesis and taking the determinant of their difference it is readily seen that S is singular, if and only if γ is a solution of (3.7). The Schur complement S vanishes completely, if and only if $\gamma \in \mathbb{N}$ and $\gamma\alpha$ is an integer multiple of π, in which case $y(\theta) = \cos\gamma\theta$ is one associated eigenfunction of (3.3), and $y(\theta) = B^{\pm}\sin\gamma\theta$ with $B^{+} = kB^{-}$ is a second one. □

As mentioned in [4] the series (3.2) can be differentiated termwise to any order with respect to θ and r (with one-sided derivatives at $\theta = \alpha_i, 0, 2\pi$, respectively). We thus compute

$$\frac{\partial}{\partial\nu}u^{-}(x) \;=\; -\frac{1}{r}\frac{\partial}{\partial\theta}u^{-}(x) \;=\; -\sum_{j=1}^{\infty}\beta_{ij}\gamma_{ij}B_{ij}^{-}r^{\gamma_{ij}-1}\,, \qquad x \in \mathit{\Gamma}_{i+1}\,, \tag{3.8a}$$

and

$$\frac{\partial}{\partial\nu}u^{-}(x) \;=\; \frac{1}{r}\frac{\partial}{\partial\theta}u^{-}(x) \;=\; \sum_{j=1}^{\infty}\beta_{ij}\gamma_{ij}(B_{ij}^{-}c_{ij} - A_{ij}^{-}s_{ij})r^{\gamma_{ij}-1}\,, \qquad x \in \mathit{\Gamma}_{i}\,, \tag{3.8b}$$

for x close to x_i, where we have set

$$c_{ij} = \cos\gamma_{ij}\alpha_i \qquad \text{and} \qquad s_{ij} = \sin\gamma_{ij}\alpha_i\,.$$

Likewise, we have

$$\frac{\partial}{\partial\tau}u(x) \;=\; \frac{\partial}{\partial r}u(x) \;=\; \sum_{j=1}^{\infty}\beta_{ij}\gamma_{ij}A_{ij}^{-}r^{\gamma_{ij}-1}\,, \qquad x \in \mathit{\Gamma}_{i+1}\,, \tag{3.9a}$$

and

$$\frac{\partial}{\partial\tau}u(x) \;=\; -\frac{\partial}{\partial r}u(x) \;=\; -\sum_{j=1}^{\infty}\beta_{ij}\gamma_{ij}(A_{ij}^{-}c_{ij} + B_{ij}^{-}s_{ij})r^{\gamma_{ij}-1}\,, \qquad x \in \mathit{\Gamma}_{i}\,, \tag{3.9b}$$

near $x = x_i$.

From (3.5), (3.8), (3.9), and (2.3), and from the fact that the gradient of u has continuous extensions from either side to every edge of $\partial\mathscr{D}$ it follows that

$$\frac{\partial}{\partial\nu}u^{\pm}\Big|_{\partial\mathscr{D}}\,,\ \frac{\partial}{\partial\tau}u\Big|_{\partial\mathscr{D}}\ \in\ L^2(\partial\mathscr{D})\,, \tag{3.10}$$

and, of course, the same is true for every solution v_g of (2.9); compare also Escauriaza, Fabes, and Verchota [11]. Accordingly, the right-hand side of (2.8) is well-defined.

4. The shape derivative of $u = u(\mathscr{D})$ for a polygonal inclusion in Ω. In the sequel we investigate the existence of the shape derivative of the solution $u = u(\mathscr{D})$ of the conductivity equation (2.2) at a polygonal inclusion $\mathscr{D}$. We also derive bounds for the corresponding shape derivative $\partial\Lambda_f(\mathscr{D})$ and its associated Taylor remainder, the latter improving upon the estimate (2.7) provided in [5].

We start by making the assumption that a vector field $h : \Omega \to \mathbb{R}^2$ with

$$h\ \in\ W_c^{1,\infty}(\Omega)\ =\ \big\{\, h \in W^{1,\infty}(\Omega)\ :\ \operatorname{supp} h \subset \Omega \,\big\}$$

is given, and define

$$\Phi_h(x)\ =\ x\ +\ h(x)\,, \qquad x \in \Omega\,. \tag{4.1}$$

Note that $\Phi_h : \Omega \to \Omega$ is bijective, if, for example,

$$\|h\|_{W^{1,\infty}(\Omega)}\ =\ \sup_{x\in\Omega}\big(\|h(x)\|_2\ +\ \|h'(x)\|_2\big)\ \leqslant\ 1/2\,, \tag{4.2}$$

by virtue of Banach's fixed point theorem.

We now follow the standard argumentation from [29], which has been worked out in [6] for the present application. Let h satisfy (4.2), define $\mathscr{D}_h = \Phi_h(\mathscr{D})$, and denote by $u_h \in H^1_\diamond(\Omega)$ the solution of the forward problem

$$\nabla\cdot(\sigma_h\nabla u_h)\ =\ 0\quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial\nu}u_h\ =\ f\quad \text{on } \partial\Omega\,,$$

where

$$\sigma_h(x)\ =\ \begin{cases} k\,, & \text{for } x \in \mathscr{D}_h\,,\\ 1\,, & \text{for } x \in \Omega\backslash\overline{\mathscr{D}}_h\,.\end{cases}$$

Introducing

$$\widetilde{u}_h\ =\ u_h\circ\Phi_h\ :\ \Omega \to \mathbb{R}\,,$$

it has been shown in [6] that the *material derivative*

$$\dot{u}_h\ =\ \lim_{t\to 0}\frac{\widetilde{u}_{th} - u}{t}\ \in\ H^1_\diamond(\Omega) \tag{4.3}$$

is well-defined, the convergence being in $H^1(\Omega)$, and that $\dot{u}_h$ satisfies the variational problem

$$\int_\Omega \sigma\,\nabla\dot{u}_h\cdot\nabla w\,\mathrm{d}x\ =\ -\int_\Omega \sigma\,\nabla u\cdot(\mathcal{A}_h\nabla w)\,\mathrm{d}x \qquad \text{for every } w \in H^1(\Omega)\,, \tag{4.4}$$

where σ is as in (2.1) and

$$\mathcal{A}_h \;=\; (\nabla\cdot h)I \;-\; h' \;-\; {h'}^T$$

is an L^∞ function of 2×2 matrices with

$$\|\mathcal{A}_h\|_{L^\infty(\Omega)} \;=\; \sup_{x\in\Omega}\|\mathcal{A}_h(x)\|_2 \;\leqslant\; 4\,\|h\|_{W^{1,\infty}(\Omega)}\,. \tag{4.5}$$

It has further been proved in [6] that $\dot u_h|_{\partial\Omega}$ coincides with the shape derivative of $\Lambda_f(\mathscr{D})$ in the direction of $h|_{\partial\mathscr{D}}$, i.e.,

$$\langle\, \dot u_h, g\,\rangle_{L^2(\partial\Omega)} \;=\; \langle\, \partial\Lambda_f(\mathscr{D}_t)h|_{\partial\mathscr{D}}, g\,\rangle_{L^2(\partial\Omega)} \tag{4.6}$$

is given by (2.8) for every $g\in L^2_\diamond(\partial\Omega)$.

This approach via the material derivative allows for a stronger result than (2.7).

THEOREM 4.1. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}}\subset\Omega$. Then the material derivative in $\mathcal{L}(W^{1,\infty}_c(\Omega), H^1_\diamond(\Omega))$ is a Fréchet derivative. Moreover, there are constants $c, C>0$, which only depend on Ω, $\mathscr{D}$, f, and k, such that*

$$\|\Lambda_f(\mathscr{D}_h) - \Lambda_f(\mathscr{D}) - \partial\Lambda_f(\mathscr{D})h\|_{L^2(\partial\Omega)} \;\leqslant\; C\,\|h\|^2_{W^{1,\infty}(\partial\mathscr{D})} \tag{4.7}$$

for every $h\in W^{1,\infty}(\partial\mathscr{D})$ with $\|h\|_{W^{1,\infty}(\partial\mathscr{D})}\leqslant c$.

