Random compressible Euler flows

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We propose a finite volume stochastic collocation method for the random Euler system. We rigorously prove the convergence of random finite volume solutions under the assumption that the discrete differential quotients remain bounded in probability. Convergence analysis combines results on the convergence of a deterministic FV method with stochastic compactness arguments due to Skorokhod and Gyöngy-Krylov.

Keywords: random compressible flows, Euler system of gas dynamics, convergence in probability

1. Introduction

The random compressible Euler equations of gas dynamics arise in many practical applications, such as meteorology, physics, engineering or medicine. In practice, model data are typically uncertain since they arise from measurements and may be influenced by various errors. Consequently, data uncertainties propagate and lead to a random PDE system. Several numerical methods have been developed in the literature to approximate random PDE equations.

The Monte Carlo method is often used, but may be very expensive due to its slow convergence and large number of required samples. Alternatively, stochastic spectral methods, such as the stochastic Galerkin and stochastic collocation methods, are used to approximate random PDE systems efficiently. While the stochastic Galerkin method is intrusive, the stochastic collocation method is nonintrusive. It only requires the application of a deterministic numerical scheme at certain collocation nodes. We refer to the monographs by Le Maître and Knio[1], Pettersson et al.[2], Xiu[3], Zhang and Karniadakis[4].

Rigorous convergence analysis of these uncertainty quantification methods typically requires uniqueness and continuous dependence of solutions on random parameters. However, continuity with respect to the random parameter may be a rather strong assumption for hyperbolic problems, and we only require Borel measurability of the data $\to$ solution mapping. Convergence of the stochastic collocation method for random elliptic and parabolic equations was studied, e.g., in Babuška et al.[5], Nobile et al.[6], Tang and Zhou[7].

In this paper, we study a stochastic collocation finite volume method applied to the *random compressible Euler system.* As shown in our recent work Chertock et al.[8] global statistical spectral methods may not be suitable for random hyperbolic conservation laws since discontinuities usually propagates also in the random direction. Global interpolation methods then yield oscillations on discontinuities due to the Gibbs phenomenon. Therefore, we use a stochastic collocation method that works with a piecewise continuous approximation in the deterministic and the random space. We aim to rigorously prove the convergence of the stochastic collocation finite volume method. To this end, we combine deterministic analysis of a finite volume method with stochastic compactness arguments. We refer to our recent work Feireisl and Lukáčová[9], where similar arguments have been used for the random Navier-Stokes equations, see also Feireisl et al.[10,11] for the error analysis of the Monte Carlo finite volume method.

We start with a deterministic model. The Euler equations of gas dynamics describe the conservation of mass, momentum and energy

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\mathbf{m}) &= 0, \\ \partial_t \mathbf{m} + \operatorname{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla p &= 0, \\ \partial_t E + \operatorname{div}((E+p)\mathbf{u}) &= 0. \end{aligned} \tag{1}$$

The dependent variables ϱ, $\mathbf{m}$ and E denote the density, momentum and energy of the fluid, respectively. The pressure p, temperature ϑ, velocity $\mathbf{u}$, internal energy e and entropy s are given by

$$p = \varrho\vartheta = (\gamma-1)\varrho e, \ \mathbf{u} = \mathbf{m}/\varrho, \ e = \varrho^{-1}(E - |\mathbf{m}|^2/(2\varrho)), \ s = \log(\vartheta^{c_v}/\varrho).$$

The adiabatic coefficient is $\gamma \in (1, \infty)$ and the heat capacity at constant volume is $c_v = (\gamma-1)^{-1}$. System (1) is considered on a time-space cylinder

$(0,T) \times Q$, where $T > 0$ is a final time and $Q \subset \mathbb{R}^d$, $d = 2, 3$, a physical domain. System (1) is equipped with the initial data $(\varrho_0, \mathbf{m}_0, E_0)$ and impermeability boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $(0,T) \times \partial Q$.

We say that $(\varrho,\, \mathbf{m},\, E) \in C^1([0,T] \times \overline{Q}, \mathbb{R}^{d+2})$ is a classical solution of (1) if (1) is satisfied pointwise. We will work with the following notion of *generalized weak solution.*

Definition 1.1 (Weak solution). *We call a tuple* $(\varrho,\, \mathbf{m},\, E) \in L^\infty((0,T); L^\gamma(Q) \times L^{2\gamma/(\gamma+1)}(Q;\, \mathbb{R}^d) \times L^1(Q))$ *a weak solution of (1) with initial data* $(\varrho_0,\, \mathbf{m}_0,\, E_0) \in L^\gamma(Q) \times L^{2\gamma/(\gamma+1)}(Q;\, \mathbb{R}^d) \times L^1(Q)$

$$\varrho_0 > 0,\;\; E_0 - \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} > 0 \;\; \textit{for a.a. } \mathbf{x} \in Q$$

if

$$\varrho > 0,\;\; E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} > 0 \;\; \textit{a.e. in } [0,T] \times Q$$

and

- *for every* $\varphi \in C^\infty([0,T] \times \overline{Q})$ *and every* $\tau \in [0,T]$
$$\left[\int_Q \varrho \varphi \,\mathrm{d}\mathbf{x}\right]_{t=0}^{t=\tau} = \int_{[0,\tau]\times Q} \varrho\, \partial_t \varphi + \mathbf{m} \cdot \nabla \varphi \;\mathrm{d}(t,\mathbf{x});$$
- *for every* $\boldsymbol{\varphi} \in C^\infty([0,T] \times \overline{Q}, \mathbb{R}^d)$ *with* $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ *a.e. on* $[0,T] \times \partial Q$ *and every* $\tau \in [0,T]$
$$\left[\int_Q \mathbf{m} \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}\right]_{t=0}^{t=\tau} = \int_{[0,\tau]\times Q} \mathbf{m}\, \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla \boldsymbol{\varphi} + p \,\mathrm{div}_x \,\boldsymbol{\varphi} \;\mathrm{d}(t,\mathbf{x}),$$
where $p = p(\varrho, \mathbf{m}, E) = (\gamma - 1)\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right)$*;*
- *for every* $\varphi \in C^\infty([0,T] \times \overline{Q})$ *with* $\varphi \geq 0$*, any* $\chi \in C^1(\mathbb{R})$ *nondecreasing, concave and bounded from above and every* $\tau \in [0,T]$
$$\left[\int_Q \varrho \chi(s) \varphi \,\mathrm{d}\mathbf{x}\right]_{t=0}^{t=\tau} \geq \int_{[0,\tau]\times Q} \varrho \chi(s)\, \partial_t \varphi + \varrho \chi(s) \mathbf{u} \cdot \nabla \varphi \;\mathrm{d}(t,\mathbf{x}),$$
where $s = s(\varrho, \mathbf{m}, E) = \frac{1}{\gamma - 1} \log\left((\gamma - 1)\frac{1}{\varrho}\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right)\right) - \log(\varrho)$*;*

- *for every* $\tau \in [0, T]$

$$\left[\int_Q E \, \mathrm{d}\mathbf{x}\right]_{t=0}^{t=\tau} \leq 0.$$

Note that this notion of weak solution matches the consistency formulation of the viscous finite volume scheme, cf.[12] Theorem 10.3. A Lipschitz continuous tuple $(\varrho, \mathbf{m}, E) \colon [0, T] \times \overline{Q} \to \mathbb{R}^{d+2}$ which satisfies (1) pointwise almost everywhere is called a *strong solution* of (1). It is straightforward to check that every Lipschitz continuous weak solution with strictly positive density and temperature is in fact a strong solution. Moreover, the strong solution is unique.

Theorem 1.1 (Uniqueness of strong solutions). *If* $(\varrho_1, \mathbf{m}_1, E_1)$ *and* $(\varrho_2, \mathbf{m}_2, E_2)$ *are strong solutions of (1) with the same initial data, then* $(\varrho_1, \mathbf{m}_1, E_1) = (\varrho_2, \mathbf{m}_2, E_2)$.

Proof. Only straightforward changes of the proof of Theorem 6.2 in Ref. 12 are necessary. Note that strong solutions satisfy the equations almost everywhere. Thus, the class of admissible test functions in the weak formulation is large enough to enable testing with another strong solution. Therefore we are not restricted to domains Q with boundary of class C^2. □

2. Discretization of the deterministic problem

We proceed with the description of a numerical method for the deterministic Euler equations (1) and assume that the domain $Q \subset \mathbb{R}^d$ is open, connected, bounded and has a Lipschitz boundary.

2.1. *The viscous finite volume method*

For the space discretization, we use the numerical scheme introduced in Ref. 13, now referred to as *viscous finite volume (VFV) method*[12]. The following standard notation is used.

