LIPSCHITZ STABILITY OF A NONLINEAR INVERSE CONDUCTIVITY PROBLEM WITH TWO CAUCHY DATA PAIRS

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Abstract. In 1996 Seo proved that two appropriate pairs of current and voltage data measured on the surface of a planar homogeneous object are sufficient to determine a conductive polygonal inclusion with known deviating conductivity. Here we show that the corresponding linearized forward map is injective, and from this we deduce Lipschitz stability of the solution of the original nonlinear inverse problem. We also treat the case of an insulating polygonal inclusion, in which case a single pair of Cauchy data is already sufficient for the same purpose.

Key words. polygonal inclusion, conductivity equation, shape derivative

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1. Introduction. We consider the boundary value problem

$$\nabla \cdot (\sigma \nabla u) \,=\, 0 \quad \text{in } \Omega\,, \qquad \frac{\partial}{\partial \nu} u \,=\, f \quad \text{on } \partial\Omega\,, \tag{1.1}$$

for the electric potential u in a planar object Ω, when a (quasi-)static boundary current f with vanishing mean is applied on its boundary. To be specific we focus on the situation that the object contains a so-called inclusion $\mathscr{D}$ of some other conducting material, such that the spatial conductivity distribution is given by

$$\sigma \,=\, \begin{cases} k & \text{in } \mathscr{D}\,, \\ 1 & \text{in } \Omega \backslash \overline{\mathscr{D}}\,, \end{cases} \tag{1.2}$$

with a nonnegative value $k \neq 1$.

The inverse conductivity problem that we are interested in seeks to recover the inclusion (i.e., location and shape of $\mathscr{D}$) from measurements of the potential on the boundary of the object. The question whether this inverse problem is uniquely solvable has a long-standing history. In the formal (degenerate) case $k = 0$, i.e., when the inclusion is taken to be insulating (cf. the discussion in Sect. 5) and its complement is a connected set, then it is known that the Cauchy data of u on $\partial\Omega$ do indeed uniquely determine the inclusion; for a proof of this result, cf., e.g., Beretta and Vessella [13].

Less is known, however, when the conductivity in the inclusion is a nonzero constant different from one. For this case Friedman and Isakov [15] proved that a convex polygonal inclusion is uniquely determined by this data, provided its conductivity is known and the diameter of $\mathscr{D}$ is smaller than its distance to the boundary of the object. Barceló, Fabes, and Seo [7], and Alessandrini and Isakov [3] were able to drop the size constraint on the polygonal inclusion for the prize of accepting only certain admissible boundary currents f for probing the object. Finally, Seo [19] proved that the convexity assumption on the polygon can also be omitted when the boundary potentials for two appropriate probing currents f_1, f_2 are given; see Assumption (3.1) in Section 3.

It is known that inverse conductivity problems in general are badly ill-posed in the sense that the solution lacks continuous dependence on the given data; some

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conditional stability – generically of logarithmic type – can be restored by providing further a priori information on the conductivity, cf., e.g., [8, 13].

The past twenty years have seen increasing activities in deriving Lipschitz stability estimates for inverse conductivity problems under very restrictive conditions on the set of admissible conductivities. One of the first results in this direction was obtained by Alessandrini and Vessella [4] who showed that if *all* Cauchy pairs for u in (1.1) are known, i.e., if the full (or the local) Neumann-Dirichlet operator associated with the differential equation in (1.1) is given, then the *values* of a piecewise constant conductivity σ with respect to a known partitioning of Ω into finitely many subdomains depend Lipschitz continuously on these data. Later it was proved by Harrach [17] that already a finite number of Cauchy pairs is sufficient for this property; see also Alberti and Santacesaria [1].

Whereas these results assume the spatial structure of σ to be known and the quantitative details are being searched for, the conductivity Ansatz (1.2) with known k but unknown form and location of the inclusion, has recently been treated by Beretta and Francini [12]. They established Lipschitz stability in terms of the Hausdorff distance between the boundaries of two admissible inclusions, if it is a priori known that they have the shape of a polygon. In contrast to the aforementioned uniqueness results by Beretta/Vessella and by Seo, however, the stability result in [12] again requires the full Neumann-Dirichlet map as data. See also [5, 11] for extensions of this finding to layered – instead of homogeneous – background media and to polyhedral inclusions in three space dimensions, respectively.

Here we extend the result by Beretta and Francini in a different direction: We assume that only two pairs of Cauchy data are given which fulfill Seo's uniqueness assumptions for a conductive inclusion; we also treat the degenerate case of an insulating polygonal inclusion, in which case we only assume a single (nontrivial) Cauchy pair to be available. We show that these minimal datasets suffice to have Lipschitz stability. We mention in passing that for a somewhat related setting, namely the reconstruction of a linear crack within a homogeneous planar object, Lipschitz stability with only two pairs of Cauchy data had been established by Alessandrini, Beretta, and Vessella [2] back in 1996.

In contrast to the analysis in the above works which, in principle, allow for an evaluation of the corresponding Lipschitz constant, our method is non-quantitative. Rather, we use a general methodology worked out by Bourgeois [14], building on earlier work by Bacchelli and Vessella [6]. As is transparent from their results, the key ingredients for Lipschitz stability in general nonlinear inverse problems are

1. a specification of the quantity of interest in terms of finitely many parameters,
2. the restriction of these parameters to a compact set,
3. injectivity of the forward operator on this compact set,
4. continuous differentiability of the forward operator, and
5. injectivity of the Jacobian of the forward operator.

Since we can build on the uniqueness results by Seo and Beretta/Vessella, respectively, and since differentiability results are also available, it remains for us to investigate the injectivity of the associated linearized problems.

Accordingly, the outline of this paper is as follows. We start in Section 2 by reviewing known properties of the solution u of (1.1), when the inclusion $\mathscr{D}$ is a conductive polygon, including the corresponding differentiability result with respect to its shape. In Section 3 we specify the associated inverse problem as it has been introduced by Seo, and we prove that the respective linearized forward problem is injective. Then, in Section 4, we adapt the method from [6, 14] to our needs and establish the corresponding Lipschitz stability result (Theorem 4.2). We conclude the paper by treating the case of an insulating polygonal inclusion in Section 5.

2. The forward problem for a conductive polygonal inclusion. We assume throughout that Ω is a two-dimensional bounded domain with smooth boundary, and that the inclusion $\mathscr{D}$ is a polygonal domain with simply connected closure $\overline{\mathscr{D}} \subset \Omega$. We denote by ν the outer normal vector on the boundaries of $\mathscr{D}$ and of Ω, respectively. Concerning the spatial conductivity distribution (1.2) we make the assumption that the constant conductivity $k \in \mathbb{R}^+\backslash\{1\}$ in $\mathscr{D}$ is known and fixed; see Section 5 for the case when $k = 0$. When the probing boundary current f satisfies

$$f \in L^2_\diamond(\partial\Omega) \;=\; \left\{ f \in L^2(\partial\Omega) \,:\, \int_{\partial\Omega} f \,\mathrm{d}s = 0 \right\},$$

then the corresponding electric potential u is the unique weak solution

$$u \in H^1_\diamond(\Omega) \;=\; \left\{ u \in H^1(\Omega) \,:\, \int_{\partial\Omega} u \,\mathrm{d}s = 0 \right\}$$

of (1.1).

Let x_i and Γ_i, $i = 1, \dots, n \geqslant 3$, denote the vertices and (relatively open) edges of $\mathscr{D}$, respectively, where we assume that x_i connects Γ_i and Γ_{i+1}. Here and throughout we identify Γ_{n+1} with Γ_1, and also x_{i+n} with x_i for $i = 1, \dots, n$, respectively. On Γ_i we let the unit tangent vector τ point in the direction of x_i. We stipulate the general assumption that the induced orientation of $\partial\mathscr{D}$ is counterclockwise, and that the interior angles $\alpha_i \in (0, 2\pi)$, $i = 1, \dots, n$, are all different from π. A polygon which satisfies all the above requirements will subsequently be called *admissible*.