Proof. Let $h\in W^{1,\infty}_c(\Omega)$ satisfy (4.2). Then the Jacobian Φ_h' is invertible, and it follows from [6, Eq. (2.11)] that the auxiliary function $w_h = \widetilde u_h - u$ solves the variational problem

$$\int_\Omega \sigma\,\nabla w_h\cdot(A_h\nabla w)\,\mathrm{d}x \;=\; \int_\Omega \sigma\,\nabla u\cdot\big((I-A_h)\nabla w\big)\,\mathrm{d}x \tag{4.8}$$

for every $w\in H^1(\Omega)$, where

$$A_h \;=\; {\Phi_h'}^{-1}{\Phi_h'}^{-T}\det\Phi_h'\,. \tag{4.9}$$

From (4.8) we deduce that

$$\begin{aligned}\int_\Omega \sigma\,\nabla w_h\cdot\nabla w\,\mathrm{d}x \;&=\; \int_\Omega \sigma\,\nabla(\widetilde u_h - u)\cdot\big((I-A_h)\nabla w\big)\,\mathrm{d}x \;+\; \int_\Omega \sigma\,\nabla w_h\cdot(A_h\nabla w)\,\mathrm{d}x\\ &=\; \int_\Omega \sigma\,\nabla\widetilde u_h\cdot\big((I-A_h)\nabla w\big)\,\mathrm{d}x\,,\end{aligned}$$

and hence, (4.4) implies that the Taylor remainder

$$e_h \;=\; \widetilde u_h - u - \dot u_h \;=\; w_h - \dot u_h$$

of the material derivative satisfies

$$\begin{aligned}\int_\Omega \sigma\,\nabla e_h\cdot\nabla w\,\mathrm{d}x \;&=\; \int_\Omega \sigma\,\nabla\widetilde u_h\cdot\big((I-A_h)\nabla w\big)\,\mathrm{d}x \;+\; \int_\Omega \sigma\,\nabla u\cdot(\mathcal{A}_h\nabla w)\,\mathrm{d}x\\ &=\; \int_\Omega \sigma\,\nabla w_h\cdot\big((I-A_h)\nabla w\big)\,\mathrm{d}x \;+\; \int_\Omega \sigma\,\nabla u\cdot\big((I-A_h+\mathcal{A}_h)\nabla w\big)\,\mathrm{d}x\end{aligned} \tag{4.10}$$

for every $w\in H^1(\Omega)$.

From (4.1) and (4.9) we have

$$A_h \;=\; (I + h')^{-1}(I + {h'}^T)^{-1}(1 + \nabla\cdot h + \det h')\,,$$

and hence, using (4.2), it follows that there exists some constant C', which we take to be greater than one, such that

$$\|A_h - I\|_{L^\infty(\Omega)} \;\leqslant\; C'\,\|h\|_{W^{1,\infty}(\Omega)} \quad \text{and} \quad \|A_h - I - \mathcal{A}_h\|_{L^\infty(\Omega)} \;\leqslant\; C'\,\|h\|^2_{W^{1,\infty}(\Omega)}\,,$$

independent of the particular choice of h. Accordingly, the right-hand side of (4.10) can be estimated by

$$C'\bigl(\|\sqrt{\sigma}\,\nabla w_h\|_{L^2(\Omega)}\|h\|_{W^{1,\infty}(\Omega)} \;+\; \|\sqrt{\sigma}\,\nabla u\|_{L^2(\Omega)}\|h\|^2_{W^{1,\infty}(\Omega)}\bigr)\,\|\sqrt{\sigma}\,\nabla w\|_{L^2(\Omega)}\,,$$

and using $w = e_h$ in (4.10) we thus obtain the estimate

$$\begin{aligned}\|\sqrt{\sigma}\,\nabla e_h\|_{L^2(\Omega)} \;&\leqslant \\ &C'\bigl(\|\sqrt{\sigma}\,\nabla w_h\|_{L^2(\Omega)}\|h\|_{W^{1,\infty}(\Omega)} \;+\; \|\sqrt{\sigma}\,\nabla u\|_{L^2(\Omega)}\|h\|^2_{W^{1,\infty}(\Omega)}\bigr)\end{aligned} \tag{4.11}$$

for the Taylor remainder.

Similarly, we conclude from (4.8) and (4.9) that

$$\frac{1}{2}\,\|\sqrt{\sigma}\,\nabla w_h\|_{L^2(\Omega)} \;\leqslant\; C'\,\|\sqrt{\sigma}\,\nabla u\|_{L^2(\Omega)}\|h\|_{W^{1,\infty}(\Omega)}$$

for every h satisfying

$$\|h\|_{W^{1,\infty}(\Omega)} \;\leqslant\; \frac{1}{2C'}\,, \tag{4.12}$$

whereas the weak form (2.4) of (2.2) implies that

$$\begin{aligned}\|\sqrt{\sigma}\,\nabla u\|^2_{L^2(\Omega)} \;&\leqslant\; \|f\|_{L^2(\partial\Omega)}\|u\|_{L^2(\partial\Omega)} \\ &\leqslant\; \frac{c_\Omega}{\min\{1,\sqrt{k}\}}\,\|f\|_{L^2(\partial\Omega)}\|\sqrt{\sigma}\,\nabla u\|_{L^2(\Omega)}\,,\end{aligned} \tag{4.13}$$

where c_Ω depends on the norm of the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and the Poincaré constant for $H^1_\diamond(\Omega)$. With these two estimates and another use of the Poincaré inequality it follows from (4.11) that

$$\|e_h\|_{H^1(\Omega)} \;\leqslant\; C''\|h\|^2_{W^{1,\infty}(\Omega)} \tag{4.14}$$

with a constant C'' which is independent of h, as long as h satisfies (4.12). This shows that the material derivative in $\mathcal{L}(W^{1,\infty}_c(\Omega), H^1_\diamond(\Omega))$ is a Fréchet derivative.

Next, let $h \in W^{1,\infty}(\partial\mathscr{D})$ be given. Then we construct an extension $h_e \in W^{1,\infty}_c(\Omega)$ of this given function in the following way. We first choose a polygonal domain $\mathscr{D}_1$ with vertices $x_i^{(1)}$ in the vicinity of x_i, $i = 1,\dots,n$, respectively, such that $\overline{\mathscr{D}}_1 \subset \mathscr{D}$ and each (open) quadrangle $\mathscr{Q}_i$ with vertices $x_i, x_{i+1}, x_i^{(1)}$, and $x_{i+1}^{(1)}$ is convex; using the associated convex representation

$$x \;=\; c(1-d)x_i \;+\; cd\,x_{i+1} \;+\; (1-c)(1-d)x_i^{(1)} \;+\; (1-c)d\,x_{i+1}^{(1)}$$

with $0 < c, d < 1$ of a general point $x \in \mathscr{Q}_i$ we can extend h to $\mathscr{Q}_i$ via

$$h_e(x) \,=\, c\, h\big((1-d)x_i \,+\, d\, x_{i+1}\big)\,, \qquad x \in \mathscr{Q}_i\,,$$

and in a second step, extend h_e continuously by zero to $\overline{\mathscr{D}}_1$. In the same manner we can construct a polygonal domain $\mathscr{D}_2$ with $\overline{\mathscr{D}} \subset \mathscr{D}_2$ and $\overline{\mathscr{D}}_2 \subset \Omega$ and a corresponding extension of h_e to $\mathscr{D}_2$, and finally extend h_e by zero to the rest of Ω. Note that this construction makes sure that there exists a constant $C_{\mathscr{D}} \geqslant 1$, independent of h, such that the extended vector field satisfies

$$\|h_e\|_{W^{1,\infty}(\Omega)} \,\leqslant\, C_{\mathscr{D}}\, \|h\|_{W^{1,\infty}(\partial\mathscr{D})}\,. \tag{4.15}$$

In particular, h_e satisfies (4.12) if

$$\|h\|_{W^{1,\infty}(\partial\mathscr{D})} \,\leqslant\, c \,:=\, \frac{1}{2C'C_{\mathscr{D}}}\,.$$

Using this extended vector field the trace of the associated material derivative $\dot{u}_{h_e}$ coincides with $\partial\Lambda_f(\mathscr{D}_t)h$, and hence, the left hand side of (4.7) is the norm of the trace on $\partial\Omega$ of

$$e_{h_e} \,=\, \widetilde{u}_{h_e} \,-\, u \,-\, \dot{u}_{h_e}\,.$$

Accordingly, the desired inequality (4.7) is a consequence of (4.14), (4.15), and the trace theorem. ∎

We also have the following estimate that is of independent interest.