We will consider a sequence of meshes with increasing mesh resolution. For simplicity, we assume that each mesh consists of shape-regular triangles in 2D and tetrahedra in 3d. We roughly follow the space discretization presented in[14] Section 7.2.1. The set of all elements (triangles or tetrahedra) of the n-th triangulation is denoted by $\mathcal{T}_n$. For $K \in \mathcal{T}_n$ the set of boundary elements of K is denoted $\mathcal{E}$(K). The set of all boundary elements of the n-th mesh is denoted as $\mathcal{E}_n = \cup\{\mathcal{E}(K) \mid K \in \mathcal{T}_n\}$. The exterior boundary elements are given by $\mathcal{E}_n^{\text{ext}} = \{\sigma \in \mathcal{E}_n \mid \sigma \subset \partial Q\}$. Likewise, the set of all interior boundary elements is given by $\mathcal{E}_n^{\text{int}} = \mathcal{E}_n \setminus \mathcal{E}_n^{\text{ext}}$. We assume that there are no hanging nodes, i.e. for each $\sigma \in \mathcal{E}_n^{\text{int}}$ there are exactly two elements $K, L \in \mathcal{T}_n$ such that $\sigma \in \mathcal{E}(K) \cap \mathcal{E}(L)$. Each boundary element $\sigma \in \mathcal{E}_n^{\text{int}}$ is associated with an arbitrary fixed normal vector $\mathbf{n} = \mathbf{n}_\sigma$. For $\sigma \in \mathcal{E}_n^{\text{ext}}$ we fix $\mathbf{n}$ to denote the outer normal unit vector of Q.

Further, we assume that there exist constants $0 < c_1 \le C_1$, $0 < c_2 \le C_2$ such that for all $K \in \mathcal{T}_n$, $\sigma \in \mathcal{E}_n, n \in \mathbb{N}$

$$c_1 h_n^d \le |K| \le C_1 h_n^d, \qquad c_2 h_n^{d-1} \le |\sigma| \le C_2 h_n^{d-1}.$$

Here, $h_n \coloneqq \max_{K \in \mathcal{T}_n} \operatorname{diam}(K)$ is assumed to vanish for $n \to \infty$. We also assume that computational domain $Q_n = (\cup \mathcal{T}_n)^\circ$ coincides with the physical domain Q for all $n \in \mathbb{N}$. Additionally, following[12] Definition 1, we suppose that there exist control points $\mathbf{x}_K$ associated to each $K \in \mathcal{T}_n$ such that for any $\sigma = K \cap L \in \mathcal{E}_n^{\text{int}}$, $K, L \in \mathcal{T}_n$, $\mathbf{x}_K - \mathbf{x}_L = \alpha \mathbf{n}_\sigma$ with $\alpha \in \mathbb{R}$.

We denote by $Q_n(Q, \mathbb{R}^k)$ the corresponding set of all piecewise constant functions and set $Q_n(Q) \coloneqq Q_n(Q, \mathbb{R})$. We define the value of $\mathbf{a}_n \in \mathcal{Q}(Q; \mathbb{R}^k)$ on $K \in \mathcal{T}$ by piecewise constant projection

$$\mathbf{a}_n^K = \frac{1}{|K|} \int_K \mathbf{a}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Further, for $\sigma \in \mathcal{E}_n$ and $\mathbf{a}_n \in Q_n(Q, \mathbb{R}^k)$ we define the average and jump of $\mathbf{a}_n$ in $\mathbf{x} \in \sigma \in \mathcal{E}_n$ by

$$\{\!\{\mathbf{a}_n\}\!\} = \frac{1}{2} \left(\mathbf{a}_n^{\text{out}} + \mathbf{a}_n^{\text{in}} \right), \qquad [\![\mathbf{a}_n]\!] = \mathbf{a}_n^{\text{out}} - \mathbf{a}_n^{\text{in}},$$

respectively. Here $\mathbf{a}_n^{\text{out,in}}$ are the outward/inner limits in the normal direction $\mathbf{n}_\sigma$. The test functions in the momentum equation will be taken from the following subspace of piecewise constant functions

$$\mathcal{D}_n(\mathbf{m}) \coloneqq \left\{ \boldsymbol{\varphi} \in Q_n(Q, \mathbb{R}^d) \mid \boldsymbol{\varphi}^{\text{in}} \cdot \mathbf{n}_\sigma = 0 \text{ on all } \sigma \in \mathcal{E}_n^{\text{ext}} \right\}.$$

The discretization of the convective terms is based on the upwind flux

$$\mathrm{UP}[\mathbf{a}_n, \mathbf{u}_n] \coloneqq \mathbf{a}_n^{\text{in}} [\{\!\{\mathbf{u}_n\}\!\} \cdot \mathbf{n}]^+ + \mathbf{a}_n^{\text{out}} [\{\!\{\mathbf{u}_n\}\!\} \cdot \mathbf{n}]^-.$$

Here, $[\cdot]^+$, $[\cdot]^-$ denote the positive and negative parts, respectively. The numerical flux function for the convective terms is augmented by an additional artificial viscosity term,

$$F_n^{\text{up}}[a_n, \mathbf{u}_n] \coloneqq \mathrm{UP}[a_n, \mathbf{u}_n] - h_n^\epsilon \, [\![a_n]\!], \qquad \epsilon > -1.$$

For a given $r \in Q_n$ we define $\widetilde{[\![r_n]\!]} := r_n^{\mathrm{up}} - r_n^{\mathrm{down}}$, where r_n^{up}, r_n^{down} are the upwind, downwind values of r^n on the cell interface σ. There exists $M \geq 0$ such that for all nonnegative $r \in \mathcal{Q}_n$

$$h \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{r\}\!\} \,\mathrm{d}S(\mathbf{x}) \leq M \int_Q r \,\mathrm{d}\mathbf{x}. \tag{2}$$

The semidiscrete VFV method[12,13] can be formulated in the following way

$$\int_Q \partial_t \varrho_n \varphi \,\mathrm{d}\mathbf{x} - \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma F_n^{\mathrm{up}}[\varrho_n, \mathbf{u}_n] \,[\![\varphi]\!] \,\mathrm{d}S = 0,$$

for every $\varphi \in \mathcal{Q}_n(Q)$;

$$\int_Q \partial_t \mathbf{m}_n \cdot \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} - \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma F_n^{\mathrm{up}}[\mathbf{m}_n, \mathbf{u}_n] \cdot [\![\boldsymbol{\varphi}]\!] \,\mathrm{d}S$$
$$+ \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma h^{\alpha-1} [\![\mathbf{u}_n]\!] \cdot [\![\boldsymbol{\varphi}]\!] \,\mathrm{d}S - \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{p_n\}\!\} \mathbf{n} \cdot [\![\boldsymbol{\varphi}]\!] \,\mathrm{d}S = 0,$$

for every $\boldsymbol{\varphi} \in \mathcal{D}_n(\mathbf{m})$, $\quad \mathbf{m}_n^{\mathrm{in}} \cdot \mathbf{n}_\sigma = 0 \quad$ for all $\sigma \in \mathcal{E}_n^{\mathrm{ext}}$;

$$\int_Q \partial_t E_n \varphi \,\mathrm{d}\mathbf{x} - \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma F_n^{\mathrm{up}}[E_n, \mathbf{u}_n] \,[\![\varphi]\!] \,\mathrm{d}S + h^{\alpha-1} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \left[\!\!\left[\frac{\mathbf{u}_n^2}{2} \right]\!\!\right] [\![\varphi]\!] \,\mathrm{d}S$$
$$- \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \Big(\{\!\{p_n\}\!\} [\![\mathbf{u}_n \varphi]\!] - \{\!\{p_n \varphi\}\!\} [\![\mathbf{u}_n]\!] \Big) \cdot \mathbf{n} \,\mathrm{d}S = 0$$

for every $\varphi \in \mathcal{Q}_n(Q)$. (3)

We will require the following regularity of the discrete initial data.

Definition 2.1 (Admissible deterministic discrete initial data). *We say that the initial data* $(\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}) \in \mathcal{Q}_n([0,T] \times Q; \mathbb{R}^{d+2})$ *are admissible if the following holds:*

(i) $\liminf_n \operatorname{ess\,inf}_{\mathbf{x} \in Q} \varrho_{0,n}(\mathbf{x}) > 0$ *and* $E_{0,n}(\mathbf{x}) > |\mathbf{m}_{0,n}(\mathbf{x})|^2/(2\varrho_{0,n}(\mathbf{x}))$ *for almost all* $\mathbf{x} \in Q$ *and for all* $n \in \mathbb{N}$,
(ii) $\limsup_n \operatorname{ess\,sup}_{x \in Q} E_{0,n}(\mathbf{x}) - |\mathbf{m}_{0,n}(\mathbf{x})|^2/(2\varrho_{0,n}(\mathbf{x})) < \infty$,
(iii) *There exists a constant* $\underline{s} \in \mathbb{R}$ *such that* $\liminf_n \operatorname{ess\,inf}_{\mathbf{x}} s_{0,n}(\mathbf{x}) \geq \underline{s}$,
(iv) $(\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}) \to (\varrho_0, \mathbf{m}_0, E_0)$ *in* $L^\gamma(Q) \times L^{\frac{2\gamma}{\gamma+1}}(Q; \mathbb{R}^d) \times L^1(Q)$ *for* $n \to \infty$.

We sum up several properties of the viscous finite volume scheme proven in Chapter 10 of Ref. 12. For convenience, we denote the initial energy by $\mathcal{E}_{0,n} := \int_Q E_{0,n}(\mathbf{x})\,\mathrm{d}\mathbf{x}$.