The two components $u_- = u|_{\mathscr{D}}$ and $u_+ = u|_{\Omega\backslash\overline{\mathscr{D}}}$ of u are harmonic functions. Moreover, they satisfy the transmission conditions

$$u_- \;=\; u_+ \qquad \text{and} \qquad k\,\frac{\partial}{\partial\nu}u_- \;=\; \frac{\partial}{\partial\nu}u_+$$

on every edge Γ_i. Therefore both components can be extended by reflection across each of the edges of $\mathscr{D}$, and hence, they both are infinitely smooth and all their derivatives extend continuously onto the edges. Concerning the behavior of u at the vertices we introduce a local coordinate system for

$$x \in \mathcal{B}_{r_0}(x_i) \;=\; \left\{ x \in \mathbb{R}^2 \,:\, |x - x_i| < r_0 \right\},$$

with r_0 sufficiently small, namely

$$x \;=\; x_i \;+\; \big(r\cos(\theta_i + \theta), r\sin(\theta_i + \theta)\big)\,, \qquad 0 < r < r_0\,,\; 0 \leqslant \theta < 2\pi\,, \tag{2.1}$$

where θ_i is such that the values $\theta = 0$, $\theta \in (0, \alpha_i)$, $\theta = \alpha_i$, and $\theta \in (\alpha_i, 2\pi)$ correspond to points on Γ_{i+1}, in $\mathscr{D}$, on Γ_i, and in $\Omega\backslash\overline{\mathscr{D}}$, respectively. It has been shown in [9] that in this coordinate system the potential u has an asymptotic expansion

$$u(x) \;=\; u(x_i) \;+\; \sum_{j=1}^{\infty} \beta_{ij} y_{ij}(\theta) r^{\gamma_{ij}}\,, \tag{2.2}$$

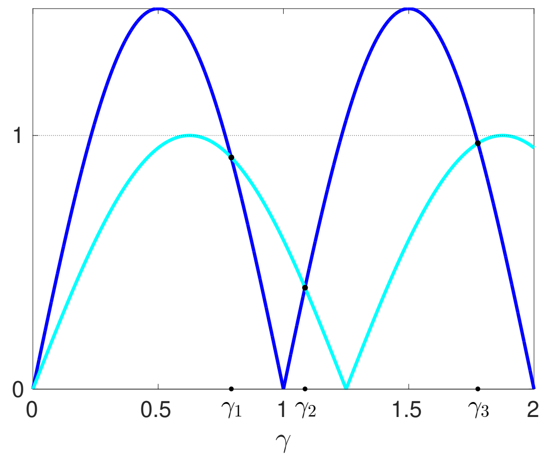

FIG. 2.1. *The graphs of the two functions of γ in* (2.3)*: The lighter curve corresponds to the function on the left-hand side, the darker one is the graph of the function on the right-hand side. The marked abscissae of the three intersection points of the two graphs are the solutions of* (2.3) *in the interval* $(0, 2]$.

where y_{ij} are continuous functions of the polar angle, given by (in general different) nontrivial linear combinations of $\cos\gamma_{ij}\theta$ and $\sin\gamma_{ij}\theta$ in $(0, \alpha_i)$ and in $(\alpha_i, 2\pi)$, respectively, and the exponents γ_{ij}, $j = 1, 2, \ldots$, are all the positive solutions γ of

$$\bigl|\sin\gamma(\alpha_i - \pi)\bigr| \;=\; \lambda\bigl|\sin\gamma\pi\bigr|\,, \qquad \lambda \;=\; \Bigl|\frac{k+1}{k-1}\Bigr|\,, \tag{2.3}$$

which we assume to be in increasing order. Both, y_{ij} and γ_{ij} are independent of the probing current; only the expansion coefficients $\beta_{ij} = \beta_{ij}[f] \in \mathbb{R}$ depend linearly on f. See [9], [19], or [16] for further details. We refer to Figure 2.1 for a graphical illustration of equation (2.3). Since the amplitude λ of the sine wave on the right-hand side of (2.3) – the darker graph in Figure 2.1 – is always greater than one, and since $0 < |\pi - \alpha_i| < \pi$, it is not difficult to see that

$$\frac{1}{2} \;<\; \gamma_{i1} \;<\; 1\,, \qquad 1 \;<\; \gamma_{i2} \;<\; \frac{3}{2}\,, \qquad \text{and} \qquad \gamma_{ij} \;>\; \frac{3}{2} \quad \text{for } j \geqslant 3\,. \tag{2.4}$$

We denote by

$$\Lambda_f \;:\; \mathscr{D} \;\mapsto\; u|_{\partial\Omega}\,, \tag{2.5}$$

the map, which takes an admissible polygon $\mathscr{D}$ onto the trace of the solution u of (1.1) on $\partial\Omega$. Let $d_i \in \mathbb{R}^2$, $i = 1, \ldots, n$, be given. Then we define a vector field $h : \partial\mathscr{D} \to \mathbb{R}^2$ by a piecewise linear interpolation of the data

$$h(x_i) \;=\; d_i\,, \qquad i = 1, \ldots, n\,,$$

i.e., both vector components of h belong to the space of linear splines over $\partial\mathscr{D}$ with the vertices x_i as its nodes. If $h \to 0$ in any norm on this (finite-dimensional) linear space denoted by $\mathcal{S}^2_{\partial\mathscr{D}}$, e.g., with respect to the norm

$$\|h\| \;=\; \max_{i=1,\ldots,n} |h(x_i)|\,, \tag{2.6}$$

then it has been shown in [10] that the operator in (2.5) is Fréchet differentiable with a (shape) derivative $\partial\Lambda_f(\mathscr{D}) \in \mathcal{L}(\mathcal{S}^2_{\partial\mathscr{D}}, L^2_\diamond(\partial\Omega))$. An explicit representation of the derivative $\partial\Lambda_f(\mathscr{D})h$ in the direction of h is given by the trace on $\partial\Omega$ of the solution u' of the inhomogeneous transmission problem

$$\Delta u' = 0 \quad \text{in } \Omega\backslash\partial\mathscr{D}\,, \qquad \frac{\partial}{\partial\nu}u' = 0 \quad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} u'\,\mathrm{d}s = 0\,,$$
$$u'_+ - u'_- = (1-k)(h\cdot\nu)\frac{\partial}{\partial\nu}u_- \qquad \text{on } \partial\mathscr{D}\,, \tag{2.7}$$
$$\frac{\partial}{\partial\nu}u'_+ - k\frac{\partial}{\partial\nu}u'_- = (1-k)\,\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big) \qquad \text{on } \partial\mathscr{D}\,,$$

where $u'_- = u'|_{\mathscr{D}}$ and $u'_+ = u'|_{\Omega\backslash\overline{\mathscr{D}}}$. This connection was first established by Hettlich and Rundell [18] for smooth inclusions, and subsequently seen to hold for admissible polygons $\mathscr{D}$ as well, cf. [16]. Take note that the inhomogeneous transmission data in (2.7) are infinitely smooth on $\partial\mathscr{D}$ except for the vertices, where in general $h\cdot\nu$ is discontinuous and the directional derivatives of u tend to infinity; compare (2.2) and (2.4).

3. The inverse problem for a conductive polygonal inclusion. In [19] Seo investigated the foward map (2.5) for two linearly independent piecewise continuous probing currents $f_1, f_2 \in L^2_\diamond(\partial\Omega)$ under the assumption that the set

$$\big\{\, x \in \partial\Omega \,:\, f(x) \geqslant 0 \,\big\} \tag{3.1}$$

is connected for every linear combination $f = \mu_1 f_1 + \mu_2 f_2$ of them; see [19] for examples, how to choose f_1 and f_2 this way. This somewhat strange assumption arises in the context of an auxiliary result by Seo, the proof of which can also be found in [19]:

Theorem A. *Let $f \in L^2_\diamond(\partial\Omega)\backslash\{0\}$ be a piecewise continuous function, and assume that the weak solution $u \in H^1_\diamond(\Omega)$ of (1.1) satisfies*

$$\big|u(x) - u(x_0)\big| \leqslant C|x - x_0|^{3/2}$$

for some $x_0 \in \Omega$, $C > 0$, and all $x \in \Omega$. Then the set $\{x \in \partial\Omega \,:\, f(x) \geqslant 0\}$ is not connected.

Based on Theorem A Seo showed that the corresponding operator

$$\Lambda_{f_1,f_2} \,:\, \mathscr{D} \mapsto \big(\Lambda_{f_1}(\mathscr{D}), \Lambda_{f_2}(\mathscr{D})\big) \tag{3.2}$$

is injective, i.e., the traces on $\partial\Omega$ of the two potentials (1.1) corresponding to the boundary data $f = f_{1,2}$ uniquely determine an admissible polygonal inclusion. Obviously, the operator Λ_{f_1,f_2} of (3.2) is also Fréchet differentiable with shape derivative

$$\partial\Lambda_{f_1,f_2}(\mathscr{D})h = \big(\partial\Lambda_{f_1}(\mathscr{D})h, \partial\Lambda_{f_2}(\mathscr{D})h\big) \in \mathcal{L}(\mathcal{S}^2_{\partial\mathscr{D}}, (L^2_\diamond(\partial\Omega))^2)$$

for every $h \in \mathcal{S}^2_{\partial\mathscr{D}}$. In the sequel we investigate the linearized inverse problem.