Theorem 4.2. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}} \subset \Omega$. Then there is a constant $C > 0$, depending only on Ω, f, and k, such that*

$$\|\partial\Lambda_f(\mathscr{D}')\|_{\mathcal{L}(W^{1,\infty}(\partial\mathscr{D}'),L^2(\partial\Omega))} \,\leqslant\, C$$

for all polygons $\mathscr{D}'$ with the same number of vertices, which are sufficiently close to those of $\mathscr{D}$.

Proof. For the given polygon $\mathscr{D}$ we choose $\mathscr{D}_1$ and $\mathscr{D}_2$ as in the proof of Theorem 4.1. Then every polygon $\mathscr{D}'$ which is a sufficiently small perturbation of $\mathscr{D}$ as specified in the statement of this theorem satisfies $\partial\mathscr{D}' \subset \mathscr{D}_2 \backslash \overline{\mathscr{D}}_1$, and every vector field $h \in W^{1,\infty}(\partial\mathscr{D}')$ can be extended to a vector field $h_e \in W_c^{1,\infty}(\Omega)$ supported in $\overline{\mathscr{D}}_2$ as described in the previous proof. Moreover, the corresponding extension satisfies

$$\|h_e\|_{W^{1,\infty}(\Omega)} \,\leqslant\, C_{\mathscr{D}'} \|h\|_{W^{1,\infty}(\partial\mathscr{D}')} \,\leqslant\, 2C_{\mathscr{D}}\, \|h\|_{W^{1,\infty}(\partial\mathscr{D}')}\,, \tag{4.16}$$

provided $\mathscr{D}'$ is sufficiently close to $\mathscr{D}$. In particular, if we stipulate that $\|h\|_{W^{1,\infty}(\partial\mathscr{D}')} = 1/(4C_{\mathscr{D}})$, then h_e satisfies (4.2).

It therefore follows from (4.6) and (4.4) with $w = \dot{u}_{h_e}$ that

$$\begin{aligned}\|\partial\Lambda_f(\mathscr{D}')h\|_{L^2(\partial\Omega)} \,&\leqslant\, \frac{c_\Omega}{\min\{1,\sqrt{k}\}}\, \|\sqrt{\sigma}\, \nabla \dot{u}_{h_e}\|_{L^2(\Omega)} \\ &\leqslant\, \frac{c_\Omega}{\min\{1,\sqrt{k}\}}\, \|\sqrt{\sigma}\, \nabla u\|_{L^2(\Omega)} \|\mathcal{A}_{h_e}\|_{L^\infty(\Omega)}\,,\end{aligned}$$

where c_Ω is the same constant as in (4.13). From (4.13), (4.5), and (4.16) we therefore deduce that

$$\begin{aligned}\|\partial\Lambda_f(\mathscr{D}')h\|_{L^2(\partial\Omega)} \,&\leqslant\, \frac{4c_\Omega^2}{\min\{1,k\}}\, \|f\|_{L^2(\Omega)}\, \|h_e\|_{W^{1,\infty}(\Omega)} \\ &\leqslant\, \frac{8c_\Omega^2\, C_{\mathscr{D}}}{\min\{1,k\}}\, \|f\|_{L^2(\Omega)}\, \|h\|_{W^{1,\infty}(\partial\mathscr{D}')}\,.\end{aligned}$$

This proves the assertion. $\square$

REMARK 4.3. In the same way one can show that the constants c and C of Theorem 4.1 can be chosen in such a way that the same Taylor remainder estimate (4.7) is valid for all polygons $\mathscr{D}'$ sufficiently close to $\mathscr{D}$. $\diamond$

Now we turn to the existence of the shape derivative u'_h of u, formally defined in (2.5).

THEOREM 4.4. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}} \subset \Omega$. Then the solution $u = u(\mathscr{D})$ of (2.2) has a Fréchet shape derivative $\partial u(\mathscr{D}) \in \mathcal{L}(W_c^{1,\infty}(\Omega), L^2(\Omega))$. The derivative $u'_h = \partial u(\mathscr{D})h$ of u in the direction of $h \in W_c^{1,\infty}(\Omega)$ is given by*

$$u'_h \;=\; \dot{u}_h \;-\; h \cdot \nabla u\,. \tag{4.17}$$

This function is harmonic in $\Omega\backslash\partial\mathscr{D}$, and it belongs to $H^1(\Omega\backslash \operatorname{supp} h)$. The Neumann boundary values of u'_h on $\partial\Omega$ vanish, and the trace $u'_h|_{\partial\Omega} \in L^2_\diamond(\Omega)$ is the shape derivative $\partial\Lambda_f(\mathscr{D})h|_{\partial\mathscr{D}}$ of the given measurements on the boundary.

Proof. The statement on the Fréchet derivative of $u = u(\mathscr{D})$ and its representation (4.17) follows with the aid of, e.g., [16, Lemme 5.3.3] from the fact that the material derivative is a Fréchet derivative.

Let $h \in W_c^{1,\infty}(\Omega)$ be fixed, and consider any domain Ω' with $\overline{\Omega}' \subset \mathscr{D}$, and let $w \in C_0^\infty(\Omega')$. Then $\Omega' \subset \mathscr{D}_{th}$ for every $0 < t < t'$ with t' sufficiently small, and hence, both u_{th} and u are harmonic in Ω' for $0 < t < t'$. Accordingly,

$$\int_{\Omega'} \frac{u_{th} - u}{t}\,\Delta w \,\mathrm{d}x \;=\; 0 \qquad \text{for } 0 < t < t'\,,$$

and in the limit $t \to 0$ this yields

$$\int_{\Omega'} u'_h \Delta w \,\mathrm{d}x \;=\; 0 \qquad \text{for every } w \in C_0^\infty(\Omega')\,.$$

It thus follows from Weil's lemma that u'_h is harmonic in Ω', and hence in $\mathscr{D}$. The same argument applies for any domain Ω' with $\overline{\Omega}' \subset \Omega\backslash\overline{\mathscr{D}}$, showing that u'_h is also harmonic in $\Omega\backslash\overline{\mathscr{D}}$.

By virtue of (4.17) u'_h coincides with $\dot{u}_h$ on $\Omega\backslash \operatorname{supp} h$. Accordingly, $\dot{u}_h$ is a harmonic function in $\Omega\backslash \operatorname{supp} h$, u'_h belongs to $H^1(\Omega\backslash \operatorname{supp} h)$, and there holds $u'_h|_{\partial\Omega} = \dot{u}_h|_{\partial\Omega} = \partial\Lambda_f(\mathscr{D})h|_{\partial\mathscr{D}}$, and

$$\frac{\partial}{\partial\nu} u'_h \;=\; \frac{\partial}{\partial\nu}\dot{u}_h \;\in\; H^{-1/2}(\partial\Omega)\,.$$

This Neumann derivative can be determined from the variational definition (4.4) of $\dot{u}_h$: Choosing an arbitrary test function $w \in C^\infty(\mathbb{R}^2)$, which vanishes on $\mathscr{D} \cup \operatorname{supp} h$, it follows from (4.4) that

$$0 \;=\; \int_{\Omega\backslash\overline{\mathscr{D}}} \nabla\dot{u}_h \cdot \nabla w \,\mathrm{d}x \;=\; \int_{\partial\Omega} w\,\frac{\partial}{\partial\nu}\dot{u}_h \,\mathrm{d}s\,.$$

Since the traces on $\partial\Omega$ of all these admissible test functions are dense in $L^2(\partial\Omega)$, the Neumann derivative on $\partial\Omega$ of $\dot{u}_h$, and hence of u'_h, must vanish. $\square$

We will see below (in Theorem 6.1) that u'_h actually only depends on $h|_{\partial\mathscr{D}}$.

5. The transmission problem for polygonal inclusions. We now investigate the transmission problem (2.6) for a polygonal inclusion.