Theorem 2.1. *If the initial data $(\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n})$ are admissible then the VFV method (3) has the following properties:*

(i) (Existence of approximate solutions) There exist $(\varrho_n, \mathbf{m}_n, E_n)$ satisfying (3) for all $t \in [0, \infty)$;
(ii) (Positivity of the density and temperature) $\varrho_n, \vartheta_n > 0$;
(iii) (Minimal entropy principle) $s_n(t) \geq \underline{s}$;
(iv) (Stability)

(a) $\|\varrho_n\|_{L^\infty([0,T];\, L^\gamma(Q))} \leq (\gamma - 1) \exp\Big(-(\gamma-1) \inf_Q s_0 \Big) \mathcal{E}_{0,n}$,
(b) $\|\mathbf{m}_n\|_{L^\infty([0,T];\, L^{2\gamma/(\gamma+1)}(Q))} \leq \sqrt[\gamma+1]{(\gamma-1)\exp(-(\gamma-1)\underline{s})}\,\mathcal{E}_{0,n}$,
(c) $\|E_n\|_{L^\infty([0,T];L^1(Q))} = \mathcal{E}_{0,n}$.

Proof. Results (i), (ii) are presented in Lemma 10.3 of Ref. 12. The proof of the minimal entropy principle (iii) follows from the calculations leading to equation (10.39) of Ref. 12. The stability estimates (iv) follow from (10.40), (10.41) and (10.18) of Ref. 12, respectively. □

As shown in Theorem 10.3 of Ref. 12, the VFV method is consistent. This means that numerical solutions satisfy the generalized weak form of the Euler system up to the so-called *consistency error*, which vanishes in the limit $n \to \infty$.

2.2. *Conditional regularity of the viscous finite volume method*

It is well known that for hyperbolic conservation laws equipped with a convex entropy, strong solutions exist locally for sufficiently smooth initial data. Moreover, conditional regularity results are known which guarantee global existence of strong solutions provided derivatives of the solutions remain bounded in suitable norms, see, e.g., Ref. 15. In this section, we prove an analogous result on the discrete level for the VFV method. Instead of bounds on derivatives, we will assume bounds on discrete derivatives of the form $\limsup_n |\llbracket a_n \rrbracket / h| < \infty$ for specific choices of a_n. We omit a more precise notation of jump $\llbracket a_n \rrbracket_\sigma$ and use $\llbracket a_n \rrbracket$ for simplicity. More precisely, we will assume that the following three assumptions hold.

Assumption 2.1.

$$\limsup_{n} \max_{\sigma\in\mathcal{E}_n^{int}} |\,[\![\varrho_n]\!]\,/h_n| < \infty$$

Assumption 2.2. *One of the following statements is true.*

(i) $\limsup_n \max_{\sigma\in\mathcal{E}_n^{int}} |\,[\![\mathbf{u}_n]\!]\,/h_n| < \infty$.
(ii) *There exists a constant* $\underline{\varrho} > 0$ *such that* $\varrho_n \geq \underline{\varrho}$ *and* $\limsup_n \max_{\sigma\in\mathcal{E}_n^{int}} |\,[\![\mathbf{m}_n]\!]\,/h_n| < \infty$.

Assumption 2.3. *One of the following statements is true.*

(i) $\limsup_n \max_{\sigma\in\mathcal{E}_n^{int}} |\,[\![\vartheta_n]\!]\,/h_n| < \infty$.
(ii) $\limsup_n \max_{\sigma\in\mathcal{E}_n^{int}} |\,[\![E_n]\!]\,/h_n| < \infty$.

Under the above assumptions, using suitable parameters α, ϵ, and starting from suitable initial data, we show that the VFV approximations converge strongly to a strong solution. In particular, the strong solution exists as long as the above assumptions are satisfied on the discrete level for all $n \in \mathbb{N}$.

Theorem 2.2. *Let* $\{\varrho_n, \mathbf{m}_n, E_n\}_{n=1}^{\infty}$ *be a sequence of VFV approximations with* $0 < \alpha < 4/3$ *and* $\epsilon > -1$ *satisfying assumptions (2.1) - (2.3) and*

(i) $h_n \to 0$ *for* $n \to \infty$,
(ii) *the discrete initial data* $(\varrho_{n,0}, \mathbf{m}_{n,0}, E_{n,0})$ *are admissible in the sense of Definition 2.1;*

Then $(\varrho_n, \mathbf{m}_n, E_n) \to (\varrho, \mathbf{m}, E)$ *in* $L^q((0,T); L^{\gamma}(Q)\times L^{\frac{2\gamma}{\gamma+1}}(Q;\mathbb{R}^d)\times L^1(Q))$, *where* $1 \leq q < \infty$ *and* $(\varrho, \mathbf{m}, E)$ *is a strong solution of (1).*

The rest of this section deals with the proof of Theorem 2.2. Therefore, from now on until the end of this section $(\varrho_n, \mathbf{m}_n, E_n)$ denotes a sequence of VFV approximations with $0 < \alpha < 4/3$ and $\epsilon > -1$. We assume that assumptions (2.1) - (2.3) and (i), (ii) from Theorem 2.2 hold.

2.2.1. *A priori bounds*

We show suitable a priori estimates and formulate the following auxiliary result.

Lemma 2.1. *There exists* $C_B > 0$ *independent of* n *and* $\mathbf{x} \in Q$ *such that the number of elements* $K \in \mathcal{T}_n$ *with* $B_{h_n}(\mathbf{x}) \cap K \neq \emptyset$ *is bounded by* C_B. *Further, let* $\mathbf{f}_n \in Q_n(Q, \mathbb{R}^k)$ *satisfy* $\max_{\sigma\in\mathcal{E}_n} |\,[\![\mathbf{f}_n]\!]\,| \leq Ch_n$ *for all* n *and* *let L, K be elements of $\mathcal{T}_n$ with $K \cap L \neq \emptyset$. Then there exists $C' > 0$ independent of n such that*

$$|\mathbf{f}_n^K - \mathbf{f}_n^L| \leq C' h_n.$$

Since Q is a bounded domain with Lipschitz boundary, there exists $C_Q \geq 1$ such that for each $\mathbf{x}, \mathbf{y} \in Q$ there is a Lipschitz continuous curve $\gamma \colon [0,1] \to Q$ such that $\gamma(0) = \mathbf{x}$, $\gamma(1) = \mathbf{y}$ and $\operatorname{length}(\gamma) \leq C_Q \|\mathbf{x} - \mathbf{y}\|$. As a straightforward consequence of the previous lemma, we get the following result.

Lemma 2.2. *Let $\mathbf{f}_n \in Q_n(Q, \mathbb{R}^k)$ satisfy $\max_{\sigma \in \mathcal{E}_n} | [\![\mathbf{f}_n]\!] | \leq C h_n$ and let L, K be elements of $\mathcal{T}_n$. Then there exists C'' independent of n such that*

$$|\mathbf{f}_n^K - \mathbf{f}_n^L| \leq C''(\operatorname{dist}(K, L) + h_n).$$

Proof. For $\delta_{\text{dist}} > 0$ there are $\mathbf{x} \in K^\circ$ and $\mathbf{y} \in L^\circ$ with $\|\mathbf{x} - \mathbf{y}\| \leq \operatorname{dist}(K, L) + \delta_{\text{dist}}$. Denote by $\gamma \colon [0,1] \to Q$ a Lipschitz continuous curve with $\gamma(0) = \mathbf{x}$, $\gamma(1) = \mathbf{y}$ and $\operatorname{length}(\gamma) \leq C_Q \|\mathbf{x} - \mathbf{y}\|$. Fixing $\delta_h \in (0, h_n)$ we choose $N_t = 2\lceil \operatorname{length}(\gamma([0,1]))/(h_n - \delta_h) \rceil + 1$ points $0 = t_1 < t_2 < \dots t_{N_t} = 1$ such that $\operatorname{length}(\gamma([t_i, t_{i+1}])) = (h_n - \delta_h)/2$ for all $i = 1, \dots, N_t - 2$ and $\operatorname{length}(\gamma([t_{N_t - 1}, t_{N_t}])) \leq (h_n - \delta_h)/2$. Thus, $B_{h_n}(\gamma(t_{2i}))$, $i = 1, \dots, (N_t - 1)/2$, is an open cover of $\gamma([0,1])$. For each set $B_{h_n}(\gamma(t_{2i}))$ there are at most C_B elements of $\mathcal{T}_n$ which cover $Q \cap B_{h_n}(\gamma(t_{2i}))$. Thus, we find a sequence of $C_1 \leq C_B \lceil \operatorname{length}(\gamma([0,1]))/(h_n - \delta_h) \rceil$ elements $K_1, \dots, K_{C_1}$ such that $K_1 = K$, $K_{C_1} = L$ and $K_i \cap K_{i+1} \neq \emptyset$. Applying the previous lemma with δ_h small enough such that $\lceil \operatorname{length}(\gamma([0,1]))/(h_n - \delta_h) \rceil \leq \operatorname{length}(\gamma([0,1]))/h_n + 1$ and taking the limit $\delta_{\text{dist}} \to 0$ finishes the proof. □

Remark 2.1. In particular, note that (2.2) (ii) implies (2.2) (i), since

$$[\![\mathbf{u}_n]\!] = [\![\mathbf{m}_n]\!] \{\!\{ \varrho_n^{-1} \}\!\} - \xi^{-2} [\![\varrho_n]\!] \{\!\{ m_n \}\!\}$$

with $\xi \in \operatorname{conv}\{\varrho_n^{\text{in}}, \varrho_n^{\text{out}}\}$. The next lemma shows that the reverse implication is true as well.