Theorem 3.1. *Let $\mathscr{D}$ be an admissible polygon, and let the two piecewise continuous probing currents $f_1, f_2 \in L^2_\diamond(\partial\Omega)$ satisfy Assumption (3.1). Then $\partial\Lambda_{f_1,f_2}(\mathscr{D})$ is injective.*

Proof. Assume that $\partial\Lambda_{f_1,f_2}(\mathscr{D})h = 0$ for some nontrivial $h \in \mathcal{S}^2_{\partial\mathscr{D}}$. Since the two normal vectors ν_i and ν_{i+1} of any two neighboring edges Γ_i and Γ_{i+1} of $\mathscr{D}$ form a basis of $\mathbb{R}^2$, and since $h \neq 0$, there is at least one vertex x_i of $\mathscr{D}$, where $h(x_i) \cdot \nu_i$ or $h(x_i) \cdot \nu_{i+1}$ is different from zero. Without loss of generality we can assume x_1 to be such a vertex, and that

$$h(x) \cdot \nu(x) \;=\; \begin{cases} 1 + b_2|x - x_1|\,, & x \in \Gamma_2\,, \\ a + b_1|x - x_1|\,, & x \in \Gamma_1\,, \end{cases} \tag{3.3}$$

for certain real parameters a, b_1, and b_2.

Consider now a fixed boundary current $f \in \operatorname{span}\{f_1, f_2\}$. Then

$$\partial \Lambda_f(\mathscr{D})h \;=\; 0\,, \tag{3.4}$$

and by virtue of (3.4) the associated potential u' of (2.7) has homogeneous Cauchy data on $\partial\Omega$. According to Holmgren's theorem this implies that $u' = 0$ in all of $\Omega\backslash\overline{\mathscr{D}}$. It therefore follows from (2.7) that $u'|_{\mathscr{D}}$ is a harmonic function with Cauchy data

$$u' \;=\; (k-1)(h\cdot\nu)\,\frac{\partial}{\partial\nu}u_- \tag{3.5a}$$

and

$$\frac{\partial}{\partial\nu}u' \;=\; \frac{k-1}{k}\,\frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big) \tag{3.5b}$$

on $\partial\mathscr{D}$.

Since the series (2.2) can be differentiated termwise and infinitely often with respect to r and $\theta \in [0, \alpha_i]$, compare [9], we conclude from (3.5) that u' and its Neumann derivative admit the following series expansions for $x \in \Gamma_2$ close to x_1 in the associated local coordinate system (for ease of simplicity we omit the index $i = 1$ in all terms of (2.2)):

$$\begin{aligned} u'(x) \;&=\; -(k-1)(1+b_2r)\,\frac{1}{r}\frac{\partial}{\partial\theta}u_-(x)\Big|_{\theta=0} \;=\; (1-k)(1+b_2r)\sum_{j=1}^{\infty}\beta_j y_j'(0)r^{\gamma_j-1} \\ &=\; (1-k)\sum_{j=1}^{\infty}\beta_j y_j'(0)r^{\gamma_j-1} \;+\; (1-k)b_2\sum_{j=1}^{\infty}\beta_j y_j'(0)r^{\gamma_j} \end{aligned} \tag{3.6a}$$

and

$$\begin{aligned} \frac{\partial}{\partial\nu}u'(x) \;&=\; \frac{k-1}{k}\,\frac{\partial}{\partial r}\Big((1+b_2r)\frac{\partial}{\partial r}u(x)\Big)\Big|_{\theta=0} \\ &=\; \frac{k-1}{k}\,\sum_{j=1}^{\infty}\beta_j\gamma_j(\gamma_j-1)y_j(0)r^{\gamma_j-2} \;+\; \frac{k-1}{k}\,b_2\sum_{j=1}^{\infty}\beta_j\gamma_j^2 y_j(0)r^{\gamma_j-1}\,. \end{aligned} \tag{3.6b}$$

Since the right-hand sides of (3.6a) and (3.6b) are analytic functions of $0 < r < r_0$, the local Cauchy problem (3.5) on this portion of Γ_2 has a unique harmonic solution, which can be written down explicitly in the same coordinates, i.e.,

$$\begin{aligned} u'(x) \;&=\; \frac{1-k}{k}\sum_{j=1}^{\infty}\Big(k\beta_j y_j'(0)\cos(\gamma_j-1)\theta \;+\; \beta_j\gamma_j y_j(0)\sin(\gamma_j-1)\theta\Big)r^{\gamma_j-1} \\ &\quad+\; \frac{1-k}{k}\,b_2\sum_{j=1}^{\infty}\Big(k\beta_j y_j'(0)\cos\gamma_j\theta \;+\; \beta_j\gamma_j y_j(0)\sin\gamma_j\theta\Big)r^{\gamma_j} \end{aligned} \tag{3.7}$$

for $0 \leqslant \theta \leqslant \alpha_1$ and $0 < r < r_0$.

In particular, when $\theta = \alpha_1$ it follows from (2.4) that for $x \in \mathit{\Gamma}_1$ we have

$$\begin{aligned} u'(x) \;&=\; \frac{1-k}{k}\Bigl(k\beta_1 y_1'(0)\cos(\gamma_1-1)\alpha_1 \;+\; \beta_1\gamma_1 y_1(0)\sin(\gamma_1-1)\alpha_1\Bigr)r^{\gamma_1-1} \\ &\quad+\; \frac{1-k}{k}\Bigl(k\beta_2 y_2'(0)\cos(\gamma_2-1)\alpha_1 \;+\; \beta_2\gamma_2 y_2(0)\sin(\gamma_2-1)\alpha_1\Bigr)r^{\gamma_2-1} \\ &\quad+\; O(r^{1/2}) \end{aligned}$$

as $r = |x - x_1| \to 0$, while at the same time, according to (3.5a), (3.3), and (2.2),

$$\begin{aligned} u'(x) \;&=\; (k-1)(a+b_1 r)\,\frac{1}{r}\frac{\partial}{\partial\theta}u_-(x)\Big|_{\theta=\alpha_1} \;=\; (k-1)(a+b_1 r)\sum_{j=1}^{\infty}\beta_j y_j'(\alpha_1)r^{\gamma_j-1} \\ &=\; (k-1)a\beta_1 y_1'(\alpha_1)r^{\gamma_1-1} \;+\; (k-1)a\beta_2 y_2'(\alpha_1)r^{\gamma_2-1} \;+\; O(r^{1/2}) \end{aligned}$$

for the same boundary points $x \in \mathit{\Gamma}_1$. A comparison of the leading order terms thus yields the two equations

$$ka\beta_j y_j'(\alpha_1) \;=\; -\beta_j\bigl(ky_j'(0)\cos(\gamma_j-1)\alpha_1 \;+\; \gamma_j y_j(0)\sin(\gamma_j-1)\alpha_1\bigr)\,, \tag{3.8}$$

$j = 1, 2$. Now we recall that

$$y_j(\theta) \;=\; A_j\cos\gamma_j\theta \;+\; B_j\sin\gamma_j\theta \qquad \text{for } \theta \in [0, \alpha_1] \tag{3.9}$$

and $j = 1, 2$, with certain coefficients $A_j, B_j \in \mathbb{R}$ with $A_j^2 + B_j^2 \neq 0$. Inserting this into (3.8) we arrive at

$$k(ac_j + c_j')\beta_j B_j \;=\; (kas_j - s_j')\beta_j A_j\,, \qquad j = 1, 2\,, \tag{3.10}$$

where we have introduced the abbreviations

$$c_j = \cos\gamma_j\alpha_1\,, \quad s_j = \sin\gamma_j\alpha_1\,, \quad c_j' = \cos(\gamma_j-1)\alpha_1\,, \quad s_j' = \sin(\gamma_j-1)\alpha_1$$

for general $j \in \mathbb{N}$.