THEOREM 5.1. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}} \subset \Omega$, and let $h \in W^{1,\infty}(\partial\mathscr{D})$. Then the transmission problem*

$$\begin{gathered}\Delta w \,=\, 0 \quad \text{in } \Omega\backslash\partial\mathscr{D}\,, \qquad \frac{\partial}{\partial\nu}w \,=\, 0 \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} w\,\mathrm{d}s \,=\, 0\,, \\ \big[w\big]_{\partial\mathscr{D}} \,=\, (1-k)(h\cdot\nu)\frac{\partial}{\partial\nu}u^-\,, \quad \big[D_\nu w\big]_{\partial\mathscr{D}} \,=\, (1-k)\,\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big)\,,\end{gathered} \tag{5.1}$$

has a unique solution w, which belongs to $H^\gamma(\Omega\backslash\partial\mathscr{D})$ for some $\gamma \in (1/2,1)$ and also to $H^1\big(\Omega\backslash\overline{\mathscr{D}\cup_i \mathcal{B}_{r_i}(x_i)}\big)$.

Proof. Uniqueness is obvious: If w_1 and w_2 are two solutions of (5.1), then $v = w_1 - w_2$ solves the homogeneous Neumann problem (2.9) with $g = 0$. This proves that $v = 0$.

To establish existence, let

$$h_i^- \,=\, \lim_{x\to x_i,\, x\in\Gamma_i}(h\cdot\nu)(x)\,, \qquad h_i^+ \,=\, \lim_{x\to x_i,\, x\in\Gamma_{i+1}}(h\cdot\nu)(x)\,, \tag{5.2}$$

for every $i = 1,\dots,n$; take note that $h_i^- \neq h_i^+$ in general, and that

$$h(x) \,=\, \begin{cases} h_i^- \,+\, O(|x-x_i|)\,, & x\in\Gamma_i\,,\\ h_i^+ \,+\, O(|x-x_i|)\,, & x\in\Gamma_{i+1}\,,\end{cases} \tag{5.3}$$

for x near x_i. Furthermore, for $i = 1,\dots,n$ let $A_i'^{\pm}$ and $B_i'^{\pm}$ be the entries of the solution of the linear system

$$Y(\gamma_{i1}-1,\alpha_i)\begin{bmatrix}A_i'^-\\B_i'^-\\A_i'^+\\B_i'^+\end{bmatrix} \,=\, (1-k)\gamma_{i1}\begin{bmatrix}h_i^+B_{i1}^-\\h_i^+A_{i1}^-\\h_i^-(A_{i1}^-s_{i1}-B_{i1}^-c_{i1})\\-h_i^-(A_{i1}^-c_{i1}+B_{i1}^-s_{i1})\end{bmatrix} \tag{5.4}$$

with $Y \in \mathbb{R}^{4\times4}$ defined in (3.6). Since $\gamma' = \gamma_{i1}-1 \in (-1/2,0)$ according to (3.5), γ' cannot be a solution of (3.4), and hence the inhomogeneous linear system (5.4) has a unique solution by virtue of Lemma 3.1. The way this system is set up, the function

$$w_i(x) \,=\, \begin{cases}\beta_{i1}\chi_i(x)\widetilde{y}_i(\theta)r^{\gamma_{i1}-1}\,, & x\in\mathcal{B}_{r_i}(x_i)\,,\\ 0\,, & x\in\Omega\backslash\overline{\mathcal{B}_{r_i}(x_i)}\,,\end{cases} \tag{5.5}$$

defined in terms of the local coordinate system (3.1), with

$$\widetilde{y}_i(\theta) \,=\, \begin{cases}A_i'^-\cos(\gamma_{ij}-1)\theta \,+\, B_i'^-\sin(\gamma_{ij}-1)\theta\,, & 0<\theta<\alpha_i\,,\\ A_i'^+\cos(\gamma_{ij}-1)\theta \,+\, B_i'^+\sin(\gamma_{ij}-1)\theta\,, & \alpha_i<\theta<2\pi\,,\end{cases} \tag{5.6}$$

is such that $\big[w_i\big]_{\partial\mathscr{D}}$ and $\big[D_\nu w_i\big]_{\partial\mathscr{D}}$ coincide near $x = x_i$ with the leading order terms of the prescribed transmission data in (5.1), compare (3.8) and (3.9). In fact, from (5.3) and (3.5) follows that

$$\varphi \,=\, (1-k)\,(h\cdot\nu)\frac{\partial}{\partial\nu}u^- \,-\, \sum_{i=1}^n\big[w_i\big]_{\partial\mathscr{D}} \,\in\, H^{1/2}(\partial\mathscr{D}) \tag{5.7a}$$

and

$$\psi \;=\; (1-k)\,\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big) \;-\; \sum_{i=1}^{n}\big[D_\nu w_i\big]_{\partial\mathscr{D}} \;\in\; H^{-1/2}(\partial\mathscr{D})\,. \tag{5.7b}$$

Further, since $\widetilde{y}_i r^{\gamma_{i1}-1}$ is harmonic in $\operatorname{supp}\chi_i\backslash\partial\mathscr{D}$,

$$F_i \;=\; \sigma\Delta w_i \;=\; \sigma\beta_{i1}\big(2\nabla\chi_i\cdot\nabla(\widetilde{y}_i r^{\gamma_{i1}-1}) \;+\; \widetilde{y}_i r^{\gamma_{i1}-1}\Delta\chi_i\big)\,, \tag{5.7c}$$

defined in $\Omega\backslash\partial\mathscr{D}$, is smooth in both subdomains and supported in a small annulus around x_i. In particular, $F_i \in L^2(\Omega)$, and hence the transmission problem

$$\begin{gathered} -\nabla\cdot(\sigma\nabla w_0) \;=\; \sum_{i=1}^{n} F_i \qquad \text{in } \Omega\backslash\partial\mathscr{D}\,,\\ \frac{\partial}{\partial\nu}w_0 = 0 \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} w_0\,\mathrm{d}s \;=\; 0\,,\\ \big[w_0\big]_{\partial\mathscr{D}} \;=\; \varphi\,, \qquad \big[D_\nu w_0\big]_{\partial\mathscr{D}} \;=\; \psi\,, \end{gathered} \tag{5.8}$$

has a unique solution $w_0 \in H^1(\Omega\backslash\partial\mathscr{D})$, if and only if the integrability condition

$$\sum_{i=1}^{n}\int_{\Omega\backslash\partial\mathscr{D}} F_i\,\mathrm{d}x \;-\; \int_{\partial\mathscr{D}}\psi\,\mathrm{d}s \;=\; 0 \tag{5.9}$$

is satisfied: compare Costabel and Stephan [8].

In order to check (5.9) one has to be careful, because although the two integrals in (5.9) are well-defined, the two individual terms of ψ in (5.7b) fail to be integrable in a classical sense. To take this into account we choose $\delta > 0$ so small that χ_i is equal to one in $\mathcal{B}_\delta(x_i)$ for every $i = 1,\ldots,n$, in which case

$$\frac{\partial}{\partial\nu}w_i(x) \;=\; \beta_{i1}(\gamma_{i1}-1)\,\widetilde{y}_i(\theta)\,\delta^{\gamma_{i1}-2} \qquad \text{on } \partial\mathcal{B}_\delta(x_i)\,,$$

where ν denotes the exterior normal vector on $\partial\mathcal{B}_\delta(x_i)$. Let $\mathcal{B}_\delta$ be the union of these disks $\mathcal{B}_\delta(x_i)$, $i = 1,\ldots,n$. Then Green's formula, applied in $\mathscr{D}\backslash\overline{\mathcal{B}}_\delta$ and in $\Omega\backslash\overline{\mathscr{D}\cup\mathcal{B}_\delta}$, yields

$$\begin{aligned} \int_{\Omega\backslash\partial\mathscr{D}} F_i\,\mathrm{d}x \;&=\; \int_{\Omega\backslash\overline{\mathcal{B}_\delta\cup\partial\mathscr{D}}} F_i\,\mathrm{d}x \;=\; \int_{\Omega\backslash\overline{\mathscr{D}\cup\mathcal{B}_\delta}} \Delta w_i\,\mathrm{d}x \;+\; \int_{\mathscr{D}\backslash\overline{\mathcal{B}}_\delta} k\,\Delta w_i\,\mathrm{d}x\\ &=\; \int_{\partial\Omega}\frac{\partial}{\partial\nu}w_i\,\mathrm{d}s \;-\; \int_{\partial\mathscr{D}\backslash\overline{\mathcal{B}}_\delta}\big[D_\nu w_i\big]_{\partial\mathscr{D}}\,\mathrm{d}s \;-\; \int_{\partial\mathcal{B}_\delta}\sigma\frac{\partial}{\partial\nu}w_i\,\mathrm{d}s\\ &=\; -\int_{\partial\mathscr{D}\backslash\overline{\mathcal{B}}_\delta}\big[D_\nu w_i\big]_{\partial\mathscr{D}}\,\mathrm{d}s \;-\; \beta_{i1}(\gamma_{i1}-1)\,\delta^{\gamma_{i1}-1}\Big(\int_0^{\alpha_i} k\,\widetilde{y}_i(\theta)\,\mathrm{d}\theta \;+\; \int_{\alpha_i}^{2\pi}\widetilde{y}_i(\theta)\,\mathrm{d}\theta\Big) \end{aligned}$$