Lemma 2.3. *If $\max_{\sigma \in \mathcal{E}_n} | [\![\mathbf{u}_n]\!] | \leq C h_n$ for all $n \in \mathbb{N}$, then there exists $\underline{\varrho}$ independent of n such that $\varrho_n \geq \underline{\varrho}$ for n large enough.*

To prove the above lemma, we will use the following auxiliary lemma that is a slight modification of Lemma 10.2 of Ref. 12.

Lemma 2.4. *Let $r_n\colon [0,T] \to \mathcal{Q}_n$ satisfy $r_n(0) \geq 0$, the bound $\sup_{t,\mathbf{x}} |r_n(t,\mathbf{x})| < \infty$ and for all $\varphi_n \in \mathcal{Q}_n, \varphi_n \geq 0$ holds*

$$\frac{d}{d\,t}\int_Q r_n \phi_n \,\mathrm{d}\mathbf{x} - \sum_{\sigma\in\mathcal{E}_{int}} \int_\sigma F_n^{up}[r_n,\, \mathbf{u}_n]\, [\![\varphi_n]\!]\, \mathrm{d}S(\mathbf{x}) \geq 0. \tag{4}$$

Then $r_n(t) \geq 0$ for all $t \in [0,T]$.

Poof of Lemma 2.3. As we assume in (ii) that $\liminf_n \varrho_{0,n} \geq \underline{\varrho}_0 > 0$, there exists $\underline{\varrho}(0) \in \mathbb{R}_{>0}$ such that $\varrho_n(0,\mathbf{x}) \geq \underline{\varrho}(0)$ for all $\mathbf{x} \in \Omega$ and n sufficiently large. We define for $t > 0$

$$\underline{\varrho}(t) \coloneqq \underline{\varrho}(0)\exp\left(-Lt\right)$$

with $L \geq C\,M$. Using (2), we derive the following inequality for arbitrary nonnegative $\varphi_n \in \mathcal{Q}_h(\Omega)$

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \underline{\varrho}\varphi_n \,\mathrm{d}\mathbf{x} - \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma F_h^{\mathrm{up}}[\underline{\varrho}, \mathbf{u}_n]\, [\![\varphi_n]\!]\, \mathrm{d}S(\mathbf{x}) \\
&= -L\underline{\varrho}\int_\Omega \varphi_n \,\mathrm{d}\mathbf{x} - \underline{\varrho}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{\mathbf{u}_n\}\!\}\cdot\mathbf{n}\, [\![\varphi_n]\!]\, \mathrm{d}S(\mathbf{x}) \\
&= -L\underline{\varrho}\int_\Omega \varphi_n \,\mathrm{d}\mathbf{x} + \underline{\varrho}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma [\![\mathbf{u}_n]\!]\cdot\mathbf{n}\{\!\{\varphi_n\}\!\}\, \mathrm{d}S(\mathbf{x}) \\
&\leq \underline{\varrho}\int_\Omega \varphi_n \,\mathrm{d}\mathbf{x}\,(-L + CM) \leq 0.
\end{aligned}$$

Choosing $r_n = \varrho_n - \underline{\varrho}$ in Lemma 2.4, we derive that $\varrho_n(t) \geq \underline{\varrho} \equiv \underline{\varrho}(T) > 0$ for all $t \in [0,T]$. ☐

Consequently, we get a uniform lower bound on ϑ_n for n large enough due to Theorem 2.1 (iii). Additionally, the bound of $\varrho\mathbf{u}^2$, cf. Theorem 2.1(iv)(c) and assumption (2.2), yield uniform boundedness of $u_n(t)$ in $L^\infty(Q)$.

With very similar arguments, it is straightforward to show an upper bound for the internal energy.

Lemma 2.5. *Under the assumptions of Theorem 2.2 there exists $\overline{\varrho e}$ such that $\varrho_n e(\varrho_n, \mathbf{m}_n, E_n) \leq \overline{\varrho e}$ for almost all $(t,\mathbf{x})$ and n sufficiently large.*

In particular, Theorem 2.1 (iii) implies that there also exist a uniform upper bound on ϱ_n for n sufficiently large.

2.2.2. *Lipschitz continuous approximation of numerical solutions*

Next, we show that there exists a sequence of uniformly Lipschitz continuous approximations of the numerical solutions. To this end, we first note that the cell center values $(\varrho_n^K(t), \mathbf{m}_n^K(t), E_n^K(t))$, $K \in \mathcal{T}_n$, are uniformly Lipschitz continuous in time with respect to both K and n.

Lemma 2.6 (Lipschitz continuity in time). *If assumptions 2.1, 2.2 and 2.3 are satisfied then $t \mapsto \varrho_n^K(t)$, $t \mapsto \mathbf{m}_n^K(t)$ and $t \mapsto E_n^K(t)$ are globally Lipschitz continuous uniformly in $K \in \mathcal{T}_n$ and $n \in \mathbb{N}$.*

Proof. By definition, $\{\varrho_n^K, \mathbf{m}_n^K, E_n^K\}_{K\in\mathcal{T}_n}$ form the solution of the system of ordinary differential equations (3). To prove uniform Lipschitz continuity, it is sufficient to bound the right hand side of the respective ordinary differential equation for each cell average $\{\varrho_n^K, \mathbf{m}_n^K, E_n^K\}_{K\in\mathcal{T}_n}$ in $L^\infty([0,T])$ for all $K \in \mathcal{T}_n$.

As the jumps of all quantities are bounded by $C\, h_n$ for some positive constant $C > 0$ independent of n and K, all terms containing a jump are bounded. For the other terms we note that

$$
\begin{aligned}
&\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \{\!\{a_n\}\!\}\{\!\{\mathbf{b}_n\}\!\} \cdot \mathbf{n}_K \,\mathrm{d}S(\mathbf{x}) \\
&= \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \left(a_n^{\mathrm{in}} + \frac{1}{2}[\![a_n]\!]\right)\left(\mathbf{b}_n^{\mathrm{in}} + \frac{1}{2}[\![b_n]\!]\right) \cdot \mathbf{n}_K \,\mathrm{d}S(\mathbf{x}) \\
&= \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma a_n^{\mathrm{in}}\mathbf{b}_n^{\mathrm{in}} \cdot \mathbf{n}_K \,\mathrm{d}S(\mathbf{x}) + \mathcal{O}(h_n^d) = \mathcal{O}(h_n^d),
\end{aligned}
$$

if both $\max_{\sigma\in\mathcal{E}_n} |[\![a_n]\!]| = \mathcal{O}(h_n)$ and $\max_{\sigma\in\mathcal{E}_n} |[\![b_n]\!]| = \mathcal{O}(h_n)$ for $a_n \in \mathcal{Q}_h(Q)$ and $\mathbf{b}_n \in \mathcal{Q}_h(Q;\mathbb{R}^d)$ with $\|a_n\|_{L^\infty}$ and $\|\mathbf{b}_n\|_{L^\infty}$ bounded uniformly in n. And similarly,

$$
\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \{\!\{a_n\}\!\}\mathbf{b} \cdot \mathbf{n}_K \,\mathrm{d}S(\mathbf{x}) = \mathcal{O}(h_n^d) = \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma a\{\!\{\mathbf{b}_n\}\!\} \cdot \mathbf{n}_K \,\mathrm{d}S(\mathbf{x})
$$

for $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. Therefore $\int_K \frac{d}{dt} f_n \,\mathrm{d}x = \mathcal{O}(h_n^d)$ for $f \in \{\varrho, \mathbf{m}, E\}$. □

Next, for fixed $t \in [0,T]$, we approximate the numerical solutions by uniformly Lipschitz continuous functions in space such that the approximation error vanishes in $L^\infty(Q)$ for $n \to \infty$.

Lemma 2.7. *Let $\{\mathbf{f}_n\}_{n\in\mathbb{N}}$ denote a sequence of functions $\mathbf{f}_n \in \mathcal{Q}_n(Q,\mathbb{R}^k)$ with $\max_{\sigma\in\mathcal{E}_n} |[\![\mathbf{f}_n]\!]| \le Ch_n$ for some $C > 0$. Then there exists a sequence*

$\hat{\mathbf{f}}_n$ *of uniformly Lipschitz continuous functions* $\hat{\mathbf{f}}_n \in C^{0,1}(Q, \mathbb{R}^k)$ *satisfying* $\|\mathbf{f}_n - \hat{\mathbf{f}}_n\|_{L^\infty} \to 0$ *for* $n \to \infty$.