Likewise we can use (3.7) to evaluate the Neumann derivative of u' on $\mathit{\Gamma}_1$ near x_1, which gives

$$\begin{aligned} \frac{\partial}{\partial\nu}u'(x) \;&=\; \frac{1}{r}\frac{\partial}{\partial\theta}u'(x)\Big|_{\theta=\alpha_1} \\ &=\; \frac{1-k}{k}\sum_{j=1}^{\infty}(\gamma_j-1)\Bigl(\beta_j\gamma_j y_j(0)c_j' \;-\; k\beta_j y_j'(0)s_j'\Bigr)r^{\gamma_j-2} \\ &\qquad+\; \frac{1-k}{k}\,b_2\sum_{j=1}^{\infty}\gamma_j\Bigl(\beta_j\gamma_j y_j(0)c_j \;-\; k\beta_j y_j'(0)s_j\Bigr)r^{\gamma_j-1} \\ &=\; \frac{1-k}{k}(\gamma_1-1)\Bigl(\beta_1\gamma_1 y_1(0)c_1' \;-\; k\beta_1 y_1'(0)s_1'\Bigr)r^{\gamma_1-2} \\ &\qquad+\; \frac{1-k}{k}(\gamma_2-1)\Bigl(\beta_2\gamma_2 y_2(0)c_2' \;-\; k\beta_2 y_2'(0)s_2'\Bigr)r^{\gamma_2-2} \;+\; O(r^{-1/2}) \end{aligned}$$

for $r \to 0$, and compare this with (3.5b):

$$\begin{aligned}\frac{\partial}{\partial\nu}u'(x) &= \frac{k-1}{k}\,\frac{\partial}{\partial r}\Big((a+b_1 r)\frac{\partial}{\partial r}u(x)\Big)\Big|_{\theta=\alpha_1} \\ &= \frac{k-1}{k}\,\gamma_1(\gamma_1-1)a\beta_1 y_1(\alpha_1)r^{\gamma_1-2} \;+\; \frac{k-1}{k}\,\gamma_2(\gamma_2-1)a\beta_2 y_2(\alpha_1)r^{\gamma_2-2} \\ &\quad + O(r^{-1/2})\,.\end{aligned}$$

Inserting (3.9) we thus obtain a second pair of equations,

$$(ks_j' - as_j)\beta_j B_j \;=\; (ac_j + c_j')\beta_j A_j\,, \qquad j=1,2\,. \tag{3.11}$$

The four equations in (3.10), (3.11) can be rearranged in two homogeneous linear systems

$$M_j\begin{bmatrix}\beta_j A_j\\ \beta_j B_j\end{bmatrix} = \begin{bmatrix}kas_j - s_j' & -k(ac_j + c_j')\\ ac_j + c_j' & as_j - ks_j'\end{bmatrix}\begin{bmatrix}\beta_j A_j\\ \beta_j B_j\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad j=1,2\,. \tag{3.12}$$

As mentioned before, the entries of the two matrices M_j only depend on the geometry of the problem and not on the probing current. The probing current f only enters into (3.12) via the coefficients $\beta_1 = \beta_1[f]$ and $\beta_2 = \beta_2[f]$. Further, since $A_j^2 + B_j^2 \neq 0$ for $j = 1,2$, it follows that $\beta_j[f] = 0$ for *every* probing current $f \in \operatorname{span}\{f_1, f_2\}$, if the matrix M_j happens to be nonsingular.

Let us therefore make the assumption that both matrices M_1 and M_2 are singular. Then we must have

$$\begin{aligned}0 &= (kas_j - s_j')(as_j - ks_j') \;+\; k(ac_j + c_j')^2 \\ &= k(1 + 2ac_jc_j' + a^2) \;-\; (k^2+1)as_js_j'\,, \qquad j=1,2\,,\end{aligned}$$

because $c_j^2 + s_j^2 = {c_j'}^2 + {s_j'}^2 = 1$. Since

$$\begin{aligned}c_jc_j' &= c_j\cos(\gamma_j-1)\alpha_1 \;=\; c_j(\cos\gamma_j\alpha_1\cos\alpha_1 \;+\; \sin\gamma_j\alpha_1\sin\alpha_1) \\ &= c_j^2\cos\alpha_1 \;+\; c_js_j\sin\alpha_1 \;=\; \cos\alpha_1 \;-\; s_j(s_j\cos\alpha_1 - c_j\sin\alpha_1) \\ &= \cos\alpha_1 \;-\; s_j\sin(\gamma_j-1)\alpha_1 \;=\; \cos\alpha_1 \;-\; s_js_j'\,,\end{aligned}$$

the previous equations can be rewritten as

$$0 \;=\; k(1 + 2a\cos\alpha_1 + a^2) \;-\; (k+1)^2as_js_j'\,, \qquad j=1,2\,. \tag{3.13}$$

From this we immediately deduce that a must be different from zero in this case. Accordingly, as (3.13) is bound to hold for $j=1$ and $j=2$ simultaneously, we can subtract these two equations, and conclude that

$$s_1s_1' \;=\; s_2s_2'\,. \tag{3.14}$$

To obtain a contradiction we turn to (2.3) and Figure 2.1 and distinguish two cases. If $0 < \alpha_1 < \pi$, then the appropriate instances of (2.3) take the form

$$\sin\gamma_1(\pi-\alpha_1) \;=\; \lambda\sin\gamma_1\pi\,, \qquad \sin\gamma_2(\pi-\alpha_1) \;=\; -\lambda\sin\gamma_2\pi\,,$$

which can be rewritten with the help of the angle sum formula as

$$(c_1 - \lambda)\sin\gamma_1\pi \;=\; s_1\cos\gamma_1\pi\,, \tag{3.15a}$$

$$(c_2 + \lambda)\sin\gamma_2\pi \;=\; s_2\cos\gamma_2\pi\,. \tag{3.15b}$$

Since $\lambda > 1$ and $\pi/2 < \gamma_1\pi < \pi$ by virtue of (2.4), the left-hand side of (3.15a) is negative, and so is $\cos\gamma_1\pi$. Likewise, since $\pi < \gamma_2\pi < 3\pi/2$, the left-hand side of (3.15b) is negative, and again, so is $\cos\gamma_2\pi$. Therefore, we conclude from (3.15) that

$$s_1 > 0 \qquad \text{and} \qquad s_2 > 0\,. \tag{3.16}$$

On the other hand, $-1/2 < \gamma_1 - 1 < 0$, and therefore $(\gamma_1 - 1)\alpha_1 \in (-\pi/2, 0)$ in this first case. This shows that

$$s_1' \;=\; \sin(\gamma_1 - 1)\alpha_1 \;<\; 0\,,$$

whereas

$$s_2' \;=\; \sin(\gamma_2 - 1)\alpha_1 \;>\; 0\,,$$

because $0 < \gamma_2 - 1 < 1/2$. Together with (3.16) this contradicts (3.14) in the case, where $0 < \alpha_1 < \pi$.

In the other case, where $\pi < \alpha_1 < 2\pi$, (2.3) implies that

$$\sin\gamma_1(\alpha_1 - \pi) \;=\; \lambda\sin\gamma_1\pi\,, \qquad \sin\gamma_2(\alpha_1 - \pi) \;=\; -\lambda\sin\gamma_2\pi\,,$$

and this yields

$$(c_1 + \lambda)\sin\gamma_1\pi \;=\; s_1\cos\gamma_1\pi\,,$$

$$(c_2 - \lambda)\sin\gamma_2\pi \;=\; s_2\cos\gamma_2\pi\,.$$

This shows that

$$s_1 < 0 \qquad \text{and} \qquad s_2 < 0\,,$$

while

$$s_1' < 0 \qquad \text{and} \qquad s_2' > 0$$

in this case, because $-\pi < (\gamma_1 - 1)\alpha_1 < 0$ and $0 < (\gamma_2 - 1)\alpha_1 < \pi$, which again contradicts (3.14).

We thus have brought our assumption, that both matrices M_1 and M_2 are singular, to a contradiction. But if M_1 is nonsingular, then

$$\beta_1[\mu_1 f_1 + \mu_2 f_2] \;=\; 0 \qquad \text{for every } \mu_1, \mu_2 \in \mathbb{R}\,,$$

while we can enforce

$$\beta_2[\mu_1 f_1 + \mu_2 f_2] \;=\; \mu_1\beta_2[f_1] + \mu_2\beta_2[f_2] \;=\; 0$$

by an appropriate nontrivial choice of $\mu_1, \mu_2 \in \mathbb{R}$. Likewise, if M_2 is nonsingular, then

$$\beta_2[\mu_1 f_1 + \mu_2 f_2] \;=\; 0 \qquad \text{for every } \mu_1, \mu_2 \in \mathbb{R}$$

and

$$\beta_1[\mu_1 f_1 + \mu_2 f_2] \;=\; \mu_1\beta_1[f_1] + \mu_2\beta_1[f_2] \;=\; 0\,,$$

if the nontrivial coefficients μ_1 and μ_2 are chosen appropriately. Therefore, in either case we can find a probing current $f \in \mathrm{span}\{f_1, f_2\}\backslash\{0\}$, such that the two leading expansion coefficients $\beta_1 = \beta_1[f]$ and $\beta_2 = \beta_2[f]$ of the corresponding electric potential u are both vanishing near the vertex x_1.