for every $i = 1,\ldots,n$. From (5.7b) we further have

$$\begin{aligned} \int_{\partial\mathscr{D}\backslash\overline{\mathcal{B}}_\delta}\psi\,\mathrm{d}s \;&=\; (1-k)\sum_{i=1}^{n}\int_{\Gamma_i\backslash\overline{\mathcal{B}}_\delta}\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big)\,\mathrm{d}s \;-\; \sum_{i=1}^{n}\int_{\partial\mathscr{D}\backslash\overline{\mathcal{B}}_\delta}\big[D_\nu w_i\big]_{\partial\mathscr{D}}\,\mathrm{d}s\\ &=\; (1-k)\sum_{i=1}^{n}\Big[(h\cdot\nu)\frac{\partial}{\partial\tau}u\Big]_i - \sum_{i=1}^{n}\int_{\partial\mathscr{D}\backslash\overline{\mathcal{B}}_\delta}\big[D_\nu w_i\big]_{\partial\mathscr{D}}\,\mathrm{d}s\,, \end{aligned}$$

where

$$\Big[(h\cdot\nu)\frac{\partial}{\partial\tau}u\Big]_i \;=\; -\beta_{i1}\gamma_{i1}\big(h_i^-(A_{i1}^-c_{i1}+B_{i1}^-s_{i1}) \;+\; h_i^+A_{i1}^-\big)\delta^{\gamma_{i1}-1} \;+\; o(1) \tag{5.10}$$

as $\delta\to 0$ is a short-hand notation for the difference of the values of $(h\cdot\nu)\frac{\partial}{\partial\tau}u$ at $\Gamma_i\cap\mathcal{B}_\delta(x_i)$ and $\Gamma_{i+1}\cap\mathcal{B}_\delta(x_i)$, which amounts to the right-hand side of (5.10) according to (3.9) and (5.3). Since (5.6) and the second and fourth equations of (5.4) give

$$\begin{aligned}(\gamma_{i1}-1)\Big(\int_0^{\alpha_i} k\,\widetilde{y}_i(\theta)\,\mathrm{d}\theta \;+\; \int_{\alpha_i}^{2\pi}\widetilde{y}_i(\theta)\,\mathrm{d}\theta\Big)&\\ \;=\;(1-k)\gamma_{i1}\big(h_i^+A_{i1}^- \;+\; h_i^-(A_{i1}^-c_{i1}+B_{i1}^-s_{i1})\big)\,,&\end{aligned} \tag{5.11}$$

it follows that

$$\begin{aligned}&\sum_{i=1}^n\int_{\Omega\setminus\partial\mathscr{D}} F_i\,\mathrm{d}x \;-\; \int_{\partial\mathscr{D}\setminus\overline{\mathcal{B}}_\delta}\psi\,\mathrm{d}s\\ &\;=\;(k-1)\sum_{i=1}^n\Big[(h\cdot\nu)\frac{\partial}{\partial\tau}u\Big]_i \;-\;\sum_{i=1}^n\beta_{i1}(\gamma_{i1}-1)\,\delta^{\gamma_{i1}-1}\Big(\int_0^{\alpha_i}k\,\widetilde{y}_i(\theta)\,\mathrm{d}\theta \;+\;\int_{\alpha_i}^{2\pi}\widetilde{y}_i(\theta)\,\mathrm{d}\theta\Big)\end{aligned}$$

converges to zero as $\delta\to 0$.

We thus have shown that (5.8) has a solution $w_0\in H^1(\Omega\setminus\partial\mathscr{D})$, and hence,

$$w \;=\; w_0 \;+\; \sum_{i=1}^n w_i \tag{5.12}$$

solves (5.1). Since w_i belongs to $H^\gamma(\Omega\setminus\partial\mathscr{D})$ for every

$$\gamma \;<\; \min\{\gamma_{i1}\;:\; i=1,\dots,n\}\,,$$

cf., e.g., [12, Theorem 1.2.18], this is also true for w. Finally, since the cut-off functions χ_i, $i=1,\dots,n$, are supported in $\mathcal{B}_{r_i}(x_i)$, the solution w coincides with w_0 in $\Omega\setminus\overline{\mathscr{D}\cup_i\mathcal{B}_{r_i}(x_i)}$, and hence w belongs to $H^1\big(\Omega\setminus\overline{\mathscr{D}\cup_i\mathcal{B}_{r_i}(x_i)}\big)$. □

6. The shape derivative of u solves the transmission problem. In Section 4 we have demonstrated that $u=u(\mathscr{D})$ admits a well-defined shape derivative $u_h'=\partial u(\mathscr{D})h$ in the direction of any given vector field $h\in W_c^{1,\infty}(\Omega)$. In the sequel it will be shown that u_h' coincides with the solution w of the transmission problem (5.1), if the inhomogeneous transmission data in (5.1) are defined in terms of the trace of h on $\partial\mathscr{D}$.

Theorem 6.1. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}}\subset\Omega$. Then the shape derivative of $u=u(\mathscr{D})$ in direction $h\in W_c^{1,\infty}(\Omega)$ is the solution of the transmission problem* (5.1).

Proof. Let w denote the solution of the transmission problem (5.1), where the vector field which enters the inhomogeneous transmission data in (5.1) is the trace of $h\in W_c^{1,\infty}(\Omega)$ on $\partial\mathscr{D}$. In the first and major step of this proof we show that the boundary values of w on $\partial\Omega$ coincide with $u_h'|_{\partial\Omega}=\partial\Lambda_f(\mathscr{D})h|_{\partial\mathscr{D}}$, compare Theorem 4.4.

Since w belongs to $H^1\big(\Omega\setminus\overline{\mathscr{D}\cup_i\mathcal{B}_{r_i}(x_i)}\big)$ by virtue of Theorem 5.1, its trace on $\partial\Omega$ is well-defined in $H^{1/2}(\partial\Omega)\subset L^2(\partial\Omega)$, and it has vanishing mean according to (5.1). In view of (2.8) we need to show that

$$\langle\, w,g\,\rangle_{L^2(\partial\Omega)} \;=\; (1-k)\int_{\partial\mathscr{D}}(h\cdot\nu)\,\nabla u^-\cdot(M\nabla v_g^-)\,\mathrm{d}s \tag{6.1}$$

for every $g \in L^2_\diamond(\partial\Omega)$, where v_g is the solution of (2.9) and $M \in \mathbb{R}^{2\times 2}$ is the symmetric 2×2-matrix with eigenvalues 1 and k and eigenvectors τ and ν, respectively.