Proof. We may assume $k = 1$ without loss of generality. For $\mathbf{x} \in \mathcal{E}_n^{(0)}$ we define $\hat{f}_n(\mathbf{x})$ as the average of f_n^K for all K with $\mathbf{x} \in K$, i.e.

$$\hat{f}_n(\mathbf{x}) = (\#\{K \in \mathcal{T}_n \, \mathbf{x} \in K\})^{-1} \sum_{K \in \mathcal{T}_n, \, \mathbf{x} \in K} f_n^K.$$

Note that $|\mathbf{f}_n^K - \mathbf{f}_n^L| \leq C\, C_B^2 h_n$ for all K, L with $\mathbf{x} \in K \cap L$ due to Lemma 2.1. In particular,

$$|\hat{\mathbf{f}}_n(\mathbf{x}) - \mathbf{f}_n^K| \leq C_B^2 C h_n \qquad \text{for all } K \text{ with } \mathbf{x} \in K.$$

Fixing $K = \operatorname{conv}\{\mathbf{x}_0, \dots, \mathbf{x}_d\} \in \mathcal{T}_n$, we define $\hat{f}_n(\mathbf{x})$ for $\mathbf{x} = \sum_i \lambda_i \mathbf{x}_i \in K$ by $\hat{f}_n(\sum_i \lambda_i \mathbf{x}_i) = \sum_i \lambda_i f_n(\mathbf{x}_i)$. We may assume $\mathbf{x}_0 = \mathbf{a}_K$ and $\mathbf{x}_i = h_n \mathbf{A}_K \mathbf{e}_i + \mathbf{a}_K$ for $i = 1, \dots, d$. In particular,

$$\hat{f}_n(\mathbf{x})|_K = h_n^{-1}(\hat{f}_n(\mathbf{x}_1) - \hat{f}_n(\mathbf{x}_0), \dots, \hat{f}_n(\mathbf{x}_d) - \hat{f}_n(\mathbf{x}_0)) A_K^{-1}(\mathbf{x} - \mathbf{x}_0) + f_n(\mathbf{x}_0).$$

Thus, for each $\mathbf{x} \in Q$ there exists a neighbourhood U such that $\hat{f}_n|_U$ is Lipschitz continuous with Lipschitz constant less than or equal to $2dCC_B^2/c_T$. Here, $c_T^2 \leq \lambda$ for all eigenvalues λ of $\mathbf{A}_K^T \mathbf{A}_K$ for all $K \in \mathcal{T}_n$, $n \in \mathbb{N}$. In particular, $\hat{f}_n$ is Lipschitz continuous on $\overline{Q}$ with Lipschitz constant less than or equal to $2dCC_Q C_B^2/c_T$, independent of n. □

In summary, the above two Lemmas imply that on $[0,T] \times \overline{Q}$ there exist a sequence of uniform globally Lipschitz continuous approximations of the VFV solutions such that the approximation error vanishes as $n \to \infty$.

2.2.3. *Proof of Theorem 2.2*

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Due to Lemmas 2.3 and 2.5, ϱ_n is bounded away from 0 and ϑ_n is bounded from above. Moreover, according to Lemmas 2.6 and 2.7 there exits a sequence of uniformly Lipschitz continuous functions $\{\hat{\varrho}_n, \, \hat{\mathbf{m}}_n, \, \hat{E}_n\}_{n=1}^{\infty}$ with

$$\varrho_n - \hat{\varrho}_n \to 0, \qquad \mathbf{m}_n - \hat{\mathbf{m}}_n \to 0, \qquad E_n - \hat{E}_n \to 0 \qquad \text{uniformly in } [0,T] \times \overline{Q}$$

for $n \to \infty$. Further, Lemma 2.2 implies the boundedness of $(\hat{\varrho}_n, \, \hat{\mathbf{m}}_n, \, \hat{E}_n)$. Thus, by the Arzelà-Ascoli theorem, after possibly passing to a subsequence and relabeling the sequence, $(\hat{\varrho}_n, \, \hat{\mathbf{m}}_n, \, \hat{E}_n)$ converges uniformly to the Lipschitz function $(\varrho, \mathbf{m}, E) \in \operatorname{Lip}([0,T] \times \overline{Q}; \mathbb{R}^{d+2})$. Note that this convergence is pointwise a.e. and thus, the nonlinear terms in the momentum and energy equations converge as well. We will show that $(\varrho, \mathbf{m}, E)$ is a strong solution of (1) proving that passing to subsequences is not necessary. Choosing $\chi(x) = \min\{x, a\}$ for $a \in \mathbb{R}$ large enough, Theorem 10.3 of Ref. 12 implies that $(\varrho, \mathbf{m}, E)$ is a Lipschitz continuous weak solution of the Euler system with density bounded away from 0 and initial data $(\varrho_0, \mathbf{m}_0, E_0)$. Lemma 1.1 implies that the limit is unique and there is no need to take subsequences.

□

3. The stochastic collocation viscous finite volume scheme

Let $[\Omega, \mathcal{B}(\Omega), \mathbb{P}]$ be a complete probability space with Ω a compact metric space of samples, $\mathcal{B}(\Omega)$ σ-algebra of Borel subsets of Ω, and $\mathbb{P}$ a complete Borel probability measure on Ω. Let us denote by D the set of data $D \equiv \{(\varrho, \mathbf{m}, E) \in L^\gamma(Q;\mathbb{R}) \times L^{\frac{2\gamma}{\gamma+1}}(Q;\mathbb{R}^d) \times L^1(Q;\mathbb{R}) | \inf_{x\in Q} \varrho > 0, \inf_{x\in Q} E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} > 0\}$. We assume that the randomness is enforced through the random initial data. This means that the mapping

$$(\varrho_0, \mathbf{m}_0, E_0)(\omega) \colon \Omega \longrightarrow D$$

is Borel measurable for a.a. $\omega \in \Omega$.

To approximate the random Euler system, we divide Ω into a sequence of finite partitions with shrinking maximal diameter. Having a piecewise constant deterministic approximation on Q, we also apply a piecewise constant approximation in the random space on Ω. Other choices are possible, and we refer to Ref. 8, where a higher order WENO approximation on Ω was successfully applied for hyperbolic conservation laws. The approximate random solutions are defined in the following way.

Definition 3.1 (Approximate random solution). *Given a partition $\{\Omega_m^M\}_{m=1}^{\nu(M)}$ of Ω and a set of collocation nodes $\omega_m^M \in \Omega_m^M$ we define the approximate random solution of the random Euler equations as*

$$\begin{aligned}
\varrho_n^M(t, \mathbf{x}, \omega) &= \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega) \varrho_n(t, \mathbf{x}, \omega_m) \\
\mathbf{m}_n^M(t, \mathbf{x}, \omega) &= \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega) \mathbf{m}_n(t, \mathbf{x}, \omega_m) \\
E_n^M(t, \mathbf{x}, \omega) &= \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega) E_n(t, \mathbf{x}, \omega_m) \qquad \textit{for a.a. } \omega \in \Omega,
\end{aligned} \tag{5}$$

where $(\varrho_n(t,\mathbf{x},\omega_m),\mathbf{m}_n(t,\mathbf{x},\omega_m),E_n(t,\mathbf{x},\omega_m))$ *denote the (deterministic) solution of the VFV method corresponding to the mesh* $\mathcal{T}_n$ *evaluated at* (t,x) *with the initial data* $(\varrho_0(\cdot,\omega_m),\mathbf{m}_0(\cdot,\omega_m),E_0(\cdot,\omega_m))$.

To guarantee the convergence of $(\varrho_n^M(0,\cdot,\cdot),\mathbf{m}_n^M(0,\cdot,\cdot),E_n^M(0,\cdot,\cdot))$ in the limit as $M\to\infty, n\to\infty$ we require that the initial data are admissible.

Definition 3.2 (Admissible random initial data). *The initial data* $(\varrho_0,\,\mathbf{m}_0,\,E_0)\colon\Omega\to D$ *are admissible, if*

(i) there exist a.e. continuous functions $\underline{\varrho}\colon\Omega\to\mathbb{R}_{>0}$ *and* $\overline{e}\colon\Omega\to\mathbb{R}_{>0}$ *and* $\underline{s}\colon\Omega\to\mathbb{R}$ *such that for all* $\omega\in\Omega$ *and* $\mathbf{x}\in Q$

$$\operatorname*{ess\,inf}_{\mathbf{y}}\varrho_0(\mathbf{y},\omega)>\underline{\varrho}(\omega),\qquad \operatorname*{ess\,sup}_{\mathbf{y}}E_0(\mathbf{y},\omega)<\overline{E}(\omega),$$
$$e_0(\mathbf{x},\omega)>0,\qquad \operatorname*{ess\,inf}_{\mathbf{y}}s_0(\mathbf{y},\omega)>\underline{s}(\omega);$$

(ii) there exists a.e. continuous functions $r_\varrho,\,r_{\mathbf{m}},\,r_S\colon\Omega\to\mathbb{R}_{\geq0}$ *such that*

$$\|\varrho_0(\cdot,\omega)\|_{L^\gamma}\leq r_\varrho\,,\quad \|\mathbf{m}_0(\cdot,\omega)\|_{L^{2\gamma/(\gamma+1)}}\leq r_\varrho\,,\quad \|S_0(\cdot,\omega)\|_{L^\gamma}\leq r_S\,,$$

where $S_0=\varrho_0s_0,\ s_0=s(\varrho_0,\mathbf{m}_0,E_0)$*;*

(iii) the following maps from Ω *to* $\mathbb{R}$ *a.e. continuous:*

$$\begin{aligned}
F_{\varrho,\varphi}\colon\omega\in\Omega&\mapsto\int_Q\varrho_0(\mathbf{x},\omega)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}, &&\varphi\in C_c^\infty(Q),\\
F_{\mathbf{m},\varphi}\colon\omega\in\Omega&\mapsto\int_Q\mathbf{m}_0(\mathbf{x},\omega)\cdot\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}, &&\boldsymbol{\varphi}\in C_c^\infty(Q;\mathbb{R}^d),\\
F_{S,\varphi}\colon\omega\in\Omega&\mapsto\int_Q S_0(\mathbf{x},\omega)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}, &&\varphi\in C_c^\infty(Q),\\
F_E\colon\omega\in\Omega&\mapsto\int_Q E_0(\mathbf{x},\omega)\,\mathrm{d}\mathbf{x}.
\end{aligned}\tag{6}$$

We present the following result, which strengthens the convergence of the piecewise constant discretizations of the initial data.