But then it follows from (2.4) that Seo's Assumption (3.1) concerning the choice of the two probing currents f_1 and f_2 is in contradiction to Theorem A. Thus we have proved that the null space of $\partial\Lambda_{f_1,f_2}(\mathscr{D})$ is trivial, i.e., that $\partial\Lambda_{f_1,f_2}(\mathscr{D})$ is injective. □

4. Lipschitz stability for a conductive polygonal inclusion. The idea of obtaining Lipschitz stability for Seo's inverse problem originates from the fact that each admissible polygon $\mathscr{D}$ with n vertices can be described by a $2n$-dimensional vector

$$\mathbf{x} \;=\; [x_1, \dots, x_n] \;\in\; (\mathbb{R}^2)^n \tag{4.1}$$

with the coordinates of its vertices in counterclockwise ordering. However, since we have the freedom of choosing any vertex of $\mathscr{D}$ for x_1, this vector is not uniquely specified.

To account for this problem we introduce the following metric in the set of all admissible polygons with n vertices (which corresponds to a pseudometric in $(\mathbb{R}^2)^n$): If $\mathscr{D}'$ is a second admissible polygon with n vertices x_i' in counterclockwise order, let

$$d(\mathscr{D}, \mathscr{D}') \;=\; \min_{j=0,\dots,n-1}\, \max_{i=1,\dots,n} |x_{i+j} - x_i'|\,. \tag{4.2}$$

Clearly, d is nonnegative and symmetric; it vanishes, if and only if $\mathscr{D} = \mathscr{D}'$. To see that d also satisfies the triangle inequality, let $\mathscr{D}''$ be a third admissible polygon with vertices x_i'', $i = 1, \dots, n$. Without loss of generality we can assume that the enumeration of the vertices of $\mathscr{D}$ and $\mathscr{D}'$ is chosen in such a way that

$$d(\mathscr{D}'', \mathscr{D}) \;=\; \max_{i=1,\dots,n} |x_i'' - x_i|\,,$$

and likewise, that

$$d(\mathscr{D}'', \mathscr{D}') \;=\; \max_{i=1,\dots,n} |x_i'' - x_i'|\,,$$

Then we readily conclude that

$$\begin{aligned} d(\mathscr{D}, \mathscr{D}') \;&\leqslant\; \max_{i=1,\dots,n} |x_i - x_i'| \;\leqslant\; \max_{i=1,\dots,n} \bigl(|x_i - x_i''| \,+\, |x_i'' - x_i'|\bigr) \\ &\leqslant\; \max_{i=1,\dots,n} |x_i - x_i''| \,+\, \max_{j=1,\dots,n} |x_j'' - x_j'| \;=\; d(\mathscr{D}, \mathscr{D}'') \,+\, d(\mathscr{D}'', \mathscr{D}')\,. \end{aligned}$$

Take note that $d(\mathscr{D}, \mathscr{D}')$ majorizes the Hausdorff distance $d_H(\partial\mathscr{D}, \partial\mathscr{D}')$ between $\partial\mathscr{D}$ and $\partial\mathscr{D}'$. For, if $x \in \partial\mathscr{D}$ then there is some vertex x_i of $\mathscr{D}$ and $c \in [0, 1)$, such that

$$x \;=\; c x_i \,+\, (1-c) x_{i+1}\,;$$

assuming further without loss of generality that the minimum in (4.2) is attained for $j = 0$, then it follows that

$$\big|x \,-\, (cx_i' + (1-c)x_{i+1}')\big| \;\leqslant\; c\,|x_i - x_i'| \,+\, (1-c)\,|x_{i+1} - x_{i+1}'| \;\leqslant\; d(\mathscr{D},\mathscr{D}')\,.$$

Since $cx_i' + (1-c)x_{i+1}' \in \partial\mathscr{D}'$ this shows that the distance between any $x \in \partial\mathscr{D}$ and $\partial\mathscr{D}'$ is at most $d(\mathscr{D},\mathscr{D}')$, and hence,

$$d_H(\partial\mathscr{D},\partial\mathscr{D}') \;\leqslant\; d(\mathscr{D},\mathscr{D}')\,. \tag{4.3}$$

Another useful ingredient to achieve Lipschitz stability is compactness. Therefore, following Beretta and Francini [12], we define for a given $\delta > 0$ the union $\mathcal{A}_{n,\delta}$ of all admissible polygons with n vertices, such that

(i) $|x_i - x| \geqslant \delta$ for every $i = 1,\dots,n$ and every $x \in \varGamma_j$ with $j \notin \{i, i+1\}$;

(ii) $\delta \leqslant \alpha_i \leqslant 2\pi - \delta$ and $|\alpha - \pi| \geqslant \delta$ for every $i = 1,\dots,n$;

(iii) $|x - y| \geqslant \delta$ for every $x \in \partial\mathscr{D}$ and every $y \in \partial\Omega$.

We denote by $\mathcal{X}_{n,\delta} \subset (\mathbb{R}^2)^n$ the set of coordinate vectors (4.1), which descibe some $\mathscr{D} \in \mathcal{A}_{n,\delta}$, and we emphasize that $\mathcal{X}_{n,\delta}$ is compact. Further, $\mathcal{X}_{n,\delta} \subset \mathcal{X}_{n,\delta/2}$, and when $(\mathbb{R}^2)^n$ is equipped with the norm

$$\|\mathbf{d}\| \;=\; \max_{i=1,\dots,n} |d_i| \qquad \text{for every} \;\; \mathbf{d} = [d_1,\dots,d_n]\,, \tag{4.4}$$

then there is an open set $\mathcal{U}_{n,\delta} \subset (\mathbb{R}^2)^n$, such that

$$\mathcal{X}_{n,\delta} \;\subset\; \mathcal{U}_{n,\delta} \;\subset\; \mathcal{X}_{n,\delta/2}\,.$$

In $\mathcal{U}_{n,\delta}$ we define

$$F : \begin{cases} \mathcal{U}_{n,\delta} \;\to\; (L^2_\diamond(\partial\Omega))^2\,, \\ \quad\;\; \mathbf{x} \;\mapsto\; \varLambda_{f_1,f_2}(\mathscr{D})\,, \end{cases} \tag{4.5}$$

where $\mathscr{D} \in \mathcal{A}_{n,\delta}$ is the polygon associated with $\mathbf{x}$. Further, for the same pair of $\mathbf{x}$ and $\mathscr{D}$ we introduce

$$S_{\mathbf{x}} : \begin{cases} (\mathbb{R}^2)^n \;\to\; \mathcal{S}^2_{\partial\mathscr{D}}\,, \\ \qquad\; \mathbf{d} \;\mapsto\; h\,, \end{cases}$$

where for $\mathbf{d} = [d_1,\dots,d_n]$ the vector field $h \in \mathcal{S}^2_{\partial\mathscr{D}}$ is defined by piecewise linear interpolation:

$$h(x_i) \;=\; d_i\,, \qquad i = 1,\dots,n\,. \tag{4.6}$$

With the norms introduced in (2.6) and (4.4) $S_{\mathbf{x}}$ is an isometry.

Proposition 4.1. *The operator F of (4.5) belongs to $C^1\big(\mathcal{U}_{n,\delta},(L^2_\diamond(\Omega))^2\big)$, and for $\mathbf{x} \in \mathcal{X}_{n,\delta}$ and the associated $\mathscr{D} \in \mathcal{A}_{n,\delta}$ its derivative is given by*

$$F'(\mathbf{x})\mathbf{d} \;=\; \partial\varLambda_{f_1,f_2}(\mathscr{D})S_{\mathbf{x}}\mathbf{d}\,, \qquad \mathbf{d} \in (\mathbb{R}^2)^n\,. \tag{4.7}$$

Proof. Let $\mathbf{x} \in \mathcal{U}_{n,\delta}$ and $\mathbf{d} \in (\mathbb{R}^2)^n$ be so small that $\mathbf{x} + \mathbf{d}$ also belongs to $\mathcal{U}_{n,\delta}$. Let $\mathscr{D}$ and $\mathscr{D}'$ be the polygons associated with $\mathbf{x}$ and $\mathbf{x} + \mathbf{d}$, respectively. Then, for F' defined as in (4.7) there holds

$$\begin{aligned} &\|F(\mathbf{x}+\mathbf{d}) \,-\, F(\mathbf{x}) \,-\, F'(\mathbf{x})\mathbf{d}\|_{(L^2(\partial\Omega))^2} \;= \\ &\quad \|\varLambda_{f_1,f_2}(\mathscr{D}') \,-\, \varLambda_{f_1,f_2}(\mathscr{D}) \,-\, \partial\varLambda_{f_1,f_2}(\mathscr{D})h\|_{(L^2(\partial\Omega))^2}\,, \end{aligned} \tag{4.8}$$

where $h \in \mathcal{S}^2_{\partial\mathscr{D}}$ is defined as in (4.6). As shown in [16] the right-hand side of (4.8) can be bounded by $C\|h\|^2$ for h sufficiently small, i.e., for $\mathbf{d}$ sufficiently small. The constant C depends on the conductivity k and on the probing currents $f_{1,2}$, and also on Ω and on $\mathscr{D}$, but this constant can be chosen in such a way that

$$\|F(\mathbf{x}' + \mathbf{d}) \,-\, F(\mathbf{x}') \,-\, F'(\mathbf{x}')\mathbf{d}\|_{(L^2(\partial\Omega))^2} \;\leqslant\; C\|\mathbf{d}\|^2 \tag{4.9}$$

holds true for $\mathbf{d}$ sufficiently small and all $\mathbf{x}'$ within a certain neighborhood of $\mathbf{x}$. This proves that F is differentiable in $\mathcal{U}_{n,\delta}$.