As in the proof of Theorem 5.1 let $\delta > 0$ be so small that $\chi_i(x) = 1$ for every $x \in \mathcal{B}_\delta(x_i)$ and every $i = 1, \ldots, n$, set $\mathcal{B}_\delta = \cup_i \mathcal{B}_\delta(x_i)$, and denote by ν the exterior normal vector on $\partial\mathcal{B}_\delta$. Further, let $w^- = w|_{\mathscr{D}}$ and $w^+ = w|_{\Omega\setminus\overline{\mathscr{D}}}$. Since the Neumann derivative of w vanishes on $\partial\Omega$ by virtue of (5.1), Green's formula in $\Omega\setminus\overline{\mathscr{D}\cup\mathcal{B}_\delta}$ and in $\mathscr{D}\setminus\overline{\mathcal{B}}_\delta$ yields

$$\begin{aligned}\langle w, g\rangle_{L^2(\partial\Omega)} &= \int_{\partial\Omega} w\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s - \int_{\partial\Omega} v_g\,\frac{\partial}{\partial\nu}w\,\mathrm{d}s\\ &= -\int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta} v_g\big[D_\nu w\big]_{\partial\mathscr{D}}\,\mathrm{d}s + \int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta} w^+\,\frac{\partial}{\partial\nu}v_g^+\,\mathrm{d}s - \int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta} kw^-\,\frac{\partial}{\partial\nu}v_g^-\,\mathrm{d}s\\ &\quad - \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w\,\mathrm{d}s + \int_{\partial\mathcal{B}_\delta}\sigma w\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s\,.\end{aligned}$$

Using

$$w^-|_{\partial\mathscr{D}} = w^+|_{\partial\mathscr{D}} - \big[w\big]_{\partial\mathscr{D}}$$

and the transmission conditions (5.1) as well as the fact that $\big[D_\nu v_g\big]_{\partial\mathscr{D}} = 0$ we can transform this equation further into

$$\begin{aligned}\langle w, g\rangle_{L^2(\partial\Omega)} &= (k-1)\int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta} v_g\,\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big)\,\mathrm{d}s\\ &\quad + k(1-k)\int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta}(h\cdot\nu)\frac{\partial}{\partial\nu}u^-\,\frac{\partial}{\partial\nu}v_g^-\,\mathrm{d}s - \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w\,\mathrm{d}s + \int_{\partial\mathcal{B}_\delta}\sigma w\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s\,.\end{aligned}$$

The first term on the right-hand side consists of n integrals over the individual edges of $\mathscr{D}$, and partial integration on each of these edges gives

$$\begin{aligned}\langle w, g\rangle_{L^2(\partial\Omega)} &= (k-1)\sum_{i=1}^n\Big[(h\cdot\nu)v_g\frac{\partial}{\partial\tau}u\Big]_i + (1-k)\int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta}(h\cdot\nu)\frac{\partial}{\partial\tau}u\,\frac{\partial}{\partial\tau}v_g\,\mathrm{d}s\\ &\quad + k(1-k)\int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta}(h\cdot\nu)\frac{\partial}{\partial\nu}u^-\,\frac{\partial}{\partial\nu}v_g^-\,\mathrm{d}s - \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w\,\mathrm{d}s + \int_{\partial\mathcal{B}_\delta}\sigma w\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s\,,\end{aligned}$$

where we utilize the notation introduced in (5.10). Note that this identity can be rewritten as

$$\begin{aligned}\langle w, g\rangle_{L^2(\partial\Omega)} &= (1-k)\int_{\partial\mathscr{D}\setminus\mathcal{B}_\delta}(h\cdot\nu)\,\nabla u^-\cdot(M\nabla v_g^-)\,\mathrm{d}s\\ &\quad + (k-1)\sum_{i=1}^n\Big[(h\cdot\nu)v_g\frac{\partial}{\partial\tau}u\Big]_i - \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w\,\mathrm{d}s + \int_{\partial\mathcal{B}_\delta}\sigma w\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s\end{aligned}\tag{6.2}$$

with M as in (2.8), and the first integral converges to the right-hand side of (6.1) as $\delta \to 0$ by virtue of (3.10). It therefore remains to show that the sum of terms in the second line of (6.2) converges to zero as $\delta \to 0$.

We now investigate these three summands individually. For the first one we utilize (3.2), (3.9), and (5.2) to obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[(h\cdot\nu)v_g\frac{\partial}{\partial\tau}u\Big]_i \\
&\quad= \sum_{i=1}^{n}\big(h_i^- + O(\delta)\big)\big(v_g(x_i)+O(\delta^{\gamma_{i1}})\big)\big(-\beta_{i1}\gamma_{i1}(A_{i1}^- c_{i1}+B_{i1}^- s_{i1})\delta^{\gamma_{i1}-1}+O(\delta^{\gamma_{i2}-1})\big) \\
&\qquad - \sum_{i=1}^{n}\big(h_i^+ + O(\delta)\big)\big(v_g(x_i)+O(\delta^{\gamma_{i1}})\big)\big(\beta_{i1}\gamma_{i1}A_{i1}^-\delta^{\gamma_{i1}-1}+O(\delta^{\gamma_{i2}-1})\big) \\
&\quad= -\sum_{i=1}^{n}\beta_{i1}\gamma_{i1}v_g(x_i)\big(h_i^-(A_{i1}^- c_{i1}+B_{i1}^- s_{i1}) \;+\; h_i^+A_{i1}^-\big)\delta^{\gamma_{i1}-1} \;+\; o(1)
\end{aligned} \tag{6.3a}$$

as $\delta\to 0$, where the estimate of the remainder follows from (3.5).

Concerning the two integrals over the boundary of $\mathcal{B}_\delta$ in (6.2) we decompose w as in (5.12) and investigate the corresponding parts separately. Consider w_0 first, i.e., the solution of the transmission problem (5.8), and let $w_0^- = w_0|_{\mathscr{D}}$ and $w_0^+ = w_0|_{\Omega\setminus\overline{\mathscr{D}}}$. Since the source terms F_i of (5.8) vanish in $\mathcal{B}_\delta$, compare (5.7c), w_0 and v_g are both harmonic in $\mathcal{B}_\delta\backslash\partial\mathscr{D}$. Since both functions also belong to $H^1(\mathcal{B}_\delta\backslash\partial\mathscr{D})$ we can apply Green's formula in $\mathcal{B}_\delta\cap\mathscr{D}$ and in $\mathcal{B}_\delta\backslash\overline{\mathscr{D}}$ to obtain

$$\begin{aligned}
&\int_{\partial\mathcal{B}_\delta}\sigma w_0\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s \;-\; \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w_0\,\mathrm{d}s \\
&\qquad= -\int_{\partial\mathscr{D}\cap\mathcal{B}_\delta} v_g\big[D_\nu w_0\big]_{\partial\mathscr{D}}\,\mathrm{d}s \;-\; \int_{\partial\mathscr{D}\cap\mathcal{B}_\delta} k\,w_0^-\,\frac{\partial}{\partial\nu}v_g^-\,\mathrm{d}s \;+\; \int_{\partial\mathscr{D}\cap\mathcal{B}_\delta} w_0^+\,\frac{\partial}{\partial\nu}v_g^+\,\mathrm{d}s\,.
\end{aligned}$$

Using (5.8) and the fact that

$$w_0^+|_{\partial\mathscr{D}} \;=\; w_0^-|_{\partial\mathscr{D}} \;+\; \big[w_0\big]_{\partial\mathscr{D}}\,,$$

we thus arrive at

$$\begin{aligned}
&\int_{\partial\mathcal{B}_\delta}\sigma w_0\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s \;-\; \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w_0\,\mathrm{d}s \\
&\quad= -\int_{\partial\mathscr{D}\cap\mathcal{B}_\delta} v_g\big[D_\nu w_0\big]_{\partial\mathscr{D}}\,\mathrm{d}s \;+\; \int_{\partial\mathscr{D}\cap\mathcal{B}_\delta} w_0^-\big[D_\nu v_g\big]_{\partial\mathscr{D}}\,\mathrm{d}s \;+\; \int_{\partial\mathscr{D}\cap\mathcal{B}_\delta}\big[w_0\big]_{\partial\mathscr{D}}\frac{\partial}{\partial\nu}v_g^+\,\mathrm{d}s \\
&\quad= -\int_{\partial\mathscr{D}\cap\mathcal{B}_\delta} v_g\psi\,\mathrm{d}s \;+\; \int_{\partial\mathscr{D}\cap\mathcal{B}_\delta}\varphi\,\frac{\partial}{\partial\nu}v_g^+\,\mathrm{d}s\,.
\end{aligned}$$

The two products $v_g\psi$ and $\varphi\frac{\partial}{\partial\nu}v_g^+$ are integrable over $\partial\mathscr{D}$ because $\psi\in H^{-1/2}(\partial\mathscr{D})$ and $\varphi\in H^{1/2}(\partial\mathscr{D})$, cf. (5.7), and from this we conclude that

$$\int_{\partial\mathcal{B}_\delta}\sigma w_0\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s \;-\; \int_{\partial\mathcal{B}_\delta}\sigma v_g\,\frac{\partial}{\partial\nu}w_0\,\mathrm{d}s \;=\; o(1)\,, \qquad \delta\to 0\,. \tag{6.3b}$$

Next, consider

$$\int_{\partial\mathcal{B}_\delta}\sigma w_i\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s \;=\; \int_{\partial\mathcal{B}_\delta(x_i)}\sigma w_i\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s$$

for any fixed $i \in \{1, \ldots, n\}$. It follows from (5.5) and (3.2) that

$$|w_i(x)| \;=\; O(\delta^{\gamma_{i1}-1})$$

and

$$\left|\frac{\partial}{\partial\nu} v_g(x)\right| \;=\; \left|\frac{\partial}{\partial r} v_g(x)\right| \;=\; O(\delta^{\gamma_{i1}-1})$$

on $\partial\mathcal{B}_\delta(x_i)$. Inserting these estimates into the integral in question shows that

$$\int_{\partial\mathcal{B}_\delta} \sigma w_i \,\frac{\partial}{\partial\nu} v_g \,\mathrm{d}s \;=\; O(\delta^{2\gamma_{i1}-1}) \;=\; o(1)\,, \qquad \delta \to 0\,, \tag{6.3c}$$

by virtue of (3.5).