Lemma 3.1. *Let* $(\varrho_0, \mathbf{m}_0, E_0)$ *be admissible random initial data. Then*

$$\sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)\varrho_0(\cdot,\omega_m) \to \varrho_0(\omega) \text{ in } L^\gamma(Q;\mathbb{R}),$$
$$\sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)\mathbf{m}_0(\cdot,\omega_m) \to \mathbf{m}_0(\omega) \text{ in } L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d),$$
$$\sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)E_0(\cdot,\omega_m) \to E_0(\omega) \text{ in } L^1(Q;\mathbb{R}) \quad \text{as } M\to\infty, \mathbb{P}-a.s. \tag{7}$$

Proof. Let $\{\varphi_k\}_{k=1}^\infty \subset C_c^\infty(Q)$ and $\{\boldsymbol{\varphi}_k\}_{k=1}^\infty \subset C_c^\infty(Q;\mathbb{R}^d)$ denote dense sequences in the respective space of smooth functions. As the initial data are admissible, there exists a set $\mathcal{N}\subset\Omega$ with $\mathbb{P}(\mathcal{N})=0$ such that the maps in (6) are continuous in each point of the complement $\mathcal{N}^c$ of $\mathcal{N}$ for test functions φ_k and $\boldsymbol{\varphi}_k$. Clearly, in every point of continuity ω of $F\colon\Omega\to\mathbb{R}$,

$$\sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m^M}(\omega)F(\omega_m^M) \to F(\omega).$$

This implies that for $M\to\infty$ we have

$$\int_Q \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)\varrho_0(\mathbf{x},\omega_m)\varphi_k(\mathbf{x})\,\mathrm{d}\mathbf{x} \to \int_Q \varrho_0(\omega)\varphi_k(\mathbf{x})\,\mathrm{d}\mathbf{x},$$
$$\int_Q \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)\mathbf{m}_0(\mathbf{x},\omega_m)\cdot\boldsymbol{\varphi}_k(\mathbf{x})\,\mathrm{d}\mathbf{x} \to \int_Q \mathbf{m}_0(\omega)\cdot\boldsymbol{\varphi}_k(\mathbf{x})\,\mathrm{d}\mathbf{x},$$
$$\int_Q \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)S_0(\mathbf{x},\omega_m)\varphi_k(\mathbf{x})\,\mathrm{d}\mathbf{x} \to \int_Q S_0(\omega)\varphi_k(\mathbf{x})\,\mathrm{d}\mathbf{x},$$
$$\int_Q \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)E_0(\mathbf{x},\omega_m)\,\mathrm{d}\mathbf{x} \to \int_Q E_0(\mathbf{x},\omega)\,\mathrm{d}\mathbf{x}$$

for all $\omega\in\mathcal{N}^c$.

Now, let us use the following notation

$$f_0^M(\omega) = \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega)f_0(\cdot,\omega_m) \tag{8}$$

for any $f_0\in\{\varrho_0,\mathbf{m}_0,E_0,S_0\}$. The uniform bounds of $\{\varrho_0^M,\mathbf{m}_0^M,S_0^M\}_{M=1}^\infty$ in $L^\gamma(Q)\times L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d)\times L^\gamma(Q)$ imply weak convergence of $(\varrho_0^M,\mathbf{m}_0^M,S_0^M)$ as $M\to\infty$. Following similar arguments as in Section 6 of

Ref. 16 we apply the convexity arguments for $E(\varrho_0, \mathbf{m}_0, S_0)$ and obtain the strong convergence claimed in (7). □

Lemma 3.1 yields $\mathbb{P}$-a.s. convergence of discrete admissible random initial data $(\varrho_0^M, \mathbf{m}_0^M, S_0^M)$ as $M \to \infty$. To discretize the initial data on Q, we apply the projection operator

$$\Pi_n \colon L^p(Q) \to L^p(Q), \qquad f \mapsto \sum_{K \in \mathcal{T}_n} \frac{\mathbb{1}_K}{|K|} \int_K f(\mathbf{y}) \,\mathrm{d}\mathbf{y}.$$

Since the projection error is controlled in the L^p-norm, we also get a.e. convergence in Q of fully discretized initial data. Choosing the space discretization parameter $n = n(M)$, such that $n \to \infty$ as $M \to \infty$, we now denote for any $f_0 \in \{\varrho_0, \mathbf{m}_0, E_0, S_0\}$ its piecewise constant projection by

$$f_{0,n}^M(\omega) = \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega) \Pi_n f_0(\cdot, \omega_m).$$

Lemma 3.2. *Let $(\varrho_0,\, \mathbf{m}_0,\, E_0)$ be admissible random initial data. Then there exists a Borel measurable map*

$$\omega \in \Omega \mapsto (\hat{\varrho}_0(\omega),\, \hat{\mathbf{m}}_0(\omega),\, \hat{E}_0(\omega)) \in D$$

such that $(\hat{\varrho}_0,\, \hat{\mathbf{m}}_0\, \hat{E}_0)$ are Lipschitz continuous on Q

$$\varrho_0 = \hat{\varrho}_0, \quad \mathbf{m}_0 = \hat{\mathbf{m}}_0, \quad E_0 = \hat{E}_0 \qquad \mathbb{P} - a.s.$$

and

$$\begin{aligned} \varrho_{0,n}^M(\omega) \to \hat{\varrho}_0(\omega) \text{ in } L^\gamma(Q), \quad \mathbf{m}_{0,n}^M(\omega) \to \hat{\mathbf{m}}_0(\omega) \text{ in } L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d), \\ E_{0,n}^M(\omega) \to \hat{E}_0(\omega) \text{ in } L^1(Q) \quad \text{ as } M, n = n(M) \to \infty \qquad \mathbb{P} - a.s. \end{aligned} \tag{9}$$

Further, for all $n \in \mathbb{N}$, $M \in \mathbb{N}$, $m \leq \nu(M)$ and all $\omega \in \Omega_m^M$

$$\begin{aligned} &\operatorname*{ess\,inf}_{\mathbf{x}\in Q} \varrho_{0,n}^M(\mathbf{x},\omega) \geq \underline{\varrho}(\omega_m^M) > 0, \qquad \operatorname*{ess\,sup}_{\mathbf{y}} E_{0,n}^M(\mathbf{y},\omega) \leq \overline{E}(\omega), \\ &\operatorname*{ess\,inf}_{\mathbf{x}\in Q} e_{0,n}^M(\mathbf{x},\omega) > 0, \qquad \operatorname*{ess\,inf}_{\mathbf{y}} s_{0,n}^M(\mathbf{y},\omega) > \underline{s}(\omega_m^M). \end{aligned} \tag{10}$$

Proof. As the pointwise limit of a Borel measurable map ranging in a metric space is again Borel measurable, it suffices to prove (9) with ϱ_0, $\mathbf{m}_0$, E_0 instead of it Lipschitz continuous representation $\hat{\varrho}_0$, $\hat{\mathbf{m}}_0$, $\hat{E}_0$. This will then immediately imply the existence of $\hat{\varrho}_0$, $\hat{\mathbf{m}}_0$, $\hat{E}_0$.

To show (9) let us consider $f_0 \in \{\varrho_0, \mathbf{m}_0, E_0\}$. We fix an arbitrary positive $\varepsilon > 0$ and choose $g \in C_c^\infty(Q)$ with $\|f_0 - g\|_{L^p} \leq \varepsilon$ for the corresponding $p \in \{\gamma, 2\gamma/(\gamma+1), 1\}$. Thus, using the notation from (8) and Jensen's inequality we derive

$$\begin{aligned}
&\|f_{0,n}^M - f_0\|_{L^p} \leq \|\Pi_n[f_0^M - f_0]\|_{L^p(Q)} + \|\Pi_n[f_0 - g]\|_{L^p(Q)} \\
&+ \|\Pi_n[g] - g\|_{L^p(Q)} + \|g - f_0\|_{L^p(Q)} \\
&\leq \|f_0^M - f_0\|_{L^p} + 2\|g - f_0\|_{L^p} + \|\Pi_n[g] - g\|_{L^p}
\end{aligned}$$

with the last term vanishing for $n(M) \to \infty$ due to the smoothness of g.