To show that F is C^1 we let $\mathbf{x}$ and $\mathscr{D}$ be defined as before, and we quote from [16] that

$$\|F'(\mathbf{x}')\|_{\mathcal{L}((\mathbb{R}^2)^n,(L^2(\partial\Omega))^2)} \;=\; \|\partial\Lambda_{f_1,f_2}(\mathscr{D}')\|_{\mathcal{L}(\mathcal{S}^2_{\partial\mathscr{D}'},(L^2(\partial\Omega))^2)}$$

is uniformly bounded for all $\mathbf{x}'$ sufficiently close to $\mathbf{x}$ and the associated polygons $\mathscr{D}'$. From this it follows immediately that

$$\|F'(\mathbf{x} + \mathbf{d})\|_{\mathcal{L}((\mathbb{R}^2)^n,(L^2(\partial\Omega))^2)} \;\leqslant\; L$$

for some $L > 0$, provided $\mathbf{d}$ is sufficiently small. Accordingly, if $\mathbf{x}' \in \mathcal{U}_{n,\delta}$ is sufficiently close to $\mathbf{x}$ then we can estimate

$$\begin{aligned}\|F(\mathbf{x}') - F(\mathbf{x})\|_{(L^2(\partial\Omega))^2} \;&\leqslant\; \int_0^1 \|F'\big(\mathbf{x} + t(\mathbf{x}' - \mathbf{x})\big)(\mathbf{x}' - \mathbf{x})\|_{(L^2(\partial\Omega))^2}\,\mathrm{d}t \\ &\leqslant\; \int_0^1 L\,\|\mathbf{x}' - \mathbf{x}\|\,\mathrm{d}t \;=\; L\,\|\mathbf{x}' - \mathbf{x}\|\,.\end{aligned} \tag{4.10}$$

Now let $\mathbf{d}$ be an arbitrary unit vector in $(\mathbb{R}^2)^n$. Then (4.9) and (4.10) imply that

$$\begin{aligned}\|F'(\mathbf{x}')\mathbf{d} - F'(\mathbf{x})\mathbf{d}\|_{(L^2(\partial\Omega))^2} \;&\leqslant\; \Big\|F'(\mathbf{x}')\mathbf{d} \,-\, \frac{1}{t}\big(F(\mathbf{x}' + t\mathbf{d}) - F(\mathbf{x}')\big)\Big\|_{(L^2(\partial\Omega))^2} \\ &\quad+\; \Big\|\frac{1}{t}\big(F(\mathbf{x} + t\mathbf{d}) - F(\mathbf{x})\big) \,-\, F'(\mathbf{x})\mathbf{d}\,\Big\|_{(L^2(\partial\Omega))^2} \\ &\quad+\; \frac{1}{t}\,\big\|F(\mathbf{x}' + t\mathbf{d}) - F(\mathbf{x} + t\mathbf{d})\big\|_{(L^2(\partial\Omega))^2} \,+\, \frac{1}{t}\,\big\|F(\mathbf{x}') - F(\mathbf{x})\big\|_{(L^2(\partial\Omega))^2} \\ &\leqslant\; 2Ct\,\|\mathbf{d}\|^2 \,+\, \frac{2L}{t}\,\big\|\mathbf{x}' - \mathbf{x}\big\|\,,\end{aligned}$$

provided that $t > 0$ is sufficiently small. In particular, for $t = \|\mathbf{x}' - \mathbf{x}\|^{1/2}$ we obtain

$$\|F'(\mathbf{x}')\mathbf{d} - F'(\mathbf{x})\mathbf{d}\|_{(L^2(\partial\Omega))^2} \;\leqslant\; (2C + 2L)\|\mathbf{x}' - \mathbf{x}\|^{1/2}\,,$$

independent of the particular choice of $\mathbf{d}$. This shows that $F \in C^1(\mathcal{U}_{n,\delta}, (L^2_\diamond(\Omega))^2)$. ☐

Now we can formulate our inverse Lipschitz stability result.

THEOREM 4.2. *Let $\delta > 0$ and $\mathcal{A}_{n,\delta}$ be defined as above. Further, let the two piecewise constant probing currents $f_1, f_2 \in L^2_\diamond(\partial\Omega)$ fulfill Seo's Assumption (3.1). Then there is a Lipschitz constant $\ell_{n,\delta}$, depending only on n, δ, k, Ω, and $f_{1,2}$, such that*

$$d(\mathscr{D}, \mathscr{D}') \;\leqslant\; \ell_{n,\delta}\,\|\Lambda_{f_1,f_2}(\mathscr{D}) - \Lambda_{f_1,f_2}(\mathscr{D}')\|_{(L^2(\partial\Omega))^2} \tag{4.11}$$

for every pair of polygons $\mathscr{D}, \mathscr{D}' \in \mathcal{A}_{n,\delta}$, where the metric d is defined in (4.2).

Proof. We assume (4.11) to be wrong, i.e., we assume that there exist two sequences $(\mathscr{D}_m)_m, (\mathscr{D}'_m)_m \subset \mathcal{A}_{n,\delta}$ with vertex coordinates $\mathbf{x}_m$ and $\mathbf{x}'_m$ in $(\mathbb{R}^2)^n$, such that

$$d(\mathscr{D}_m, \mathscr{D}'_m) \;>\; \eta_m \, \| F(\mathbf{x}_m) - F(\mathbf{x}'_m) \|_{(L^2(\partial\Omega))^2} \,, \tag{4.12}$$

where $\eta_m \to \infty$ for $m \to \infty$. Since $\mathcal{X}_{n,\delta}$ is compact we can find a subsequence $(m_l)_{l\in\mathbb{N}}$ of indices, such that the associated subsequences $(\mathbf{x}_{m_l})_l$ and $(\mathbf{x}'_{m_l})_l$ converge, i.e., there exist $\mathbf{x}, \mathbf{x}' \in \mathcal{X}_{n,\delta}$ with

$$\mathbf{x}_{m_l} \to \mathbf{x} \quad \text{and} \quad \mathbf{x}'_{m_l} \to \mathbf{x}' \qquad \text{for } l \to \infty \,.$$

For ease of notation we consider these subsequences to be the original sequences that we have started with. For the two polygons $\mathscr{D}$ and $\mathscr{D}'$ in $\mathcal{A}_{n,\delta}$ associated with $\mathbf{x}$ and $\mathbf{x}'$, respectively, we then have

$$d(\mathscr{D}_m, \mathscr{D}) \;\to\; 0 \quad \text{and} \quad d(\mathscr{D}'_m, \mathscr{D}') \;\to\; 0 \qquad \text{for } m \to \infty \,,$$

and hence,

$$d(\mathscr{D}_m, \mathscr{D}'_m) \;\to\; d(\mathscr{D}, \mathscr{D}') \;<\; \infty \,. \tag{4.13}$$

We further conclude that

$$F(\mathbf{x}_m) \;\to\; F(\mathbf{x}) \quad \text{and} \quad F(\mathbf{x}'_m) \;\to\; F(\mathbf{x}') \qquad \text{for } m \to \infty \,,$$

because F is differentiable. Together with (4.12) and (4.13) this implies that $F(\mathbf{x}) = F(\mathbf{x}')$. In other words, $\Lambda_{f_1,f_2}(\mathscr{D}) = \Lambda_{f_1,f_2}(\mathscr{D}')$, and by Seo's uniqueness result we necessarily have $\mathscr{D} = \mathscr{D}'$. Without loss of generality we can assume in the sequel that $\mathbf{x} = \mathbf{x}'$; otherwise, we reenumerate the vertices of $\mathscr{D}'_m$ for every $m \in \mathbb{N}$ in such a way that the enumeration of the vertices of $\mathscr{D}$ and $\mathscr{D}'$ is the same.