Finally, since δ is so small that $\chi_i = 1$ in $\mathcal{B}_\delta(x_i)$ the Neumann boundary derivative of w_i satisfies

$$\frac{\partial}{\partial\nu} w_i \;=\; \begin{cases} \beta_{i1}(\gamma_{i1}-1)\widetilde{y}_i(\theta)\,\delta^{\gamma_{i1}-2}\,, & \text{on } \partial\mathcal{B}_\delta(x_i)\,, \\ 0\,, & \text{on } \partial\mathcal{B}_\delta(x_j),\; j \neq i\,, \end{cases}$$

while

$$v_g \;=\; v_g(x_i) \;+\; O(\delta^{\gamma_{i1}}) \qquad \text{on } \partial\mathcal{B}_\delta(x_i)\,,$$

and hence,

$$\begin{aligned} \int_{\partial\mathcal{B}_\delta} \sigma v_g \,\frac{\partial}{\partial\nu} w_i \,\mathrm{d}s \;&=\; \beta_{i1}(\gamma_{i1}-1) v_g(x_i) \left(\int_0^{\alpha_i} k\,\widetilde{y}_i(\theta)\,\mathrm{d}\theta + \int_{\alpha_i}^{2\pi} \widetilde{y}_i(\theta)\,\mathrm{d}\theta \right) \delta^{\gamma_{i1}-1} \\ &\quad + \; O(\delta^{2\gamma_{i1}-1})\,. \end{aligned} \tag{6.3d}$$

Assembling the pieces of (6.3) and inserting them into (6.2) we thus obtain

$$\begin{aligned} \langle\, w, g \,\rangle_{L^2(\partial\Omega)} \;&=\; (1-k) \int_{\partial\mathscr{D}} (h\cdot\nu)\,\nabla u^- \cdot (M \nabla v_g^-)\,\mathrm{d}s \\ &\quad - \; (k-1) \sum_{i=1}^n \beta_{i1}\gamma_{i1} v_g(x_i) \big(h_i^- (A_{i1}^- c_{i1} + B_{i1}^- s_{i1}) \;+\; h_i^+ A_{i1}^- \big) \delta^{\gamma_{i1}-1} \\ &\quad - \sum_{i=1}^n \beta_{i1}(\gamma_{i1}-1) v_g(x_i) \left(\int_0^{\alpha_i} k\,\widetilde{y}_i(\theta)\,\mathrm{d}\theta \;+\; \int_{\alpha_i}^{2\pi} \widetilde{y}_i(\theta)\,\mathrm{d}\theta \right) \delta^{\gamma_{i1}-1} \;+\; o(1)\,. \end{aligned}$$

The two integrals over $\widetilde{y}_i$ in the bottom line have already been evaluated in the course of the proof of Theorem 5.1, compare (5.11). Their value is such that all terms of order $\delta^{\gamma_{i1}-1}$ cancel, proving that

$$\langle\, w, g \,\rangle_{L^2(\partial\Omega)} \;=\; (1-k) \int_{\partial\mathscr{D}} (h\cdot\nu)\,\nabla u^- \cdot (M \nabla v_g^-)\,\mathrm{d}s \;+\; o(1)\,, \qquad \delta \to 0\,.$$

Since the left-hand side of this equation is independent of δ, the desired identity (6.1) must hold true.

As shown in Theorem 4.4 the shape derivative u_h' of u is harmonic in $\mathscr{D}$ and in $\Omega\backslash\overline{\mathscr{D}}$ and has homogeneous Neumann boundary values. Accordingly, u_h' and w share the same Cauchy data on $\partial\Omega$. Since they are both harmonic in $\Omega\backslash\overline{\mathscr{D}}$ they coincide up to the boundary of $\mathscr{D}$ by virtue of Holmgren's theorem. In particular, their restrictions to $\Omega\backslash\overline{\mathscr{D}}$ have the same trace on $\partial\mathscr{D}$. Since the material derivative $\dot{u}_h$ belongs to $H^1(\Omega)$ and the traces of $u^\pm$ (and their tangential derivatives) coincide on $\partial\mathscr{D}$ it follows from (4.17) and (2.3) that

$$\big[u_h'\big]_{\partial\mathscr{D}} \;=\; -h\cdot\big[\nabla u\big]_{\partial\mathscr{D}} \;=\; -(h\cdot\nu)\Big(\frac{\partial}{\partial\nu}u^+ \;-\; \frac{\partial}{\partial\nu}u^-\Big) \;=\; (1-k)(h\cdot\nu)\frac{\partial}{\partial\nu}u^-\,.$$

According to (5.1) this shows that w and u_h' experience the same jump across $\partial\mathscr{D}$, and therefore the traces on $\partial\mathscr{D}$ of $w|_{\mathscr{D}}$ and $u_h'|_{\mathscr{D}}$ also coincide. Both being harmonic functions we thus conclude that $w = u_h'$ in $\mathscr{D}$, and hence, in all of Ω. ◻

Note that it follows from Theorem 6.1 that the shape derivative $\partial u(\mathscr{D})h$ of u in direction $h \in W_c^{1,\infty}(\Omega)$ only depends on $h|_{\partial\mathscr{D}}$.

7. The case of an insulating or a perfectly conducting polygonal inclusion. So far we have assumed that the conductivity of the inclusion is a known positive value $k > 0$. Now we turn to the limiting cases which are formally described by $k = 0$ (the insulating case) and $k = +\infty$ (the perfectly conducting case).

7.1. The insulating case. The appropriate formulation of the conductivity equation (2.2) when $\mathscr{D}$ is an insulating inclusion, is in terms of the boundary value problem

$$\begin{aligned} \Delta u \;&=\; 0 \quad \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \qquad & \int_{\partial\Omega} u\,\mathrm{d}s \;&=\; 0\,,\\ \frac{\partial}{\partial\nu}u \;&=\; 0 \quad \text{on } \partial\mathscr{D}\,, \qquad & \frac{\partial}{\partial\nu}u \;&=\; f \quad \text{on } \partial\Omega\,. \end{aligned} \tag{7.1}$$

When $\mathscr{D}$ is smooth the existence of an associated shape derivative of $u = u(\mathscr{D})$ is implicit in the result of [29, Section 3.2]: this shape derivative in direction $h \in W_c^{1,\infty}(\Omega)$ is given by the solution $u_h' \in H^1(\Omega\backslash\overline{\mathscr{D}})$ of the Neumann boundary value problem

$$\begin{aligned} \Delta u_h' \;&=\; 0 \quad \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \qquad & \int_{\partial\Omega} u_h'\,\mathrm{d}s \;&=\; 0\,,\\ \frac{\partial}{\partial\nu}u_h' \;&=\; \frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big) \quad \text{on } \partial\mathscr{D}\,, \qquad & \frac{\partial}{\partial\nu}u_h' \;&=\; 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{7.2}$$

To the best of our knowledge the existence of a corresponding shape derivative for a polygonal inclusion $\mathscr{D}$ has not yet been investigated. However, the analysis in [6] can be adapted in a straightforward manner to derive the respective material derivative $\dot{u}_h \in H^1(\Omega\backslash\overline{\mathscr{D}})$ in the direction of h for the insulating case and the corresponding variational definition of $\partial\Lambda_f(\mathscr{D})h|_{\partial\mathscr{D}} = \dot{u}_h|_{\partial\Omega}$, i.e.,

$$\langle\, \partial\Lambda_f(\mathscr{D})h|_{\partial\mathscr{D}}, g\,\rangle_{L^2(\partial\Omega)} \;=\; \int_{\partial\mathscr{D}} (h\cdot\nu)\,\frac{\partial}{\partial\tau}u\,\frac{\partial}{\partial\tau}v_g\,\mathrm{d}s$$

for every $g \in L^2_\diamond(\partial\Omega)$, where v_g is the corresponding solution of (7.1) with f replaced by g. This is the natural analog of (2.8) for $k = 0$.