For every point of continuity ω of $\underline{\varrho}$, $\overline{e}$ and $\underline{s}$, the properties listed in (10) follow from the respective properties of admissible random initial data. The positivity of $\varrho_{0,n}^M$ and $\vartheta_{0,n}^M$ follows immediately from the properties of admissible random initial data. Likewise, for every point of continuity ω of $\underline{\varrho}$, $\overline{\varrho}$ and $\underline{s}$ the properties listed in (10) follow from the respective properties of admissible random initial data. Here, the lower bound on the density and upper bound on the energy are obvious. For the positivity of the initial internal energy and for the lower bound on the initial entropy, we use Jensen's inequality. More precisely, as $(\varrho, \mathbf{m}) \mapsto \mathbf{m}^2/\varrho$ is convex on $\mathbb{R}^d \times (0, \infty)$, we find that

$$\begin{aligned}
&\varrho_{0,n}^M(\omega, \cdot) e_{0,n}^M(\omega, \cdot) \\
&= \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega) \Big(\Pi_n[E_0(\omega_m)] - \Pi_n[\mathbf{m}_0(\omega_m)]^2 / \Pi_n[\varrho_0(\omega_m)] \Big) \\
&\geq \sum_{m=1}^{\nu(M)} \mathbb{1}_{\Omega_m}(\omega) \Big(\Pi_n[E_0(\omega_m)] - \Pi_n\Big[\mathbf{m}_0(\omega_m)^2 / \varrho_0(\omega_m)\Big] \Big) > 0.
\end{aligned}$$

Similarly, as $\varrho \mapsto \varrho^\gamma$ is convex and $(\varrho, \mathbf{m}, E) \mapsto E - \mathbf{m}^2/\varrho$ is concave,

$$\begin{aligned}
&\Pi_n\Big[\varrho_0(\omega_m)\Big]^\gamma \leq \Pi_n\Big[\varrho_0(\omega_m)^\gamma\Big] \\
&\leq \Pi_n\Big[(\gamma - 1)\exp(-\underline{s}/c_v)(E_0(\omega_m) - \mathbf{m}_0(\omega_m)^2/\varrho_0(\omega_m))\Big] \\
&\leq (\gamma - 1)\exp(-\underline{s}/c_v)\Big(\Pi_n[E_0(\omega_m)] - \Pi_n[\mathbf{m}_0(\omega_m)]^2/\Pi_n[\varrho_0(\omega_m)]\Big).
\end{aligned}$$

Thus

$$s_{0,n}^M(\omega, \cdot) = c_v \log\left((\gamma - 1) \frac{\varrho_{0,n}^M(\omega, \cdot) e_{0,n}^M(\omega, \cdot)}{\varrho_{0,n}^M(\omega, \cdot)^\gamma} \right) \geq \underline{s}(\omega_m)$$

for m with $\omega \in \Omega_m^M$. Using the a.e. continuity of $\underline{\varrho}$, $\overline{\varrho}$ and $\underline{s}$ in ω we conclude the statement of the lemma. □

3.1. *Main result: convergence*

This section is devoted to our main result: convergence of the stochastic collocation VFV method applied to the random Euler equations. Our working hypothesis will be that Assumptions 2.1 - 2.3 hold in probability.

Theorem 3.1 (Convergence in probability). *Let the initial data* $(\varrho_0, \mathbf{m}_0, E_0)$ *be admissible random data. Let the family of sets* $(\Omega_m^M)_{m=1,M\in\mathbb{N}}^{\nu(M)}$ *satisfy* $\Omega_m^M \cap \Omega_\ell^M = \emptyset \ \ \forall \ell \neq m$,

$$\max_{m=1,\dots,\nu(M)} \mathrm{diam}(\Omega_m^M) \to 0 \text{ as } M \to \infty, \qquad \bigcup_{m=1}^{\nu(M)} \Omega_m^M = \Omega.$$

Let $h_n > 0$ *and* $h_n \to 0$ *for* $n = n(M) \to \infty$. *Finally, let the solutions* $\{\varrho_{n(M)}^M, \mathbf{m}_{n(M)}^M, E_{n(M)}^M\}_{M=1}^{\infty}$ *obtained by the stochastic collocation VFV method* (5) *have bounded discrete gradients in probability. More precisely, this means that for each* $\varepsilon > 0$ *there exists* $N = N(\varepsilon)$ *such that*

$$\limsup_{M,n(M)\to\infty} \mathbb{P}\left(\max_{\sigma\in\mathcal{E}^{int}} \max\left\{ \left| [\![\varrho_n^M]\!] \right|, \| [\![\mathbf{u}_n^M]\!] \|, \left| [\![E_n^M]\!] \right| \right\} / h_n \geq N \right) \leq \varepsilon. \tag{11}$$

Then for $M \to \infty, n(M) \to \infty$

$$(\varrho_{n(M)}^M, \mathbf{m}_{n(M)}^M, E_{n(M)}^M) \to (\varrho, \mathbf{m}, E) \text{ strongly in}$$
$$L^q((0,T); L^\gamma(Q) \times L^{\frac{2\gamma}{\gamma+1}}(Q;\mathbb{R}^d) \times L^1(Q)), \text{ in probability, } 1 \leq q < \infty.$$

Proof. For $K > 0$ large enough, there is a compact embedding from $W_0^{K,2}$ to $C(\overline{(0,T)\times Q})$, see e.g. Theorem 4.12 of Ref. 17. Schauder's theorem implies that the adjoint of the above embedding is compact as well. Since Theorem 2.1 (iv) implies boundedness in probability of the sequence of approximate solutions in $L^q((0,T); L^\gamma(Q) \times L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d) \times L^1(Q))$, the sequence is tight in $W^{-K,2}((0,T)\times Q;\mathbb{R}^{d+2})$ for K enough large.

Further, defining

$$C(\omega) = \max_{\sigma\in\mathcal{E}^{\rm int}} \max\left\{ \left| \left[\!\left[\varrho_{n(M)}^M\right]\!\right] \right|, \quad \left\| \left[\!\left[\mathbf{u}_{n(M)}^M\right]\!\right] \right\|, \quad \left| \left[\!\left[E_{n(M)}^M\right]\!\right] \right| \right\} / h_n,$$

assumption (11) implies tightness of C in $\mathbb{R}$. Finally, by Lemma 3.1, assumptions on admissible initial data imply convergence of $(\varrho_{0,n}^M, \mathbf{m}_{0,n}^M, E_{0,n}^M)$ in $L^\gamma(Q) \times L^{2\gamma/(\gamma+1)}(Q) \times L^1(Q)$ as $M, n(M) \to \infty$ $\mathbb{P}$-a.s. Consequently, the initial data also converge in law and the application of the Prokhorov theorem, Theorem 5.2 of Ref. 18, yields tightness of the initial data.

Let us define the following separable Banach space (i.e. in particular a Polish space)

$$X = (L^\gamma(Q))^3 \times \left(L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d)\right)^3 \times \left(L^1(Q)\right)^3 \times \left(W^{-K,2}((0,T)\times Q)\right)^2 \\ \times \left(W^{-K,2}((0,T)\times Q;\mathbb{R}^d)\right)^2 \times \left(W^{-K,2}((0,T)\times Q)\right)^2 \times \mathbb{R}.$$

Given two arbitrary but fixed subsequences $\{M_{1,k}\}_{k\in\mathbb{N}}$ and $\{M_{2,k}\}_{k\in\mathbb{N}}$ of $\{M\}_{M\in\mathbb{N}}$, we define the sequence $\{U_k\}_{k\in\mathbb{N}}$ ranging in X by

$$U_k = \Big(\varrho_0,\, \varrho_0^{M_{1,k}},\, \varrho_0^{M_{2,k}},\, \mathbf{m}_0,\, \mathbf{m}_0^{M_{1,k}},\, \mathbf{m}_0^{M_{2,k}},\, E_0,\, E_0^{M_{1,k}},\, E_0^{M_{2,k}}, \\ \varrho^{M_{1,k}}_{n(M_{1,k})},\, \varrho^{M_{2,k}}_{n(M_{2,k})},\, \mathbf{m}^{M_{1,k}}_{n(M_{1,k})},\, \mathbf{m}^{M_{2,k}}_{n(M_{2,k})},\, E^{M_{1,k}}_{n(M_{1,k})},\, E^{M_{2,k}}_{n(M_{2,k})},\, C\Big).$$

The Prohorov theorem implies convergence in law of $\{U_k\}_{k\in\mathbb{N}}$ in X. Using the Skorokhod representation theorem, Section 3.1.1 of Ref. 19, possibly going to a subsequence and relabeling the sequence, we find that there exist $\tilde{U}_k\colon [0,1]\to X$ and $\tilde{U}\colon [0,1]\to X$,

$$\tilde{U}_k = \Big(\tilde{\varrho}_0^{0,k},\, \tilde{\varrho}_0^{1,k},\, \tilde{\varrho}_0^{2,k},\, \tilde{\mathbf{m}}_0^{0,k},\, \tilde{\mathbf{m}}_0^{1,k},\, \tilde{\mathbf{m}}_0^{2,k},\, \tilde{E}_0^{0,k},\, \tilde{E}_0^{1,k},\, \tilde{E}_0^{2,k}, \\ \tilde{\varrho}^{1,k},\, \tilde{\varrho}^{2,k},\, \tilde{\mathbf{m}}^{1,k},\, \tilde{\mathbf{m}}^{2,k},\, \tilde{E}^{1,k},\, \tilde{E}^{2,k},\, \tilde{C}^k\Big),$$

$$\tilde{U} = \Big(\tilde{\varrho}_0^0,\; \tilde{\varrho}_0^1,\; \tilde{\varrho}_0^2,\; \tilde{\mathbf{m}}_0^0,\; \tilde{\mathbf{m}}_0^1,\; \tilde{\mathbf{m}}_0^2,\; \tilde{E}_0^0,\; \tilde{E}_0^1,\; \tilde{E}_0^2, \\ \tilde{\varrho}^1,\; \tilde{\varrho}^2,\; \tilde{\mathbf{m}}^1,\; \tilde{\mathbf{m}}^2,\; \tilde{E}^1,\; \tilde{E}^2,\; \tilde{C}\Big),$$

such that $\tilde{U}_k(\tilde{\omega}) \to \tilde{U}(\tilde{\omega})$ in X as $k\to\infty$ for a.a. $\tilde{\omega}\in[0,1]$ and

$$\mathbb{P}(U_k \in A) = \lambda(\tilde{U}_k \in A) \qquad \text{for all } k\in\mathbb{N},\, A\in\mathcal{B}(X). \tag{12}$$

Note that in the original sequence U_k, the initial data ϱ_0, $\mathbf{m}_0$, E_0 and C did not depend on k. The corresponding entries

$$\tilde{\varrho}_0^{0,k},\, \tilde{\mathbf{m}}_0^{0,k},\, \tilde{E}_0^{0,k},\, \tilde{C}^k$$

of $\tilde{U}_k$ however might depend on k and need not be equal to the corresponding entries of $\tilde{U}$. The transformation of, e.g., ϱ_0 to the map $\tilde{\varrho}_0^{0,k}$ from $[0,1]$ to $L^\gamma(Q)$ might be different for each k.