For m sufficiently large we now rewrite

$$\begin{aligned} F(\mathbf{x}_m) - F(\mathbf{x}'_m) \;&=\; \int_0^1 F'\big(\mathbf{x}'_m + t(\mathbf{x}_m - \mathbf{x}'_m)\big)(\mathbf{x}_m - \mathbf{x}'_m)\,\mathrm{d}t \\ &=\; F'(\mathbf{x})(\mathbf{x}_m - \mathbf{x}'_m) \;+\; \int_0^1 \Big(F'\big(\mathbf{x}'_m + t(\mathbf{x}_m - \mathbf{x}'_m)\big) - F'(\mathbf{x})\Big)(\mathbf{x}_m - \mathbf{x}'_m)\,\mathrm{d}t \,, \end{aligned}$$

and hence, introducing

$$\mathbf{d}_m \;=\; \frac{\mathbf{x}_m - \mathbf{x}'_m}{\|\mathbf{x}_m - \mathbf{x}'_m\|}$$

and

$$\varepsilon_m \;=\; \int_0^1 \big\| F'\big(\mathbf{x}'_m + t(\mathbf{x}_m - \mathbf{x}'_m)\big) - F'(\mathbf{x}) \big\| \,\mathrm{d}t \,,$$

we arrive at

$$\| F(\mathbf{x}_m) - F(\mathbf{x}'_m) \|_{(L^2(\partial\Omega))^2} \;\geqslant\; \Big(\| F'(\mathbf{x})\mathbf{d}_m \|_{(L^2(\partial\Omega))^2} \;-\; \varepsilon_m \Big) \| \mathbf{x}_m - \mathbf{x}'_m \| \,. \tag{4.14}$$

Take note that $\mathbf{x}_m \neq \mathbf{x}'_m$ because $d(\mathscr{D}_m, \mathscr{D}'_m) > 0$ according to (4.12), and that $\varepsilon_m \to 0$ as $m \to \infty$ by the continuity of F' and the fact that

$$\left\|\mathbf{x}'_m + t(\mathbf{x}_m - \mathbf{x}'_m) - \mathbf{x}\right\| \leqslant t\,\|\mathbf{x}_m - \mathbf{x}\| + (1-t)\|\mathbf{x}'_m - \mathbf{x}\| \to 0$$

as $m \to \infty$, uniformly for $t \in [0,1]$. Inserting (4.14) into (4.12) we thus conclude that

$$\|F'(\mathbf{x})\mathbf{d}_m\|_{(L^2(\partial\Omega))^2} < \varepsilon_m + \frac{1}{\eta_m}\frac{d(\mathscr{D}_m, \mathscr{D}'_m)}{\|\mathbf{x}_m - \mathbf{x}'_m\|} \tag{4.15}$$

for m sufficiently large. Since $\mathbf{x}_m - \mathbf{x}'_m \to 0$ and the vertices of $\mathscr{D}_m$ (and of $\mathscr{D}'_m$, respectively) are at least δ apart by requirement (i) in the definition of $\mathcal{A}_{n,\delta}$, we conclude from (4.2) that

$$d(\mathscr{D}_m, \mathscr{D}'_m) = \|\mathbf{x}_m - \mathbf{x}'_m\|, \qquad \text{if } \|\mathbf{x}_m - \mathbf{x}'_m\| < \delta/2,$$

and hence, the right-hand side of (4.15) goes to zero as $m \to \infty$.

Since $(\mathbf{d}_m)_m$ consists of unit vectors from $(\mathbb{R}^2)^n$ we can find a convergent subsequence – again denoted by $(\mathbf{d}_m)_m$ – with

$$\mathbf{d}_m \to \mathbf{d}, \qquad m \to \infty,$$

where the limit $\mathbf{d} = [d_1, \ldots, d_n] \in (\mathbb{R}^2)^n$ also has norm one. It thus follows from (4.15) that

$$0 = F'(\mathbf{x})\mathbf{d} = \partial\Lambda_{f_1,f_2}(\mathscr{D})h,$$

where $0 \neq h \in \mathcal{S}^2_{\partial\mathscr{D}}$ is given by (4.6). But this violates our injectivity result in Theorem 3.1, and hence, we have the desired contradition to (4.12). □

As we have mentioned in the introduction the idea of this proof is borrowed from [6, 14]. We had to rearrange the argumentation, though, because the operator F of (4.5) fails to be injective, and because its domain $\mathcal{X}_{n,\delta}$ is not convex.

For ease of completeness we also state the following Lipschitz stability result for admissible polygons with *at most* N vertices.

Corollary 4.3. *Let $\delta > 0$ and $\mathcal{B}_{N,\delta} = \bigcup_{n=3}^N \mathcal{A}_{n,\delta}$ for some $N \geqslant 3$. Further, let the two piecewise constant probing currents $f_1, f_2 \in L^2_\diamond(\partial\Omega)$ fulfill Seo's assumption (3.1). Then there is a Lipschitz constant $L_{N,\delta}$, depending only on N, δ, k, Ω, and $f_{1,2}$, such that*

$$d_H(\partial\mathscr{D}, \partial\mathscr{D}') \leqslant L_{N,\delta}\,\|\Lambda_{f_1,f_2}(\mathscr{D}) - \Lambda_{f_1,f_2}(\mathscr{D}')\|_{L^2(\partial\Omega)^2}$$

for every pair of polygons $\mathscr{D}, \mathscr{D}' \in \mathcal{B}_{N,\delta}$, where d_H denotes the Hausdorff metric.

Proof. As in the previous proof we assume to the contrary that there are sequences $(\mathscr{D}_m)_m, (\mathscr{D}'_m)_m \subset \mathcal{B}_{N,\delta}$ with

$$d_H(\partial\mathscr{D}_m, \partial\mathscr{D}'_m) > \eta_m\,\|\Lambda_{f_1,f_2}(\mathscr{D}_m) - \Lambda_{f_1,f_2}(\mathscr{D}'_m)\|_{L^2(\partial\Omega)^2}, \tag{4.16}$$

where $\eta_m \to \infty$ as $m \to \infty$. Then there are infinitely many indices m_l, $l \in \mathbb{N}$, and two natural numbers $n, n' \in \{3, \ldots, N\}$ such that all $\mathscr{D}_{m_l}$ are n-gons and all $\mathscr{D}'_{m_l}$ are n'-gons. Again we assume that these two subsequences have been the original ones, and as in the proof of Theorem 4.2 we can further assume without loss of generality that the corresponding vectors $\mathbf{x}_m \in \mathcal{X}_{n,\delta}$ and $\mathbf{x}'_m \in \mathcal{X}_{n',\delta}$ with the coordinates of the vertices of $\mathscr{D}_m$ and $\mathscr{D}'_m$, respectively, converge. If we denote the polygons corresponding to the two limit vectors by $\mathscr{D} \in \mathcal{A}_{n,\delta}$ and $\mathscr{D}' \in \mathcal{A}_{n',\delta}$ this implies that

$$\|\Lambda_{f_1,f_2}(\mathscr{D}) - \Lambda_{f_1,f_2}(\mathscr{D}')\|_{L^2(\partial\Omega)^2} \;=\; \lim_{m\to\infty} \|\Lambda_{f_1,f_2}(\mathscr{D}_m) - \Lambda_{f_1,f_2}(\mathscr{D}'_m)\|_{L^2(\partial\Omega)^2}$$
$$\leqslant \lim_{m\to\infty} \frac{1}{\eta_m}\, d_H(\partial\mathscr{D}_m, \partial\mathscr{D}'_m) \;=\; 0\,,$$

because $d_H(\partial\mathscr{D}_m, \partial\mathscr{D}'_m) \to d_H(\partial\mathscr{D}, \partial\mathscr{D}')$. From Seo's uniqueness result therefore follows that $\mathscr{D} = \mathscr{D}'$, and in particular, that $n = n'$. We can thus apply Theorem 4.2 to conclude that

$$d(\mathscr{D}_m, \mathscr{D}'_m) \;\leqslant\; \ell_{n,\delta}\, \|\Lambda_{f_1,f_2}(\mathscr{D}_m) - \Lambda_{f_1,f_2}(\mathscr{D}'_m)\|_{L^2(\partial\Omega)^2} \tag{4.17}$$

for all $m \in \mathbb{N}$. Combined with (4.16) and (4.3) this implies that the sequence $(\eta_m)_m$ is bounded, which is the desired contradiction. ∎

5. The case of an insulating polygonal inclusion. So far we have assumed that the conductivity k of the inclusion is positive. The limiting case $k = 0$ corresponds to an insulating inclusion, which is more adequately modeled by the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \qquad \frac{\partial}{\partial\nu}u = 0 \quad \text{on } \partial\mathscr{D}\,, \qquad \frac{\partial}{\partial\nu}u = f \quad \text{on } \partial\Omega\,. \tag{5.1}$$

If $\mathscr{D}$ is an admissible polygon, then (5.1) has a unique weak solution in

$$u \in H^1_\diamond(\Omega\backslash\overline{\mathscr{D}}) \;=\; \Big\{\, u \in H^1(\Omega\backslash\overline{\mathscr{D}}) \,:\, \int_{\partial\Omega} u \,\mathrm{d}s = 0 \Big\}\,,$$

and we now use

$$\Lambda_f \;:\; \mathscr{D} \;\mapsto\; u|_{\partial\Omega} \;\in\; L^2_\diamond(\partial\Omega)$$

as the associated forward operator. Recall from the introduction that $\mathscr{D}$ is uniquely determined by $\Lambda_f(\mathscr{D})$, cf. [13].