For the proof of the corresponding version of Theorem 5.1 analogous properties of the solution of the forward problem (7.1), as recollected in Section 3 for the case

$k > 0$, are needed. The corresponding results can be found, e.g., in [12]: In the case $k = 0$ the eigenvalues γ_{ij}^2 and eigenfunctions y_{ij} which enter into the expansion of u near any of the vertices, are known explicitly, cf. [12, p. 50], namely

$$\gamma_{ij} \;=\; j\pi/(2\pi - \alpha_i)\,, \qquad j \in \mathbb{N}_0\,, \tag{7.3}$$

and, for $j \in \mathbb{N}$,

$$y_{ij}(\theta) \;=\; \Bigl(\frac{2}{2\pi - \alpha_i}\Bigr)^{1/2} \cos\gamma_{ij}(\theta - \alpha_i)\,, \qquad \alpha_i < \theta < 2\pi\,.$$

Take note that the nonnegative roots γ_{ij} of these eigenvalues still satisfy (3.5) when $0 < \alpha_i < \pi$, whereas they are all greater than one, if $\pi < \alpha_i < 2\pi$. Using these properties of the solution of the forward problem the proof of Theorem 5.1 carries over to the insulating case with straightforward modifications. Since at least one of the interior angles of $\mathscr{D}$ is smaller than π, the corresponding singular function w_i only belongs to $H^\gamma(\Omega\backslash\overline{\mathscr{D}})$ for some $\gamma \in (1/2, 1)$ according to [12, Theorem 1.2.18], and hence the overall smoothness of the solution u_h' of (7.2) is of similar type than in the case of a polygonal inclusion with positive conductivity.

If the inhomogeneous Neumann boundary data on $\partial\mathscr{D}$ in (7.2) are defined by some vector field $h \in W_c^{1,\infty}(\Omega)$, a little extra care is necessary to show that the corresponding solution of (7.2) is the shape derivative of u in direction h, because the function $(u_{th} - u)/t$ is not defined in all of Ω; in fact, the two functions u_{th} and u live on different domains. To overcome this problem one can consider an arbitrary domain $\Omega' \subset \Omega\backslash\overline{\mathscr{D}}$ which satisfies $\partial\Omega' \cap \partial\mathscr{D} = \varnothing$, in which case

$$\frac{u_{th} - u}{t} \,:\, \Omega' \to \mathbb{R}$$

is well-defined for t sufficiently close to zero. The analog of Theorems 4.4, 5.1, and 6.1 then reads as follows.

THEOREM 7.1. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}} \subset \Omega$ and $h \in W_c^{1,\infty}(\Omega)$. Then the boundary value problem (7.2) has a unique solution $u' \in H^\gamma(\Omega\backslash\overline{\mathscr{D}})$ for some $\gamma \in (1/2, 1)$. This solution is the shape derivative of $u = u(\mathscr{D})$ in the direction of h, i.e.,*

$$\frac{u_{th} - u}{t} \;\to\; u_h' \;=\; \dot{u}_h \;-\; h \cdot \nabla u\,, \qquad t \to 0\,,$$

in $L^2(\Omega')$ for every domain $\Omega' \subset \Omega$ with $\partial\Omega' \cap \partial\mathscr{D} = \varnothing$. This shape derivative also belongs to $H^1\bigl(\Omega\backslash\overline{\mathscr{D} \cup_i \mathcal{B}_{r_i}(x_i)}\bigr)$, and its trace on $\partial\Omega$ is the shape derivative of $\Lambda_f(\mathscr{D})$ in direction $h|_{\partial\mathscr{D}}$.

The other results of Section 4 apply verbatim to the insulating case; the corresponding modifications of the proofs are straightforward.

7.2. The perfectly conducting case. When k becomes infinitely large, on the other hand, the transmission condition (2.3) forces u^- to approximate a constant but unknown value; this is in agreement with our intuition that in a perfect conductor potential differences equilibrate immediately. Up to an additive constant the corresponding potential $u \in H^1(\Omega\backslash\overline{\mathscr{D}})$ is therefore given by the solution of the boundary value problem

$$\Delta u \;=\; 0 \quad \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \qquad u \;=\; 0 \quad \text{on } \partial\mathscr{D}\,, \qquad \frac{\partial}{\partial\nu}u \;=\; f \quad \text{on } \partial\Omega\,. \tag{7.4}$$

Note that u can be extended by zero to a function in $H^1(\Omega)$. Ito, Kunisch, and Li [18] pointed out that the solution of the boundary value problem

$$\begin{aligned} \Delta u_h' &= 0 \quad \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \\ u_h' = -(h\cdot\nu)\frac{\partial}{\partial\nu}u \quad \text{on } \partial\mathscr{D}\,, &\qquad \frac{\partial}{\partial\nu}u_h' = 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{7.5}$$

is the shape derivative of $u = u(\mathscr{D})$ of (7.4) in the direction of $h \in W_c^{1,\infty}(\Omega)$, provided that $\mathscr{D}$ has a smooth boundary.

When $\mathscr{D}$ is a polygon the solution of the forward problem (7.4) admits a similar expansion near the vertices of $\mathscr{D}$ as in the insulating case; again, see [12] for details. The eigenvalues γ_{ij}^2 which are relevant for this expansion are the same as in the insulating case, cf. (7.3), this time the eigenfunctions being the corresponding sine functions

$$y_{ij}(\theta) \;=\; \Bigl(\frac{2}{2\pi-\alpha_i}\Bigr)^{1/2}\sin\gamma_{ij}(\theta-\alpha_i)\,, \qquad \alpha_i < \theta < 2\pi\,,$$

where the index j runs through the natural numbers only; all eigenvalues are strictly positive.

The analysis in [6] also extends to the perfectly conducting case, showing that the shape derivative $\partial\Lambda_f(\mathscr{D})$ of a polygonal perfect conductor $\mathscr{D}$ exists, and that it satisfies

$$\langle\, \partial\Lambda_f(\mathscr{D})h|_{\partial\mathscr{D}}, g\,\rangle_{L^2(\partial\Omega)} \;=\; -\int_{\partial\mathscr{D}} (h\cdot\nu)\,\frac{\partial}{\partial\nu}u\,\frac{\partial}{\partial\nu}v_g\,\mathrm{d}s$$

for every $g \in L^2_\diamond(\partial\Omega)$, where v_g solves the boundary value problem (7.4) with f replaced by g. Note that the projection of $\partial\Lambda_f(\mathscr{D})h$ onto $L^2_\diamond(\partial\Omega)$ is independent of the chosen grounding in the definition (7.4) of u and v_g, respectively.

Further, the analysis from Sections 4 to 6 can be modified in a straightforward way to achieve the following result.

THEOREM 7.2. *Let $\mathscr{D}$ be a polygon with simply connected closure $\overline{\mathscr{D}} \subset \Omega$ and $h \in W_c^{1,\infty}(\Omega)$. Then the shape derivative of the solution $u = u(\mathscr{D})$ of (7.4) in direction h is given by the unique solution u_h' of the boundary value problem (7.5). This solution belongs to $H^\gamma(\Omega\backslash\overline{\mathscr{D}})$ for some $\gamma \in (1/2, 1)$ and to $H^1\bigl(\Omega\backslash\overline{\mathscr{D}\cup_i \mathcal{B}_{r_i}(x_i)}\bigr)$. Its trace on $\partial\Omega$ is the shape derivative of $\Lambda_f(\mathscr{D})$ in direction $h|_{\partial\mathscr{D}}$.*

Like in the insulating case the other results of Section 4 are valid in the perfectly conducting case, too.

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