Still, using (10) we find that $(\tilde{\varrho}_0^\ell, \tilde{\mathbf{m}}_0^\ell, \tilde{E}_0^\ell)$, $\ell=1,2$, are elements of the set

$$\{(\varrho,\mathbf{m},E)\colon [0,1]\to L^\gamma(Q)\times L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d)\times L^1(Q) \text{ Borel measurable} \mid \\ \operatorname*{ess\,inf}_Q \varrho > 0,\, \operatorname*{ess\,inf}_Q p(\varrho,\mathbf{m},E) \geq 0,\, \operatorname*{ess\,sup}_Q p(\varrho,\mathbf{m},E) < \infty, \\ \operatorname*{ess\,inf}_Q s(\varrho,\mathbf{m},E) \geq -\infty\}.$$

Thus, for a.a. $\omega \in [0,1]$, the initial data $(\tilde{\varrho}_0^{\ell,k}(\omega), \tilde{\mathbf{m}}_0^{\ell,k}(\omega), \tilde{E}_0^{\ell,k}(\omega))$ are admissible deterministic initial data for $\ell = 1, 2$.

For $i \in \{1, \dots, \nu(M_{1,k})\}$, $j \in \{1, \dots, \nu(M_{2,k})\}$ let $\omega \in \Omega_i^{M_{1,k}} \cap \Omega_j^{M_{2,k}}$. Clearly, for every fixed ω the set $A \subset X$ given by,

$$A \coloneqq \Big\{ \Big(x,\, \varrho_0^{M_{1,k}}(\omega),\, \varrho_0^{M_{2,k}}(\omega),\, y,\, \mathbf{m}_0^{M_{1,k}}(\omega),\, \mathbf{m}_0^{M_{2,k}}(\omega),\, z,\, E_0^{M_{1,k}}(\omega),$$
$$E_0^{M_{2,k}}(\omega),\, \varrho_{n(M_{1,k})}^{M_{1,k}}(\omega),\, \varrho_{n(M_{2,k})}^{M_{2,k}}(\omega),\, \mathbf{m}_{n(M_{1,k})}^{M_{1,k}}(\omega),\, \mathbf{m}_{n(M_{2,k})}^{M_{2,k}}(\omega),\, E_{n(M_{1,k})}^{M_{1,k}}(\omega),$$
$$E_{n(M_{2,k})}^{M_{2,k}}(\omega),\, C(\omega)\Big) \mid x \in L^\gamma(Q),\, y \in L^{2\gamma/(\gamma+1)}(Q),\, z \in L^1(Q) \Big\},$$

is Borel measurable and thus there exists $\tilde{\Omega}_{i,j}^k = (\tilde{U}^k)^{-1}(A) \subset [0,1)$ with $\lambda(\tilde{\Omega}_{i,j}^k) = \mathbb{P}(U_k \in \Omega_i^{M_{1,k}} \cap \Omega_j^{M_{2,k}})$. In particular, all components of $\tilde{U}_k$ other than $\tilde{\varrho}_0^{0,k}$, $\tilde{\mathbf{m}}_0^{0,k}$, $\tilde{E}_0^{0,k}$ are a.e. equal to piecewise constant functions. As a consequence, $(\tilde{\varrho}^{1,k}, \tilde{\mathbf{m}}^{1,k}, \tilde{E}^{1,k})$ satisfy the numerical scheme (3) for the initial data $(\tilde{\varrho}_0^{1,k}, \tilde{\mathbf{m}}_0^{1,k}, \tilde{E}_0^{1,k})$ a.e. in $[0,1]$. The analogous statement is true for $(\tilde{\varrho}^{2,k}, \tilde{\mathbf{m}}^{2,k}, \tilde{E}^{2,k})$.

Now, we fix $\tilde{\omega} \in [0,1]$ such that both $(\tilde{\varrho}^{1,k}, \tilde{\mathbf{m}}^{1,k}, \tilde{E}^{1,k})(\tilde{\omega})$ and $(\tilde{\varrho}^{2,k}, \tilde{\mathbf{m}}^{2,k}, \tilde{E}^{2,k})(\tilde{\omega})$ are solutions of the numerical scheme (3) and such that $\tilde{U}_k(\tilde{\omega}) \to \tilde{U}(\tilde{\omega})$ in X. As a consequence $(\tilde{\varrho}_0^{1,k}(\tilde{\omega}), \tilde{\mathbf{m}}_0^{1,k}(\tilde{\omega}), \tilde{E}_0^{1,k}(\tilde{\omega}))$ and $(\tilde{\varrho}_0^{2,k}(\tilde{\omega}), \tilde{\mathbf{m}}_0^{2,k}(\tilde{\omega}), \tilde{E}_0^{2,k}(\tilde{\omega}))$ converge in $L^\gamma(Q) \times L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d) \times L^1(Q)$. Moreover, $\{\tilde{C}^k(\tilde{\omega})\}_{k\in\mathbb{N}}$ converges and thus is bounded. Thus, Theorem 2.2 implies that the limits $(\tilde{\varrho}^\ell,\, \tilde{\mathbf{m}}^\ell,\, \tilde{E}^\ell)$ are strong solutions of the Euler equations corresponding to the initial data $(\tilde{\varrho}_0^\ell,\, \tilde{\mathbf{m}}_0^\ell,\, \tilde{E}_0^\ell)$ for $\ell = 1, 2$. More precisely, for $k \to \infty$

$$(\tilde{\varrho}^{\ell,k}, \tilde{\mathbf{m}}^{\ell,k}, \tilde{E}^{\ell,k})(\tilde{\omega}) \to (\tilde{\varrho}^{\ell}, \tilde{\mathbf{m}}^{\ell}, \tilde{E}^{\ell})(\tilde{\omega})$$
$$\text{in } L^q((0,T); L^\gamma(Q) \times L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d) \times L^1(Q)),\ 1 \le q < \infty.$$

As the strong solution is unique, see Theorem1.1, we only need to show that the initial data $\tilde{f}^1(0,\cdot)$ and $\tilde{f}^2(0,\cdot)$ coincide for $f = \varrho$, $\mathbf{m}$, E to apply the Gyöngy-Krylov lemma, Lemma 1.1 of Ref. 20.

From now on we denote by f_0 either ϱ_0, $\mathbf{m}_0$ or E_0. Note that $f_0^{M_{1,k}}$ and $f_0^{M_{2,k}}$ converge $\mathbb{P}$-a.s. to f_0 in the respective L^p space. In particular, by Egorov's theorem, see Theorem 5.1.4 of Ref. 21, there exist sets $A_k \in \mathcal{B}(\Omega)$ and an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\mathbb{P}(A_k) < 2^{-k}$ and $\|f_0(\omega) - f_0^{M_{n_k,\ell}}(\omega)\|_{L^p} < 1/k$ for all $\omega \in A_k$ and $\ell = 1, 2$. In particular, after possibly going to subsequences and relabeling them, we find that

$$\mathbb{P}\left(\bigcup_{m}\bigcap_{k\geq m}\left\{\|\tilde{f}_0^{\ell,k}-f_0^{0,k}\|<\epsilon\right\}\right)=1,\quad \ell=1,2.$$

Here we have used that the laws of $\tilde{U}_k$ and U_k coincide. Thus, the Gyöngy-Krylov lemma implies that

$$\begin{aligned}&(\varrho^M_{n(M)},\mathbf{m}^M_{n(M)},E^M_{n(M)})\to(\varrho,\mathbf{m},E)\\ &\text{ in } L^q((0,T);L^\gamma(Q)\times L^{2\gamma/(\gamma+1)}(Q;\mathbb{R}^d)\times L^1(Q)),\ 1\leq q<\infty,\end{aligned}$$

for $M\to\infty, n(M)\to\infty$ in probability. Applying again the Skorokhod theorem, but now for the sequence

$$\left\{\varrho_0,\varrho_0^M,\varrho^M_{n(M)},\varrho,\mathbf{m}_0,\mathbf{m}_0^M,\mathbf{m}^M_{n(M)},\mathbf{m},E_0,E_0^M,E^M_{n(M)},E,C\right\}_{M=1}^{\infty}$$

we find that $(\varrho,\mathbf{m},E)$ is the random strong solution of the Euler equations with the initial data $(\varrho_0,\mathbf{m}_0,E_0)$. This concludes the proof. □

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