Again, Λ_f turns out to be shape differentiable for polygonal inclusions, cf. [16]: For $h \in \mathcal{S}^2_{\partial\mathscr{D}}$ the shape derivative $\partial\Lambda_f(\mathscr{D})h$ is given by the trace on $\partial\Omega$ of the solution u' of the Neumann boundary value problem

$$\begin{gathered} \Delta u' \;=\; 0 \quad \text{in } \Omega\backslash\overline{\mathscr{D}}\,, \qquad \int_{\partial\Omega} u' \,\mathrm{d}s \;=\; 0\,, \\ \frac{\partial}{\partial\nu}u' \;=\; \frac{\partial}{\partial\tau}\Big((h\cdot\nu)\frac{\partial}{\partial\tau}u\Big) \quad \text{on } \partial\mathscr{D}\,, \qquad \frac{\partial}{\partial\nu}u' \;=\; 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{5.2}$$

Note that u is defined only in the exterior of $\mathscr{D}$, but using the reflection principle it can be extended as a harmonic function across any edge of $\mathscr{D}$ into the interior of $\mathscr{D}$. Therefore u is smooth on $\partial\mathscr{D}$, except for the vertices of $\mathscr{D}$.

We now establish injectivity of $\partial\Lambda_f(\mathscr{D})$.

THEOREM 5.1. *Let $\mathscr{D}$ be an admissible polygon and $f \in L^2_\diamond(\partial\Omega)$ be a nontrivial probing current. Then $\partial\Lambda_f(\mathscr{D})$ is injective.*

Proof. Again we assume to the contrary that there exists some nontrivial $h \in \mathcal{S}^2_{\partial\mathscr{D}}$, for which $\partial\Lambda_f(\mathscr{D})h = 0$. Further, as in the proof of Theorem 3.1, we stipulate without loss of generality that

$$h(x)\cdot\nu(x) \;=\; 1 + b\,|x - x_1|\,, \qquad x \in \Gamma_2\,, \tag{5.3}$$

for some $b \in \mathbb{R}$. Throughout we use the same notation for the vertices, edges, and interior angles of $\mathscr{D}$ as in Section 2, and introduce the same local coordinates (2.1) near the vertices.

Let u' be the solution of (5.2) for this particular h. Since $u'|_{\partial\Omega} = \partial\Lambda_f(\mathscr{D})h = 0$ it follows from Holmgren's theorem that u' is vanishing in all of $\Omega\backslash\overline{\mathscr{D}}$. Therefore, the Neumann boundary condition for u' on $\partial\mathscr{D}$ in (5.2) must be zero, i.e., there is some constant $c \in \mathbb{R}$, such that

$$(h \cdot \nu) \frac{\partial}{\partial \tau} u \;=\; c \qquad \text{on } \Gamma_2 \,.$$

It thus follows from (5.3) that

$$u(x) \;=\; \begin{cases} u(x_1) \;+\; c\,|x - x_1|\,, & b = 0\,, \\ u(x_1) \;+\; \dfrac{c}{b}\log\bigl|1 + b\,|x - x_1|\bigr|\,, & b \neq 0\,, \end{cases} \qquad x \in \Gamma_2\,. \tag{5.4}$$

Together with the insulating boundary condition $\partial u/\partial\nu = 0$ on Γ_2 this yields a Cauchy problem for u with real analytic data on Γ_2, which has a unique solution in $\Omega\backslash\overline{\mathscr{D}}$. Consider first the case $b = 0$. In this case the solution of the Cauchy problem is obviously given by

$$u(x) \;=\; u(x_1) + c\,r\cos\theta\,, \qquad 0 < r < r_0\,,\ \alpha_1 \leqslant \theta \leqslant 2\pi\,,$$

in the local coordinate system (2.1), and hence,

$$\frac{\partial}{\partial\nu}u(x)\bigg|_{\Gamma_1} \;=\; \frac{1}{r}\frac{\partial}{\partial\theta}u(x)\bigg|_{\theta=\alpha_1} \;=\; -c\sin\alpha_1\,, \qquad 0 < r < r_0\,.$$

But this must be zero in the insulating case, proving that $c = 0$, because $\alpha_1 \notin \{0, \pi, 2\pi\}$.

In the case when $b \neq 0$ in (5.3) we have

$$u(x) \;=\; u(x_1) \;+\; \frac{c}{b}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\,b^j|x - x_1|^j$$

according to (5.4) for x near x_1 on Γ_2, and the solution of the Cauchy problem is given by

$$u(x) \;=\; u(x_1) \;+\; c\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}b^{j-1}r^j\cos j\theta$$

in the local coordinate system (2.1) with $0 < r = |x - x_1| < r_0$ and $\alpha_1 \leqslant \theta \leqslant 2\pi$, and hence,

$$0 \;=\; \frac{\partial}{\partial\nu}u(x)\bigg|_{\Gamma_1} \;=\; c\sum_{j=0}^{\infty}(-1)^{j+1}b^jr^j\sin(j+1)\alpha_1$$

for $0 < |x - x_0| < r_0$. Again this is only possible if $c = 0$.

In either case we have shown that $c = 0$ in (5.4), which implies that u is constant in $\Omega\backslash\overline{\mathscr{D}}$. But this is a contradiction to the assumption that f is a nontrivial boundary current, i.e., that

$$f \;=\; \frac{\partial}{\partial\nu}u\Big|_{\partial\Omega} \;\neq\; 0\,.$$

We have thus established the injectivity of $\partial\Lambda_f(\mathscr{D})$. □

We can now proceed as in Section 4, define for $\delta > 0$ and $n \geqslant 3$ the operator

$$F : \begin{cases} \mathcal{U}_{n,\delta} \to\; L^2_\diamond(\partial\Omega)\,, \\ \quad\;\; \mathbf{x} \mapsto\; \Lambda_f(\mathscr{D})\,, \end{cases}$$

and use results from [16] to show that F belongs to $C^1(\mathcal{U}_{n,\delta}, L^2_\diamond(\partial\Omega))$. Without any change of proof we thus get the following result.

THEOREM 5.2. *Let $\delta > 0$ and $n, N \geqslant 3$. Furthermore, let $\mathcal{A}_{n,\delta}$ and $\mathcal{B}_{N,\delta}$ be defined as in Section 4, and let $f \in L^2_\diamond(\partial\Omega)$ be a nontrivial probing current. Then there are positive Lipschitz constants $\ell'_{n,\delta}$ and $L'_{N,\delta}$, depending only on n (resp. N), δ, Ω, and f, such that*

$$d(\mathscr{D}, \mathscr{D}') \;\leqslant\; \ell'_{n,\delta}\, \|\Lambda_f(\mathscr{D}) - \Lambda_f(\mathscr{D}')\|_{L^2(\partial\Omega)}$$

for every pair of polygons $\mathscr{D}, \mathscr{D}' \in \mathcal{A}_{n,\delta}$, and

$$d_H(\partial\mathscr{D}, \partial\mathscr{D}') \;\leqslant\; L'_{N,\delta}\, \|\Lambda_f(\mathscr{D}) - \Lambda_f(\mathscr{D}')\|_{L^2(\partial\Omega)}$$

for every pair of polygons $\mathscr{D}, \mathscr{D}' \in \mathcal{B}_{N,\delta}$. Here, d is the metric defined in (4.2) and d_H is the Hausdorff metric.

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